PSEUDO-DIFFERENTIAL OPERATORS
AND MAXIMAL REGULARITY RESULTS
FOR NON-AUTONOMOUS PARABOLIC EQUATIONS

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Abstract. In this paper, we show that a pseudo-differential operator associated to a symbol \( a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{L}(H)) \) (\( H \) being a Hilbert space) which admits a holomorphic extension to a suitable sector of \( \mathbb{C} \) acts as a bounded operator on \( L^2(\mathbb{R}, H) \). By showing that maximal \( L^p \)-regularity for the non-autonomous parabolic equation \( u'(t) + A(t)u(t) = f(t), \ u(0) = 0 \) is independent of \( p \in (1, \infty) \), we obtain as a consequence a maximal \( L^p([0, T], H) \)-regularity result for solutions of the above equation.

1. Introduction

A classical result in the theory of pseudo-differential operators states that an operator associated to a symbol belonging to the class \( S^0 \) acts as a bounded operator on \( L^2(\mathbb{R}^N) \) (see e.g. [12], Ch.VI). It was observed in recent years that pseudo-differential operators with operator-valued symbols (i.e. symbols which take values in the space of bounded linear operators on a Banach space \( X \)) are very useful in proving so-called maximal regularity results for autonomous parabolic evolution equations. For details and more information in this direction we refer to [2], [8], [4], [10], [3] and [7]. In this paper we examine maximal \( L^p \)-regularity results for non-autonomous equations of the form

\[
u'(t) + A(t)u(t) = f(t), \quad t \in [0, T],
\]

\[
u(0) = 0
\]

via the technique of pseudo-differential operators with operator-valued symbols. Since operators \( A(t) \) associated to specific boundary value problems arising in applications very often show non-smooth dependence on \( t \), we are in particular interested in symbols \( a(x, \xi) \) having non-smooth dependence on \( x \).

It is one aim of this paper to show, roughly speaking, that a pseudo-differential operator associated to a symbol \( a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{L}(H)) \), where \( H \) is a Hilbert space, which admits a bounded, holomorphic extension to a suitable sector of the complex plane, acts as a bounded operator on \( L^2(\mathbb{R}, H) \). Considering in particular the symbol \( a \) given by \( a(t, \tau) := A(t)(i\tau + A(t))^{-1} \) we obtain as a consequence a maximal \( L^2([0, T], H) \)-regularity result for (1.1) provided the family \( A(t)_{t \in [0, T]} \) satisfies the so-called Acquistapace-Terreni condition. Note that our result generalizes in particular...
the result of de Simon [5] on $L^2(0, T; H)$-regularity for the autonomous case, i.e. $A(t) = A$ for all $t \in [0, T]$, to equations of the form (1.1).

Observe that we allow that the domains $D(A(t))$ of $A(t)$ may vary with $t \in [0, T]$. Hence maximal regularity results for (1.1) cannot be obtained from those for the autonomous equation by simple perturbation techniques.

We remark that the maximal $L^2([0, T]; H)$-regularity result for (1.1) is the first cornerstone in establishing mixed $L^p([0, T]; L^q(\Omega))$-estimates ($1 < p, q < \infty$) for equations of the form (1.1). The Calderón-Zygmund theory for operator-valued kernels as developed for instance in [11] allows us to prove that, for arbitrary Banach spaces $X$ and $p \in (1, \infty)$, there is maximal $L^p(0, T; X)$-regularity for (1.1) if and only if there is maximal $L^2(0, T; X)$-regularity for (1.1). Hence we obtain maximal $L^p(0, T; H)$-regularity for (1.1).

In [6] we prove mixed $L^p - L^q$ estimates for the solution of (1.1) (under suitable assumptions on the heat kernels on the semigroups generated by $A(t)$), by interpolating between the $L^1 - L^1_w$ result proved in [6] and the $L^2 - L^2$ result stated as Theorem 2.1 below and by applying the fact that the property of maximal $L^p$-regularity is independent of $p$. We finally remark that our maximal regularity results may be used to prove existence and uniqueness results for semilinear problems of the form $u'(t) + A(t)u(t) = f(t, u(t))$, $u(0) = 0$. For details we refer to [6].

Throughout this paper we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from $X$ to $Y$, whenever $X$ and $Y$ are Banach spaces and by $H$ a Hilbert space. If $A$ is a linear operator in $X$, we denote its domain by $D(A)$, its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$. Furthermore, we denote by $S(\mathbb{R}; X)$ the space of all rapidly decreasing smooth functions on $\mathbb{R}$. The Fourier transform $\hat{f}$ of a function $f \in S(\mathbb{R}; X)$ is defined by

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi}f(x)dx, \quad \xi \in \mathbb{R}.$$ 

Finally, we denote by $C$ various constants which may differ from occurrence to occurrence but are always independent of the free variable of a given formula.

2. Pseudo-differential operators with non-smooth operator-valued symbols

For $\theta \in (0, \pi)$ set $\Sigma_{\theta} := \{z \in \mathbb{C}\setminus\{0\}; |\arg z| < \theta\}$. Let $a \in L^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(H))$ and define the pseudo-differential operator

$$Op(a) : S(\mathbb{R}, H) \to BC(\mathbb{R}, H)$$

with operator-valued symbol $a$ by

$$(Op(a)u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}a(x, \xi)\hat{u}(\xi)d\xi, \quad x \in \mathbb{R},$$

where $S(\mathbb{R}, H)$ denotes the Schwartz space of rapidly decreasing smooth $H$-valued functions on $\mathbb{R}$.

2.1. Theorem. Let $a \in L^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(H))$ and assume that $\xi \mapsto a(x, \xi)$ admits a holomorphic $\mathcal{L}(H)$-valued extension $z \mapsto a(x, z)$ to $\Sigma_\theta$ and $-\Sigma_\theta$ for some $\theta \in (0, \pi)$ such that $\sup_{\xi \in \Sigma_\theta, z \in -\Sigma_\theta} \sup_{x \in \mathbb{R}} \|a(x, z)\|_{\mathcal{L}(H)} < \infty$. Then the operator $Op(a)$, initially defined on $S(\mathbb{R}, H)$, extends to a bounded operator on $L^2(\mathbb{R}, H)$.  

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Proof. Let \( \alpha := \frac{\sin \theta}{z} \) and set \( R := 1 + \frac{\alpha}{2} \). Choose \( \varphi \in C_\infty_c(\mathbb{R}) \) with \( \text{supp} \varphi \subset (R^{-1}, R) \) such that
\[
\int_{\mathbb{R}} \frac{\varphi^2(\tau)}{|\tau|} \, d\tau = 1.
\]

For \( u \in S(\mathbb{R}, H) \) we then have
\[
(Op(a)u)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} a(x, \xi) \frac{\varphi^2(\xi)}{|\xi|} \hat{u}(\xi) \, d\xi \, d\tau.
\]

Furthermore, let \( \Gamma := \{ z \in \mathbb{C}; |z - 1| = \alpha \} \) be positively oriented. By Cauchy’s theorem we have
\[
a(x, \xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{a(x, \tau z)}{z - \xi/\tau} \, dz, \quad (x \in \mathbb{R})
\]
for those \((\xi, \tau) \in \mathbb{R} \times \mathbb{R}\) satisfying \( \varphi(z) \neq 0 \). Inserting this in (2.1) we obtain by Fubini’s theorem
\[
(Op(a)u)(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi} e^{ix\xi} \frac{\varphi^2(z)}{z - \xi/\tau} a(x, \tau z) \hat{u}(\xi) \, d\tau \, dz.
\]
Setting \( g(x, \tau) := \frac{\varphi(z)}{z - \xi/\tau} \), \( h(\xi) := \varphi(z) \) and denoting the inner integral above by \( I_{\tau, z} u \) we obtain
\[
(I_{\tau, z} u)(x) = (\mathcal{F}^{-1}(g_{x, \tau}) \ast \mathcal{F}^{-1}(h_{\tau}) \ast a(x, \tau z) u)(x), \quad x \in \mathbb{R}.
\]
Since \( \text{supp} \varphi \subset (R^{-1}, R) \) it follows from Plancherel’s theorem that
\[
(I_{\tau, z} u, I_{\rho, z} u)_{L^2(\mathbb{R}, H)} = 0
\]
provided \( z \in \mathbb{R} \setminus \{ (R^{-2}, R^2) \} \). In order to estimate \( Op(a)u \) in \( L^2(\mathbb{R}, H) \) notice that
\[
\| Op(a)u \|_{L^2(\mathbb{R}, H)} \leq \frac{1}{2\pi} \int_{\mathbb{R}} \| H_z u \|_{L^2(\mathbb{R}, H)} \, dz,
\]
where \( H_z u := \int_{\mathbb{R}} I_{\tau, z} u \frac{d\tau}{|\tau|} \) is understood as an improper integral in \( L^2(\mathbb{R}, H) \). It follows from (2.2) and the Cauchy-Schwarz inequality that
\[
\| H_z u \|_{L^2(\mathbb{R}, H)}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} (I_{\tau, z} u, I_{\rho, z} u)_{L^2(\mathbb{R}, H)} \frac{d\rho}{|\rho|} \frac{d\tau}{|\tau|}
\]
\[
= \int_{R^{-2}} \int_{R} (I_{\tau, z} u, I_{\rho, z} u)_{L^2(\mathbb{R}, H)} \frac{d\tau}{|\tau|} \frac{d\rho}{|\rho|}
\]
\[
\leq \int_{R^{-2}} \int_{\mathbb{R}} \| I_{\tau, z} u \|_{L^2(\mathbb{R}, H)}^2 \frac{d\tau}{|\tau|} \frac{d\rho}{|\rho|}
\]
\[
= 4 \log R \int_{\mathbb{R}} \| I_{\tau, z} u \|_{L^2(\mathbb{R}, H)}^2 \frac{d\tau}{|\tau|}.
\]
Observe that by Plancherel’s theorem we have
\[ \|I_{\tau,z}u\|_{L^2(\mathbb{R},H)} \leq \sup_{x \in \mathbb{R}} \|a(x,\tau z)\|_{L^2(H)} \sup_{\eta \in \mathbb{R}} \frac{|\varphi(\eta)|}{z - \eta} \|h_\tau \hat{u}\|_{L^2(\mathbb{R},H)}. \]
Therefore there exists a constant \( C > 0 \) such that
\[ \|H_u\|_{L^2(\mathbb{R},H)}^2 \leq C \int_{\mathbb{R}} \|h_\tau \hat{u}\|^2_{L^2(\mathbb{R},H)} \frac{d\tau}{|\tau|} = C \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(\frac{\xi}{\tau})|^2 \frac{d\tau}{|\tau|} \|\hat{u}(\xi)\|^2_{L^2} d\xi = C \|u\|_{L^2(\mathbb{R},H)}^2. \]
Combining this estimate with (2.3) it follows that
\[ \|Op(a)u\|_{L^2(\mathbb{R},H)} \leq C \|u\|_{L^2(\mathbb{R},H)} \]
for \( u \in S(\mathbb{R},H) \) and by density for all \( u \in L^2(\mathbb{R},H) \).

\[ \square \]

3. MAXIMAL REGULARITY FOR NON-AUTONOMOUS PARABOLIC EQUATIONS

Let \( T > 0 \) and let \( (A(t))_{t \in [0,T]} \) be a family of densely defined linear operators in \( X \) satisfying the following two assumptions:
A1) There exists \( \theta \in (0,\pi/2) \) such that \( \sigma(A(t)) \subset \Sigma_\theta \) for all \( t \in [0,T] \) and for \( \varphi \in (\theta,\pi) \) there exists \( M > 0 \) such that
\[ \|(\lambda - A(t))^{-1}\|_{L(X)} \leq \frac{M}{1 + |\lambda|}, \quad t \in [0,T], \lambda \in \mathbb{C} \setminus \Sigma_\varphi. \]
A2) There exist constants \( \alpha, \beta \in [0,1], \alpha < \beta, \omega \in (\theta,\pi/2), \epsilon > 0 \) such that
\[ \|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\|_{L(X)} \leq C \frac{|t - s|^{\beta}}{1 + |\lambda|^{1-\alpha}} \]
for \( s, t \in [0,T], \lambda \in \mathbb{C} \setminus \Sigma_\omega. \)

We remark that the above conditions A1), A2) on \( A(t) \) were introduced and investigated by Acquistapace, Terreni [1] and Yagi [13] in order to construct the evolution operator associated with \( A(t), t \in [0,T] \).

Let \( 1 < p < \infty \) and \( f : [0,T] \to X \) be a function. We consider the following non-autonomous initial value problem:
\[
\begin{align*}
(3.1) \quad u'(t) + A(t)u(t) &= f(t), \quad t \in [0,T], \\
\quad u(0) &= 0.
\end{align*}
\]
The family \( \{A(t), t \in [0,T]\} \) is said to belong to the class \( MR(p,X) \) and we say that there is maximal \( L^p \) regularity for (3.1) if for each \( f \in L^p(0,T;X) \) there exists a unique
\[ u \in W^{1,p}(0,T;X) \quad \text{with} \quad t \mapsto A(t)u(t) \in L^p(0,T;X) \]
satisfying (3.1) in the \( L^p(0,T;X) \)-sense.

The following two theorems are the main results of this section.

3.1. Theorem. Let \( X \) be a Banach space, \( T > 0 \), and assume that \( \{A(t), t \in [0,T]\} \) satisfies A1) and A2). Suppose that there exists \( p \in (1,\infty) \) such that the family \( \{A(t), t \in [0,T]\} \) belongs to the class \( MR(p,X) \). Then \( \{A(t), t \in [0,T]\} \) belongs to \( MR(q,X) \) for all \( q \in (1,\infty) \).
3.2. Theorem. Let \( H \) be a Hilbert space, \( 1 < p < \infty, T > 0 \) and assume that \( \{A(t), t \in [0, T]\} \) satisfies A1 and A2. Then \( \{A(t), t \in [0, T]\} \) belongs to \( MR(p; H) \).

We start the proof of the two theorems above with the following observation. It follows from the results in [1], [9] that if \( u \) is a solution of (3.1), then \( u \) fulfills

\[
(3.2) \quad A(t)u(t) = \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})A(s)u(s)ds + \int_0^t A(t)e^{-(t-s)A(t)}f(s)ds
\]

for \( t \in [0, T] \). For the time being let \( q \in (1, \infty) \) and define the operator \( Q \in \mathcal{L}(L^q(0, T; X)) \) by

\[
(Qg)(t) := \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})g(s)ds, \quad t \in [0, T].
\]

The results in [1] and [9] imply that \( \|Q\|_{\mathcal{L}(L^q(0, T; X))} \leq 1/2 \) provided the constant \( c \) in A2 is sufficiently small. Observe, however, that the family \( \{A(t), t \in [0, T]\} \) belongs to the class \( MR(q; X) \) if and only if this holds true for \( \{A(t)+K, t \in [0, T]\} \), where \( K \) denotes an arbitrary constant. Hence, there is no loss of generality in choosing \( c \) as small as we want. It follows that the operator \( Id - Q \) is invertible in \( L^q(0, T; X) \). Moreover, by (3.2) we know that

\[
(Id - Q)A(\cdot)u = Sf, \quad \text{where} \quad (Sf)(t) := \int_0^t A(t)e^{-(t-s)A(t)}f(s)ds
\]

provided \( u \) is a solution of (3.1). Summarizing, we proved the following fact.

3.3. Proposition. The family \( \{A(t), t \in [0, T]\} \) belongs to the class \( MR(q; X) \) if and only if \( S \) acts a bounded operator on \( L^q(0, T; X) \).

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. By assumption and Proposition 3.1 we know that \( S \) acts boundedly on \( L^p(0, T; X) \). In order to show that \( S \) is bounded on \( L^q(0, T; X) \) for \( q \in (1, \infty) \), it suffices to verify (see [11], Theorems III.1.2, III.1.3) that

\[
(3.3) \quad \sup_{s, s' \in (0, T)} \int_{|s-s'| \leq \frac{|t-s|}{2}} \|k(t, s) - k(t, s')\| dt < \infty,
\]

\[
(3.4) \quad \sup_{s, s' \in (0, T)} \int_{|s-s'| \leq \frac{|t-s|}{2}} \|k(s, t) - k(s', t)\| dt < \infty
\]

where \( k(t, s) := A(t)e^{-(t-s)A(t)}1_{(0, t)}(s) \).
To this end, note that for $s, s' \in (0, T)$ we have

\[
\int_{|s-s'| \leq \frac{1}{2}|t-s|} \|k(t, s) - k(t, s')\|dt
\]

\[
= \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|A(t)e^{-(t-s)A(t)}1_{(0,t)}(s) - A(t)e^{-(t-s')A(t)}1_{(0,t)}(s')\|dt
\]

\[
= \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|A(t)^2e^{-(t-s)A(t)}\|d\sigma\|dt
\]

\[
\leq \int_{|s-s'| \leq \frac{1}{2}|t-s|} \frac{M}{(t-s')^2}d\sigma|dt = M \int_{|s-s'| \leq \frac{1}{2}|t-s|} \frac{1}{t-s} - \frac{1}{t-s'}|dt
\]

\[
< \infty.
\]

Moreover, for $s, s' \in (0, T)$, we have

\[
\int_{|s-s'| \leq \frac{1}{2}|t-s|} \|k(s, t) - k(s', t)\|dt
\]

\[
= \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|A(s)e^{-(s-t)A(s)}1_{s \geq t} - A(s)e^{-(s'-t)A(s')}1_{s' \geq t}\|dt
\]

\[
\leq \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|A(s)e^{-(s-t)A(s)} - A(s)e^{-(s'-t)A(s)}\|dt
\]

\[
+ \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|A(s)e^{-(s'-t)A(s)} - A(s')e^{-(s'-t)A(s')}\|dt
\]

\[
\leq \int_{|s-s'| \leq \frac{1}{2}|t-s|} \|\int_{t-s}^{s'} A(s)^2e^{-(\sigma-t)A(s)}d\sigma\|dt
\]

\[
+ \int_{|s-s'| \leq \frac{1}{2}|t-s|} \left| \frac{1}{2\pi i} \int_{\Gamma_0} e^{-(s'-t)\lambda}((\lambda - A(s))^{-1} - (\lambda - A(s'))^{-1})d\lambda \right|dt
\]

\[
\leq \int_{|s-s'| \leq \frac{1}{2}|t-s|} \frac{M}{(t-s')^2}d\sigma|dt
\]

\[
+ \int_{|s-s'| \leq \frac{1}{2}|t-s|} \left( \frac{1}{\pi} \int_0^\infty r e^{-(s'-t)r \cos \theta} \frac{c(M + 1)|s - s'|^\beta}{(1 + r)^{1-\alpha}}dr \right) dt
\]

\[
< \infty.
\]

The proof is complete. \qed
Proof of Theorem 3.2. Observe that the symbol $a$ defined by
\[
a(t, \tau) := \begin{cases} A(0)(i\tau + A(0))^{-1}, & t < 0, \\ A(t)(i\tau + A(t))^{-1}, & t \in [0, T], \\ A(T)(i\tau + A(T))^{-1}, & t > T, \end{cases}
\]
satisfies, thanks to A1), the assumptions of Theorem 2.1. Hence it follows from this theorem and Proposition 3.3 that the family $\{A(t), t \in [0, T]\}$ belongs to the class $MR(2; B)$. Theorem 3.1 implies now the assertion.

References


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