

ON A POLYNOMIAL INEQUALITY OF E. J. REMEZ

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ABSTRACT. We prove a result which extends a well-known polynomial inequality of E. J. Remez and another one due to W. A. Markov.

1. INTRODUCTION

Let \mathcal{P}_n denote the class of all polynomials of degree at most n with real or complex coefficients. Polynomials in \mathcal{P}_n whose coefficients are all real will form the sub-class $\mathcal{P}_{n,\mathbb{R}}$. As usual we shall denote by T_n the n th Chebyshev polynomial of the first kind, which is given by $\cos n \arccos x$ for $-1 \leq x \leq 1$. In particular, $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$ and $T_n(\cos((n-j)\pi/n)) = (-1)^{n-j}$ for $j = 0, 1, \dots, n$. All its zeros are real and lie in the open interval $(-1, 1)$. It was observed by P.L. Chebyshev (see [7] or [9]) that if $f \in \mathcal{P}_n$ and $|f(x)| \leq 1$ for $-1 \leq x \leq 1$, then

$$(1) \quad |f(x)| \leq |T_n(x)| \quad \text{for all } x \in \mathbb{R} \setminus [-1, 1].$$

Subsequently, it was shown by W.A. Markov [5] that under the same condition on f , we have

$$(2) \quad \left| f^{(k)}(x) \right| \leq \left| T_n^{(k)}(x) \right| \quad \text{for all } x \in \mathbb{R} \setminus [-1, 1] \quad \text{and } 1 \leq k \leq n.$$

Now we must introduce a couple of additional notations. We shall write $\mu(\mathfrak{S})$ for the measure of a Lebesgue measurable subset \mathfrak{S} of \mathbb{R} . For *any* polynomial f and any subinterval \mathbb{I} of \mathbb{R} we denote the set $\{x \in \mathbb{I} : |f(x)| \leq 1\}$ by $\mathfrak{E}(f; \mathbb{I})$.

The following generalization of Chebyshev's inequality (1) is due to E.J. Remez (see [1], [2], [3, Lemma 7.3], [8]). The proof in [1] is the simplest.

Theorem A. *For all $g \in \mathcal{P}_n$, the following inequality holds:*

$$(3) \quad \max_{-1 \leq x \leq 1} |g(x)| \leq T_n \left(\frac{4}{\mu(\mathfrak{E}(g; [-1, 1]))} - 1 \right).$$

An equivalent formulation of this result stated below as Theorem A' shows clearly why it contains (1).

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For each $R \geq 1$, let

$$\pi_n(R) := \{f \in \mathcal{P}_n : \mu(\mathfrak{E}(f; [-1, R])) \geq 2\},$$

$$\pi_{n,\mathbb{R}}(R) := \{f \in \mathcal{P}_{n,\mathbb{R}} : \mu(\mathfrak{E}(f; [-1, R])) \geq 2\}.$$

If $f \in \pi_n(R)$ for some $R \geq 1$ and $g(x) := f((R + 1)x/2 + (R - 1)/2)$, then

$$\mu(\mathfrak{E}(g; [-1, 1])) = \frac{2}{R + 1} \mu(\mathfrak{E}(f; [-1, R])) \geq \frac{4}{R + 1}.$$

Hence Theorem A implies that if $f \in \pi_n(R)$, then

$$(4) \quad \max_{-1 \leq x \leq R} |f(x)| = \max_{-1 \leq x \leq 1} |g(x)| \leq T_n(R).$$

Conversely, let (4) hold for all $f \in \pi_n(R)$. If g is any polynomial of degree at most n and $f(x) := g((2x - R + 1)/(R + 1))$, then

$$\mu(\mathfrak{E}(f; [-1, R])) = \frac{R + 1}{2} \mu(\mathfrak{E}(g; [-1, 1])) \geq 2$$

if $R \geq 4/\mu(\mathfrak{E}(g; [-1, 1])) - 1$. Hence, $f \in \pi_n(R)$ for all such values of R and

$$\max_{-1 \leq x \leq 1} |g(x)| = \max_{-1 \leq x \leq R} |f(x)| \leq T_n(R),$$

that is, (3) holds. Thus, Theorem A may be reformulated as follows.

Theorem A'. *If $f \in \pi_n(R)$ for some $R \geq 1$, then*

$$(5) \quad \max_{-1 \leq x \leq R} |f(x)| \leq T_n(R).$$

It was noted by B.D. Bojanov that if $m(t) := \sup\{|f(t)| : f \in \pi_n(R)\}$, then

$$m(t) \leq m(R)$$

for all $t \in (-1, R)$. His argument goes roughly as follows.

Take an arbitrary $t \in (-1, R)$ and any $f \in \pi_n(R)$. If

$$q_1(t) := \frac{\mu(\mathfrak{E}(f; [-1, t]))}{1 + t} \quad \text{and} \quad q_2(t) := \frac{\mu(\mathfrak{E}(f; [t, R]))}{R - t},$$

then

$$\max\{q_1(t), q_2(t)\} \geq \frac{2}{1 + R},$$

since otherwise, we would have

$$\mu(\mathfrak{E}(f; [-1, R])) = \mu(\mathfrak{E}(f; [-1, t])) + \mu(\mathfrak{E}(f; [t, R])) < 2.$$

Now consider the linear transformations

$$\alpha_1(x) := \frac{(1 + t)x - R + t}{1 + R} \quad \text{and} \quad \alpha_2(x) := \frac{(t - R)x + R^2 + t}{1 + R}.$$

It is to be noted that as x increases from -1 to R , the number $\alpha_1(x)$ increases from -1 to t whereas $\alpha_2(x)$ decreases from R to t . Under the first transformation every subinterval of $[-1, R]$ shrinks by the factor $(1 + t)/(1 + R)$; under the second, they all shrink by the factor $(R - t)/(1 + R)$. This means that if \mathbb{I} is an interval contained either in $[-1, t]$ or in $[t, R]$, then $\mu(\{x \in [-1, R] : \alpha_1(x) \in \mathbb{I}\})$ is equal to $((1 + R)/(1 + t))\mu(\mathbb{I})$ in the first case and $\mu(\{x \in [-1, R] : \alpha_2(x) \in \mathbb{I}\})$ is equal

to $((1 + R)/(R - t))\mu(\mathbb{I})$ in the second. Hence, choosing $\kappa \in \{1, 2\}$ such that $q_\kappa(t) \geq 2/(1 + R)$ we obtain

$$\mu(\mathfrak{E}(f(\alpha_\kappa(\cdot)); [-1, R])) \geq 2.$$

In other words, $f(\alpha_\kappa(\cdot)) \in \pi_n(R)$. Since $t = \alpha_\kappa(R)$, we conclude that

$$|f(t)| = |f(\alpha_\kappa(R))| \leq m(R).$$

In view of this fact, Theorem A' may be stated as follows.

Theorem A''. *Let $R \geq 1$. If $f \in \pi_n(R)$, then for all $x \geq R$,*

$$(6) \quad |f(x)| \leq T_n(x).$$

We prove

Theorem 1. *Let $R \geq 1$. If $f \in \pi_{n,\mathbb{R}}(R)$, then for all $z \in \mathcal{H}_R := \{z \in \mathbb{C} : \operatorname{Re}(z) \geq R\}$ we have*

$$(7) \quad \left|f^{(k)}(z)\right| \leq \left|T_n^{(k)}(z)\right| \quad (k = 0, 1, \dots, n).$$

If k belongs to $\{1, \dots, n\}$, then equality holds in (7) for any $z \in \mathcal{H}_R$ if and only if $f(z) = \pm T_n(z)$. The same can be said when $k = 0$ if $z \in \mathcal{H}_R \setminus \{1\}$.

Theorem 1 is not only an extension of Theorem A' but also of (2).

Note that if f is of degree n , then $\mathfrak{E}(f; (-\infty, \infty))$ consists of at most n disjoint closed though possibly degenerate intervals. To see this consider the polynomial $F(z) := f(z)\overline{f(\bar{z})}$. It is non-negative on the real axis and $\mathfrak{E}(F; (-\infty, \infty)) = \mathfrak{E}(f; (-\infty, \infty))$. Suppose that $\mathfrak{E}(f; (-\infty, \infty))$ consists of N disjoint closed intervals $[a_1, b_1], \dots, [a_N, b_N]$, where $N \geq n + 1$. It is geometrically evident that the derivative F' must vanish at least once in each of the $N - 1$ open intervals $(b_1, a_2), \dots, (b_{N-1}, a_N)$ and also in each of the intervals $[a_1, b_1], \dots, [a_N, b_N]$, even in the degenerate ones. Thus F' has at least $2N - 1 (\geq 2n + 1)$ zeros, which is a contradiction since F' is of degree $2n - 1$. It follows that $\mathfrak{E}(f; [-1, \xi])$ consists of at most n disjoint, closed, possibly degenerate intervals for all $\xi \geq -1$.

Now let $x_j := \cos((n - j)\pi/n)$ for $j = 0, 1, \dots, n$ and let f be an arbitrary polynomial in $\pi_n(R)$, where $R \geq 1$. For $j = 0, 1, \dots, n$ let ξ_j be the infimum of all ξ such that $\mu(\mathfrak{E}(f; [-1, \xi])) = 1 + x_j$. The numbers $\xi_0, \xi_1, \dots, \xi_n$ are well defined and form an increasing sequence such that $\xi_{j+1} - \xi_j \geq x_{j+1} - x_j$ for $j = 0, \dots, n - 1$; in particular, $\xi_j \geq x_j$ for $j = 0, 1, \dots, n$. So, Theorem 1 is contained in the following result. This is what we shall really prove.

Theorem 1*. *Let*

$$x_j := \cos \frac{n - j}{n} \pi \quad \text{for } j = 0, 1, \dots, n.$$

Further, let $\xi_0, \xi_1, \dots, \xi_n$ be another sequence of $n + 1$ numbers in $[-1, \infty)$ such that

$$(8) \quad \xi_{j+1} - \xi_j \geq x_{j+1} - x_j \quad (j = 0, 1, \dots, n - 1),$$

and \mathcal{H}_R as in Theorem 1. If f is a real polynomial of degree at most n such that

$$(9) \quad |f(\xi_j)| \leq 1 \quad (j = 0, 1, \dots, n),$$

then for all $z \in \mathcal{H}_R$ with $R \geq \xi_n$ we have

$$(10) \quad \left|f^{(k)}(z)\right| \leq \left|T_n^{(k)}(z)\right| \quad (k = 0, 1, \dots, n),$$

where $f^{(0)}(z) \equiv f(z)$. If k belongs to $\{1, \dots, n\}$, then equality holds in (10) for any $z \in \mathcal{H}_R$ if and only if $\xi_j = x_j$ for all j and $f(z) = \pm T_n(z)$. The same can be said when $k = 0$ if $z \in \mathcal{H}_R \setminus \{1\}$.

Remark 1. Let f be any polynomial (real or not) of degree at most n satisfying (9). For any $x_0 \geq \xi_n$ and any $k \in \{0, 1, \dots, n\}$, let $f^{(k)}(x_0) = |f^{(k)}(x_0)| e^{i\gamma}$. Then $g(x) := Re(e^{-i\gamma} f(x))$ is a real polynomial of degree at most n satisfying (9) and so

$$|f^{(k)}(x_0)| = Re\left(e^{-i\gamma} f^{(k)}(x_0)\right) = |g^{(k)}(x_0)| \leq T_n^{(k)}(x_0)$$

by (10). In other words, (10) holds for any polynomial f of degree at most n satisfying (9) if $z \in \mathcal{H}_R \cap \mathbb{R}$.

2. PREPARATORY LEMMAS

The following auxiliary result is a simple principle of mechanics expressed in terms of complex numbers rather than vectors.

Lemma 1. Let $\varphi_1, \dots, \varphi_n$ and ψ_1, \dots, ψ_n be non-negative numbers with $\varphi_k \leq \psi_k$ for $k = 1, \dots, n$ and $\sum_{k=1}^n \psi_k < \pi/2$. Besides, let ρ_0, \dots, ρ_n and R_0, \dots, R_n be two other sets of positive numbers such that $\rho_k \leq R_k$ for $k = 0, 1, \dots, n$. Then

$$(11) \quad \left| \rho_0 + \sum_{k=1}^n \rho_k \exp\left(-i \sum_{j=1}^k \psi_j\right) \right| \leq \left| R_0 + \sum_{k=1}^n R_k \exp\left(-i \sum_{j=1}^k \varphi_j\right) \right|,$$

where equality holds only if $\rho_k = R_k$ for $0 \leq k \leq n$ and $\psi_k = \varphi_k$ for $1 \leq k \leq n$.

Proof. Clearly, $|\rho_0 + \sum_{k=1}^n \rho_k \exp(-i \sum_{j=1}^k \psi_j)|^2$ is equal to

$$\begin{aligned} \sum_{k=0}^n \rho_k^2 + 2\rho_0 \sum_{l=1}^n \rho_l \cos\left(\sum_{j=1}^l \psi_j\right) + \sum_{k=1}^{n-1} 2\rho_k \cos\left(\sum_{j=1}^k \psi_j\right) \left(\sum_{l=k+1}^n \rho_l \cos\left(\sum_{j=1}^l \psi_j\right)\right) \\ + \sum_{k=1}^{n-1} 2\rho_k \sin\left(\sum_{j=1}^k \psi_j\right) \left(\sum_{l=k+1}^n \rho_l \sin\left(\sum_{j=1}^l \psi_j\right)\right), \end{aligned}$$

which is in turn equal to

$$\begin{aligned} \sum_{k=0}^n \rho_k^2 + 2\rho_0 \sum_{l=1}^n \rho_l \cos\left(\sum_{j=1}^l \psi_j\right) + \sum_{k=1}^{n-1} 2\rho_k \sum_{l=k+1}^n \rho_l \\ \times \left\{ \cos\left(\sum_{j=1}^l \psi_j\right) \cos\left(\sum_{j=1}^k \psi_j\right) + \sin\left(\sum_{j=1}^l \psi_j\right) \sin\left(\sum_{j=1}^k \psi_j\right) \right\}. \end{aligned}$$

Thus, $|\rho_0 + \sum_{k=1}^n \rho_k \exp(-i \sum_{j=1}^k \psi_j)|^2$ can be written in the form

$$\sum_{k=0}^n \rho_k^2 + \sum_{k=0}^{n-1} 2\rho_k \sum_{l=k+1}^n \rho_l \cos\left(\sum_{j=k+1}^l \psi_j\right),$$

which, obviously, increases as any of the numbers $\rho_0, \rho_1, \dots, \rho_n$ increases or as any of the numbers ψ_1, \dots, ψ_n decreases. Hence (11) holds, wherein equality holds only if $\rho_k = R_k$ for $0 \leq k \leq n$ and $\psi_k = \varphi_k$ for $1 \leq k \leq n$. \square

Lemma 2. Let x_j, ξ_j be as in Theorem 1* and $z = x + iy$ where $x \geq \xi_n, y \geq 0$. Denote by A_j, B_j and P the points of the complex plane which correspond to x_j, ξ_j and z , respectively. If φ_j, ψ_j stand for the angles $\widehat{A_{j-1}PA_j}, \widehat{B_{j-1}PB_j}$, respectively, then

$$(12) \quad \psi_j \geq \varphi_j \text{ for } j = 1, \dots, n,$$

where, in the case $y > 0$, equality holds for some j if and only if $\xi_j = x_j$ for all j .

Proof. There is nothing to prove when $y = 0$ since in that case ψ_j and φ_j are all zero. So we assume $y > 0$. For $j = 1, \dots, n$ let δ_j, Δ_j denote the areas of the triangles $A_{j-1}PA_j, B_{j-1}PB_j$, respectively; then

$$\delta_j = \frac{1}{2}(x_j - x_{j-1})y, \quad \Delta_j = \frac{1}{2}(\xi_j - \xi_{j-1})y.$$

By assumption, $\xi_j - \xi_{j-1} \geq x_j - x_{j-1}$ and so

$$(13) \quad \Delta_j \geq \delta_j \text{ for } 1 \leq j \leq n.$$

Using another well-known formula for the area of a triangle we write

$$\delta_j = \frac{1}{2}|z - x_j||z - x_{j-1}|\sin \varphi_j, \quad \Delta_j = \frac{1}{2}|z - \xi_j||z - \xi_{j-1}|\sin \psi_j,$$

from which, for $j = 1, \dots, n$, we obtain

$$(14) \quad \sin \varphi_j = \frac{2\delta_j}{|z - x_j||z - x_{j-1}|}, \quad \sin \psi_j = \frac{2\Delta_j}{|z - \xi_j||z - \xi_{j-1}|}.$$

It is geometrically evident that $|z - \xi_j| \leq |z - x_j|$ for $j = 0, 1, \dots, n$. Hence, (14) combined with (13) implies that

$$\sin \varphi_j \leq \sin \psi_j \quad (1 \leq j \leq n).$$

This is equivalent to the desired result since $0 < \varphi_j, \psi_j < \pi/2$. □

Lemma 3. Let x_j, z and φ_j be as in Lemma 2. If

$$G(z) := \prod_{j=0}^n (z - x_j), \quad G_k(z) := \frac{G(z)}{z - x_k} \text{ for } k = 0, 1, \dots, n,$$

then with $\varphi_0 = 0$, we have

$$(15) \quad |T_n(z)| = \left| \sum_{k=0}^n \frac{1}{|G'(x_k)|} |G_k(z)| \exp(-i \sum_{j=0}^k \varphi_j) \right|.$$

Proof. Note that

$$G'(x_k) = \prod_{j=0, j \neq k}^n (x_k - x_j) = (-1)^{n-k} \prod_{j=0, j \neq k}^n |x_k - x_j| = (-1)^{n-k} |G'(x_k)|$$

and

$$T_n(x_k) = (-1)^{n-k}.$$

Hence, by Lagrange interpolation in the points x_0, x_1, \dots, x_n we obtain

$$(16) \quad |T_n(z)| = \left| \sum_{k=0}^n T_n(x_k) \frac{1}{G'(x_k)} G_k(z) \right| = \left| \sum_{k=0}^n \frac{1}{|G'(x_k)|} G_k(z) \right|.$$

If $\alpha := \text{Arg } G_0(z)$ and $k \in \{0, 1, \dots, n\}$, then

$$\begin{aligned}
 (17) \quad G_k(z) &= G_0(z) \frac{z - x_0}{z - x_k} \\
 &= e^{i\alpha} |G_0(z)| \left| \frac{z - x_0}{z - x_k} \right| \exp \left(-i \sum_{j=0}^k \varphi_j \right) \\
 &= e^{i\alpha} |G_k(z)| \exp \left(-i \sum_{j=0}^k \varphi_j \right).
 \end{aligned}$$

Substituting this expression for $G_k(z)$ in (16) we obtain (15). □

Remark 2. Let ξ_j, z and ψ_j be as in Lemma 2. Further, let

$$H(z) := \prod_{j=0}^n (z - \xi_j), \quad H_k(z) := \frac{H(z)}{z - \xi_k} \quad \text{for } k = 0, 1, \dots, n.$$

Arguing as for (17) we can show that if $\beta := \text{Arg } H_0(z)$, then with $\psi_0 = 0$, we have

$$(18) \quad H_k(z) = e^{i\beta} |H_k(z)| \exp \left(-i \sum_{j=0}^k \psi_j \right) \quad \text{for } k = 0, 1, \dots, n.$$

Lemma 4. For $k = 0, 1, \dots, n$ let $w_k = u_k + iv_k$, where $u_k > 0, v_k \leq 0$. If $-1 \leq t_k \leq 1$ for $k = 0, 1, \dots, n$, then

$$(19) \quad \left| \sum_{k=0}^n t_k w_k \right| \leq \left| \sum_{k=0}^n w_k \right|,$$

where equality holds if and only if the numbers t_k are all of the same sign and of modulus 1.

Proof. Since the numbers u_k are all of the same sign and so are the numbers v_k , we clearly have

$$\begin{aligned}
 \left| \sum_{k=0}^n t_k w_k \right|^2 &= \left(\sum_{k=0}^n t_k u_k \right)^2 + \left(\sum_{k=0}^n t_k v_k \right)^2 \\
 &\leq \left(\sum_{k=0}^n u_k \right)^2 + \left(\sum_{k=0}^n v_k \right)^2 \\
 &= \left| \sum_{k=0}^n u_k + i \sum_{k=0}^n v_k \right|^2 \\
 &= \left| \sum_{k=0}^n w_k \right|^2.
 \end{aligned}$$

□

3. PROOF OF THEOREM 1*

First let $k = 0$. For reasons of symmetry it is enough to prove that

$$(20) \quad |f(z)| \leq |T_n(z)| \text{ for } z = x + iy, x \geq \xi_n, y \geq 0.$$

By Lagrange interpolation in the points $\xi_0, \xi_1, \dots, \xi_n$ we have

$$f(z) = \sum_{k=0}^n f(\xi_k) \frac{1}{H'(\xi_k)} H_k(z).$$

Noting that

$$H'(\xi_k) = \prod_{j=0, j \neq k}^n (\xi_k - \xi_j) = (-1)^{n-k} \prod_{j=0, j \neq k}^n |\xi_k - \xi_j| = (-1)^{n-k} |H'(\xi_k)|$$

and taking (18) into account we obtain

$$|f(z)| = \left| e^{i\beta} \sum_{k=0}^n (-1)^{n-k} f(\xi_k) \frac{|H_k(z)|}{|H'(\xi_k)|} \exp\left(-i \sum_{j=0}^k \psi_j\right) \right|.$$

So by Lemma 4,

$$|f(z)| \leq \left| \sum_{k=0}^n \frac{|H_k(z)|}{|H'(\xi_k)|} \exp\left(-i \sum_{j=0}^k \psi_j\right) \right|.$$

From (8) it follows that $|\xi_k - \xi_j| \geq |x_k - x_j|$ and so

$$|H'(\xi_k)| = \prod_{j=0, j \neq k}^n |\xi_k - \xi_j| \geq \prod_{j=0, j \neq k}^n |x_k - x_j| = |G'(x_k)|.$$

Besides, it is geometrically evident that $|H_k(z)| \leq |G_k(z)|$. Since $\psi_j \geq \varphi_j$ by Lemma 2, we may apply Lemma 1 to conclude that for $z = x + iy$ with $x \geq \xi_n, y > 0$ we have

$$|f(z)| \leq \left| \sum_{k=0}^n \frac{|G_k(z)|}{|G'(x_k)|} \exp\left(-i \sum_{j=0}^k \varphi_j\right) \right| = |T_n(z)|$$

by Lemma 3. Since, in (11) equality holds if and only if $\rho_k = R_k, \psi_k = \varphi_k$ for $k = 0, \dots, n$, it is easily seen from the above proof that $|f(z)| < |T_n(z)|$ for all $z \in \mathcal{H}_R$ with $R \geq \xi_n$ unless $f(z) \equiv \pm T_n(z)$.

Now let $1 \leq k \leq n$. Assume that $\xi_j \neq x_j$ for some j . Then $\xi_n \neq x_n$ and $f(z)$ is not identically equal to $\pm T_n(z)$. Consequently, $|f(z)| < |\lambda T_n(z)|$ for all $z \in \mathcal{H}_R$ and all $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$. This means that the polynomial $p(z) := f(z) - \lambda T_n(z) \neq 0$ in \mathcal{H}_R . Hence, there exists a positive number δ such that all the zeros of p lie in the half-plane $Re(z) \leq \xi_n - \delta$. By the Gauss-Lucas theorem [4, p. 84] all the zeros of $p^{(k)}$, if any, also lie in the same half-plane. It follows that $f^{(k)}(z) - \lambda T_n^{(k)}(z) \neq 0$ in \mathcal{H}_R for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$. This is possible only if $|f^{(k)}(z)| < |T_n^{(k)}(z)|$ for all $z \in \mathcal{H}_R$. Indeed, if we had $|f^{(k)}(z_0)| \geq |T_n^{(k)}(z_0)|$ for some $z_0 \in \mathcal{H}_R$, then with $\lambda_0 = f^{(k)}(z_0)/T_n^{(k)}(z_0)$, which is of modulus ≥ 1 , we

would have

$$f^{(k)}(z_0) - \lambda_0 T_n^{(k)}(z_0) = 0$$

contradicting the fact that $f^{(k)}(z) - \lambda T_n^{(k)}(z) \neq 0$ in \mathcal{H}_R if $|\lambda| \geq 1$.

Remark 3. Let x_j , ξ_j and \mathcal{H}_R be as above. The argument used to prove Theorem 1* shows that if f is any polynomial of degree at most n , with real or complex coefficients, such that $|f(\xi_j)| \leq 1$ for $j = 0, 1, \dots, n$ and $z_0 \in \mathcal{H}_R$, then

$$|f(z_0)| \leq |T_{n,z_0}(z_0)|,$$

where T_{n,z_0} is the unique polynomial of degree n satisfying the interpolation condition

$$T_{n,z_0}(x_j) = (-1)^{n-j} \exp(i \arg(z_0 - x_j)) \quad (j = 0, 1, \dots, n).$$

Note that the extremal polynomial $T_{n,z_0}(z)$ may change with z_0 . But clearly, $T_{n,z_0}(z) \equiv T_n(z)$, when $z_0 \in \mathcal{H}_R \cap \mathbb{R}$.

REFERENCES

1. B. Bojanov, *Elementary Proof of the Remez Inequality*, Amer. Math. Monthly **100** (1993), pp. 483–485. CMP 93:11
2. P. Borwein & T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer - Verlag, 1995. MR **97e**:41001
3. G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford, 1971.
4. E. Hille, *Analytic Function Theory*, vol. **I**, Blaisdel Publishing Company, New York, 1959. MR **21**:6415
5. V.A. Markov (W.A. Markov), *Über polynôme die in einem gegebenem Intervalle möglichst wenig von Null abweichen*, Math. Ann. **77** (1916), pp. 213–258.
6. G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities*, World Scientific, Singapore, 1994. MR **95m**:30009
7. I.P. Natanson, *Constructive Function Theory*, vol. **I**, Frederick Ungar Publishing Co., Inc., New York, 1964. MR **33**:4529a
8. E.J. Remez, *Sur une propriété des polynômes de Tchebycheff*, Comm. Inst. Sci. Khar-kow **13** (1936), pp. 93–95.
9. T.J. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, 2nd ed., Wiley & Sons, New York, 1990. MR **92a**:41016

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