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A FREE ANALOGUE OF HINČIN'S CHARACTERIZATION OF INFINITE DIVISIBILITY

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ABSTRACT. Hinčin characterized the class of infinitely divisible distributions on the line as the class of all distributional limits of sums of infinitesimal independent random variables. We show that an analogue of this characterization is true in the addition theory of free random variables introduced by Voiculescu.

Denote by $\mathcal{M}$ the collection of all probability measures on the real line $\mathbf{R}$. We denote by $\mu * \nu$ the classical convolution of two measures $\mu, \nu \in \mathcal{M}$, and recall that in probabilistic terms $\mu * \nu$ is the distribution of $X + Y$, where X and Y are real independent random variables with distributions μ and ν, respectively. A measure $\mu \in \mathcal{M}$ is said to be $*$-infinitely divisible if, for every natural number n, μ can be written as

$$\mu = \underbrace{\nu_n * \nu_n * \cdots * \nu_n}_{n \text{ times}},$$

with $\nu_n \in \mathcal{M}$. Hinčin showed that $*$-infinitely divisible measures are the most general distributions that can be obtained as limit laws of sums of uniformly infinitesimal independent random variables. More precisely, assume we are given an array $\{X_{ij} : i \geq 1, 1 \leq j \leq k_i\}$ of real random variables, and a sequence $\{c_i : i \geq 1\}$ of real numbers, such that

(i) $X_{i1}, X_{i2}, \ldots, X_{ik_i}$ are independent for every i;
(ii) $\lim_{i\to\infty} \max_{1\leq j\leq k_i} P(|X_{ij}| > \varepsilon) = 0$ for every $\varepsilon > 0$; and
(iii) the distributions μ_i of $c_i + \sum_{j=1}^{k_i} X_{ij}$ converge to a limit $\mu \in \mathcal{M}$ in the weak topology (i.e., $\lim_{i\to\infty} \int_{-\infty}^{\infty} f(t)\, d\mu_i(t) = \int_{-\infty}^{\infty} f(t)\, d\mu(t)$ for every bounded continuous function f on $\mathbf{R}$).

Then Hinčin proved that the measure μ must be $*$-infinitely divisible. Conversely, every $*$-infinitely divisible measure μ occurs as a weak limit of this type. Indeed, if

$$\mu = \underbrace{\nu_n * \nu_n * \cdots * \nu_n}_{n \text{ times}},$$

one can choose $k_i = i$, $c_i = 0$, and $X_{i1}, X_{i2}, \ldots, X_{ik_i}$ with distribution ν_i.

Now, on $\mathcal{M}$ there is defined another operation, namely Voiculescu's free additive convolution [7]. The free convolution $\mu \boxplus \nu$ is the distribution of $X+Y$, where X and

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Y are freely independent random variables with distributions μ and ν, respectively. As in the classical case, a measure μ is $\boxplus$-infinitely divisible if, for every natural number n, μ can be written as

$$\mu = \underbrace{\nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n}_{n \text{ times}},$$

with $\nu_n \in \mathcal{M}$. These measures were first considered in [8], where the compactly supported $\boxplus$-infinitely divisible measures were characterized. The measures with finite variance were considered in [5], and [3] treats the general case of measures $\mu \in \mathcal{M}$. In [6] $\boxplus$-infinitely divisible measures are characterized as those measures possessing a non-empty domain of partial attraction; the classical version of this result is also due to Hinčin. Finally, in [2] it was shown that there exists a bijection $\mu \leftrightarrow \mu'$ between $*$-infinitely divisible measures μ and $\boxplus$-infinitely divisible measures μ' such that the respective domains of partial attraction coincide.

The main result of this paper is a free analogue of Hinčin's theorem mentioned earlier. We formulate it in terms of measures. We will denote by δ_c the Dirac measure at $c \in \mathbf{R}$.

1. Theorem. *Let $\{\mu_{ij} : i \geq 1, 1 \leq j \leq k_i\}$ be an array of measures in $\mathcal{M}$, and $\{c_i : i \geq 1\}$ a sequence of real numbers such that*

(i) $\lim_{i\to\infty} \max_{1\leq j\leq k_i} \mu_{ij}(\{t : |t| > \varepsilon\}) = 0$ *for every $\varepsilon > 0$; and*
(ii) *the measures $\mu_i = \delta_{c_i} \boxplus \mu_{i1} \boxplus \mu_{i2} \boxplus \cdots \boxplus \mu_{ik_i}$ converge in the weak topology to a measure $\mu \in \mathcal{M}$.*

Then μ is $\boxplus$-infinitely divisible.

The converse of this statement follows as in the classical case. The proof of Hinčin's classical result is achieved in two steps. First, one shows that the measures μ_{ij} can be replaced by $*$-infinitely divisible approximants in such a way that the limit μ remains unchanged. Second, one argues that the class of $*$-infinitely divisible measures is closed under weak limits. The class of $\boxplus$-infinitely divisible measures was shown to be closed under weak limits in [6]. However our approach in this paper is more direct and does not require this result. There is a good candidate for the $\boxplus$-infinitely divisible approximant of μ_{ij}, but this line of thought will be pursued in more detail elsewhere.

The proof of Theorem 1 is based on a method for the calculation of free convolution which was discovered by Voiculescu [8] (cf. also [5] and [3] for the extension of Voiculescu's result which is actually used here). Given a measure $\mu \in \mathcal{M}$, one defines the Cauchy transform

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t}.$$

We will view G_μ as a function defined in the upper half-plane $\mathbf{C}^+$ with values in the lower half-plane $\mathbf{C}^-$. The reciprocal $F_\mu(z) = 1/G_\mu(z)$ maps $\mathbf{C}^+$ to $\mathbf{C}^+$, and $\Im F_\mu(z) \geq \Im z$ for $z \in \mathbf{C}^+$. Moreover, $F_\mu(z)/z$ tends to one as $z \to \infty$ nontangentially to $\mathbf{R}$, i.e., such that $\Re z/\Im z$ stays bounded. As a consequence, for every $\alpha > 0$, F_μ has a right inverse F_μ^{-1} (relative to composition) defined in

$$\Gamma_{\alpha,\beta} = \{z \in \mathbf{C} : |\Re z| < \alpha \Im z, \Im z > \beta\}$$

provided that β is sufficiently large. Moreover, $\Im F_\mu^{-1}(z) \leq \Im z$ for $z \in \Gamma_{\alpha,\beta}$. Thus we can define $\varphi_\mu : \Gamma_{\alpha,\beta} \to \mathbf{C}^- \cup \mathbf{R}$ by the formula

$$\varphi_\mu(z) = F_\mu^{-1}(z) - z, \quad z \in \Gamma_{\alpha,\beta}.$$

Voiculescu's result mentioned above is as follows.

2. Theorem. *For $\mu, \nu \in \mathcal{M}$ we have $\varphi_{\mu \boxplus \nu} = \varphi_\mu + \varphi_\nu$ in any truncated angle $\Gamma_{\alpha,\beta}$ in which the three functions involved are defined.*

The passage from μ to φ_μ has good continuity properties. What we will need here is the following result which follows from [3] (cf. Proposition 5.7).

3. Proposition. *If $\mu_n, \mu \in \mathcal{M}$ are such that $\mu_n \to \mu$ in the weak topology as $n \to \infty$, then there exist $\alpha, \beta > 0$ with the following properties:*

(i) *φ_{μ_n} and φ_μ are defined in $\Gamma_{\alpha,\beta}$; and*
(ii) *$\varphi_{\mu_n}(z) \to \varphi_\mu(z)$ as $n \to \infty$ for every $z \in \Gamma_{\alpha,\beta}$.*

There is also an easy characterization of $\boxplus$-infinitely divisible measures μ in terms of φ_μ. This was given in [8] for compactly supported measures and extended in [5] and [3].

4. Theorem. *A measure $\mu \in \mathcal{M}$ is $\boxplus$-infinitely divisible if and only if φ_μ can be continued analytically to a function $\varphi : \mathbf{C}^+ \to \mathbf{C}^- \cup \mathbf{R}$.*

Let us notice the fact that the function ϕ_μ takes no real values unless it identically equals a real constant τ. This only happens when $\mu(\{\tau\}) = 1$.

We now state the basic ingredient in the proof of Theorem 1.

5. Lemma. *For every $\alpha, \beta > 0$ there exists $\varepsilon > 0$ with the following property. If $\mu \in \mathcal{M}$ is a measure such that $\mu((-\varepsilon, \varepsilon)) > 1 - \varepsilon$, then φ_μ is defined in $\Gamma_{\alpha,\beta}$ and $\varphi_\mu(\Gamma_{\alpha,\beta}) \subset \mathbf{C}^+ \cup \mathbf{R}$.*

Proof. Fix $\varepsilon > 0$ and $z = x + iy \in \mathbf{C}^+$. We have

$$\left| \frac{t}{z - t} \right| \leq \sqrt{1 + \left(\frac{x}{y} \right)^2}, \quad t \in \mathbf{R},$$

with equality for $t = (x^2 + y^2)/x$ if $x \neq 0$, and

$$\left| \frac{t}{z - t} \right| \leq \frac{\varepsilon}{y}, \quad t \in (-\varepsilon, \varepsilon).$$

Thus for every $\mu \in \mathcal{M}$

$$\begin{aligned} |zG_\mu(z) - 1| &= \left| \int_{-\infty}^{\infty} \left(\frac{z}{z - t} - 1 \right) d\mu(t) \right| \\ &\leq \int_{-\infty}^{\infty} \left| \frac{t}{z - t} \right| d\mu(t) \\ &\leq \frac{\varepsilon}{y} + (1 - \mu((-\varepsilon, \varepsilon))) \sqrt{1 + \left(\frac{x}{y} \right)^2}. \end{aligned}$$

If we also have $\mu((-\varepsilon, \varepsilon)) \geq 1 - \varepsilon$, then we conclude that

$$|zG_\mu(z) - 1| \leq \varepsilon \left[\frac{1}{y} + \sqrt{1 + \left(\frac{x}{y} \right)^2} \right].$$

When the right hand side is less than 1/2 we can also make the following estimate:

$$\begin{aligned}\left|\frac{F_\mu(z)}{z}-1\right| &= \left|\frac{zG_\mu(z)-1}{zG_\mu(z)}\right| \\ &\le \frac{|zG_\mu(z)-1|}{1-|zG_\mu(z)-1|} \\ &\le 2\varepsilon\left[\frac{1}{y}+\sqrt{1+\left(\frac{x}{y}\right)^2}\right].\end{aligned}$$

Let us now fix $\alpha_0, \beta_0 > 0$, and assume that $z = x + iy \in \Gamma_{\alpha_0,\beta_0}$. Also choose $\varepsilon > 0$ so that

$$\eta = 2\varepsilon\left(\frac{1}{\beta_0}+\sqrt{1+\alpha_0^2}\right) < 1,$$

and assume that $\mu \in \mathcal{M}$ is such that $\mu((-\varepsilon,\varepsilon)) > 1-\varepsilon$. The preceding estimate now yields

$$\left|\frac{F_\mu(z)}{z}-1\right| \le \eta,$$

and Rouché's theorem implies that F_μ has a right inverse defined in $\Gamma_{\alpha_0-\eta,(1+\eta)\beta_0}$. In other words, φ_μ is defined in $\Gamma_{\alpha_0-\eta,(1+\eta)\beta_0}$. The proof is concluded by noting that, given $\alpha, \beta > 0$, we can choose α_0, β_0 and ε so that $\alpha_0-\eta > \alpha$ and $(1+\eta)\beta_0 < \beta$, and hence $\Gamma_{\alpha_0-\eta,(1+\eta)\beta_0} \supset \Gamma_{\alpha,\beta}$. □

Our main result is now easily derived.

Proof of Theorem 1. Given $\alpha, \beta > 0$ we will show that φ_μ can be continued analytically to $\Gamma_{\alpha,\beta}$, and the continuation takes values in $\mathbf{C}^- \cup \mathbf{R}$. Indeed, choose $\varepsilon > 0$ so that the conclusion of the lemma holds. Assumption (i) of the theorem implies that the function $\varphi_{\mu_i} = c_i + \sum_{j=1}^{k_i} \varphi_{\mu_{ij}}$ is defined in $\Gamma_{\alpha,\beta}$ for i sufficiently large, and φ_{μ_i} maps $\Gamma_{\alpha,\beta}$ to $\mathbf{C}^- \cup \mathbf{R}$. Since $\mathbf{C}^-$ is conformally equivalent to a disk, the family φ_{μ_i} is normal on $\Gamma_{\alpha,\beta}$, and hence there exists a subsequence $\varphi_{\mu_{i_n}}$ which converges pointwise in $\Gamma_{\alpha,\beta}$ to a function φ which is either identically infinite, or analytic with values in $\mathbf{C}^- \cup \mathbf{R}$. Proposition 3 shows that $\varphi_{\mu_{i_n}}(z) \to \varphi_\mu(z)$ for z in some open subset of $\Gamma_{\alpha,\beta}$, and therefore $\varphi(z) = \varphi_\mu(z)$ for such z. It follows that $\varphi : \Gamma_{\alpha,\beta} \to \mathbf{C}^- \cup \mathbf{R}$ is an analytic continuation of φ_μ. Thus φ_μ can be continued analytically to $\bigcup_{\alpha,\beta>0} \Gamma_{\alpha,\beta} = \mathbf{C}^+$, and hence μ is $\boxplus$-infinitely divisible by Theorem 4. □

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