ASYMPTOTIC REGULARITY OF DAUBECHIES’ SCALING FUNCTIONS

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Abstract. Let \( \phi_N, N \geq 1, \) be Daubechies’ scaling function with symbol \( \left( \frac{1 + e^{-i\xi}}{2} \right)^N Q_N(\xi) \), and let \( s_p(\phi_N), 0 < p \leq \infty, \) be the corresponding \( L^p \) Sobolev exponent. In this paper, we make a sharp estimation of \( s_p(\phi_N) \), and we prove that there exists a constant \( C \) independent of \( N \) such that

\[
N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2} \leq s_p(\phi_N) \leq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}.
\]

This answers a question of Cohen and Daubechies (Rev. Mat. Iberoamericana, 12(1996), 527-591) positively.

1. Introduction

For \( N \geq 1, \) let

\[
P_N(t) = \sum_{k=0}^{N-1} \binom{N-1}{k} t^k.
\]

Then

\[
(1 - t)^N P_N(t) + t^N P_N(1 - t) = 1
\]

and \( P_N \) is the unique polynomial solution of the equation with degree not greater than \( N - 1. \)

Let \( Q_N(\xi) \) be a trigonometric polynomial with real coefficients satisfying

\[
|Q_N(\xi)|^2 = P_N(\sin^2 \frac{\xi}{2}).
\]

It is known that such \( Q_N \) exists by the Riesz Lemma, but \( Q_N \) is not unique. Set

\[
H_N(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^N Q_N(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} c_k e^{-ik\xi}.
\]

We are interested in the \( Q_N \) such that the solution \( \phi_N \) of the refinement equation

\[
\phi_N(x) = \sum_{k \in \mathbb{Z}} c_k \phi_N(2x - k)
\]

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with \( \int_{\mathbb{R}} \phi_N(x) dx = 1 \) that generates an orthonormal basis of \( L^2(\mathbb{R}) \). The functions \( \phi_N \) are the well known Daubechies’ scaling functions \( [6] \). For an integrable function \( f \), let \( \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \) be the Fourier transform of \( f \). Then

\[
\hat{\phi_N}(\xi) = H_N(\frac{\xi}{2}) \hat{\phi_N}(\frac{\xi}{2})
\]

and

\[
\hat{\phi_N}(\xi) = \prod_{j=1}^{\infty} H_N(2^{-j} \xi).
\]

The regularity of the scaling functions has central importance in the theory of wavelets. In \[14\] Volkmer proved that the Hölder index of \( \phi_N \) is \((1 - \frac{\ln 3}{2\ln 2})N + o(N)\) as \( N \) tends to infinity. Recently Bi, Dai and Sun \([1]\) improved the estimation as

\[
(1 - \frac{\ln 3}{2\ln 2})N + \frac{\ln N}{4\ln 2} + O(1).
\]

Another popular approach to the regularity is to use the Sobolev exponent. Recall that the Sobolev exponent \( s_p(f) \), \( 0 < p < \infty \), is defined by

\[
s_p(f) = \sup \{ s : \int_{\mathbb{R}} |\hat{f}(\xi)|^p (1 + |\xi|)^{ps} d\xi < \infty \},
\]

and for \( p = \infty \),

\[
s_{\infty}(f) = \sup \{ s : \hat{f}(\xi)(1 + |\xi|)^s \text{ is a bounded function} \}.
\]

There is considerable literature devoted to estimating the Sobolev exponent for scaling functions in general, for example, \[8\] and \[13\] for \( s_2(f) \), \[2\] for \( s_1(f) \), \[10\] and \[9\] for \( s_p(f) \) with \( 1 \leq p < \infty \), \[12\] for Triebel-Lizorkin space and Besov space, and \[11\] for \( L^p \) Lipschitz space. For Daubechies’ scaling functions, Volkmer \[15\] proved that

\[
N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2} - \frac{1}{2} \leq s_2(\phi_N) \leq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}.
\]

Recently, Cohen and Daubechies \([3, 7]\) computed \( s_p(\phi_N) \) for \( p = 1, 2, 4, 8 \) and \( N = 1, 2, \ldots, 19 \), and found that the difference of \( s_p(\phi_N) \) between different \( p \) becomes very small for \( N \) large. Based on this observation, they asked

**Problem.** Let \( \phi_N \) be defined by (2). For \( 0 < p, \ q < \infty \), is it true that

\[
\lim_{N \to \infty} (s_p(\phi_N) - s_q(\phi_N)) = 0?
\]

In this paper, we answer this question affirmatively and generalize the estimation in \[15\] in part.

**Theorem.** Let \( \phi_N \) be defined by (2). For \( 0 < p < \infty \), there exists a constant \( C \) independent of \( N \) such that

\[
N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2} - \frac{C}{N} \leq s_p(\phi_N) \leq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2},
\]

and for \( p = \infty \),

\[
s_{\infty}(\phi_N) = N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}.
\]
In the table, we list the approximate value of the \( L^p \) Sobolev exponent \( s_p(\phi_N) \). The first three columns \( s_p(\phi_N), p = 1, 2, 8 \), are obtained by Cohen and Daubechies in [3]. The last column \( N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2} \) is the approximate value from the theorem. Note that the numerical data matches with the theorem.

### 2. Upper bound estimation

In this section, we will prove the upper bound estimate of \( s_p(\phi_N) \).

**Proposition 1.** Let \( \phi_N \) be defined by (2). Then for \( 0 < p \leq \infty \),

\[
(5) \quad s_p(\phi_N) \leq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}.
\]

**Proof.** It follows from (3) that

\[
|\hat{\phi}_N(\frac{2^k\pi}{3})| = 2^{-(k-1)N}|Q_N(\frac{2\pi}{3})|^{|k-1|\hat{\phi}_N(\frac{2\pi}{3})|}.
\]

Hence (5) holds for \( p = \infty \).

To prove the case for \( 0 < p < \infty \), we let \( \tilde{\phi}_N \) be the compactly supported distribution defined by

\[
\tilde{\phi}_N(\xi) = \prod_{j=1}^{\infty} Q_N(\xi/2^j).
\]

Let \( n_k = (4^k - 1)/3 \); then by a similar method as used in Proposition 3 in [4], we obtain for any \( \epsilon > 0 \) there exists a constant \( C \) such that for \( \xi \in [-\pi, \pi] \) and for sufficiently large \( k \),

\[
|\tilde{\phi}_N(\xi + 2n_k\pi)| \geq C|Q_N(\frac{2\pi}{3})|^{2k}4^{-k\epsilon}.
\]
Since
\[ \hat{\phi}_N(\xi) = \prod_{j=1}^{\infty} \left( 1 + \frac{e^{-i2^{-j}\xi}}{2} \right)^N \hat{\phi}_N(\xi) = \left( \frac{1 - e^{-i\xi}}{i\xi} \right)^N \hat{\phi}_N(\xi), \]
there exists an integer \( k_0 \) such that for \( \xi \in \left[ \frac{2\pi}{3}, \frac{2\pi}{3} \right] \) and \( k \geq k_0, \)
\[ |\hat{\phi}_N(\xi + 2nk\pi)| \geq C4^{-Nk-e^k}|Q_N(\frac{2\pi}{3})|^{2k}. \]
Obviously
\[ \int_{\mathbb{R}} |\hat{\phi}_N(\xi)|^p (1 + |\xi|)^{ps} d\xi < \infty \]
implies that
\[ \int_{\left[ \frac{2\pi}{3}, \frac{2\pi}{3} \right] + 2nk\pi} |\hat{\phi}_N(\xi)|^p (1 + |\xi|)^{ps} d\xi \]
is bounded on \( k \). Hence there exists a constant \( C \) such that \( 4^{s-N-e^k}|Q_N(\frac{2\pi}{3})|^{2kp} \leq C \) for all \( k \). This implies that
\[ s - N = \frac{\ln |Q_N(2\pi/3)|}{\ln 2} - \epsilon \leq 0 \]
and (5) follows from the definition of \( s_p(\phi_N), 0 < p < \infty. \)

3. Lower bound estimation

In this section, we prove the lower bound estimate for \( s_p(\phi_N) \).

**Proposition 2.** Let \( \phi_N \) be defined by (2). Then for \( 0 < p < \infty \) and for any integer \( M \geq 2 \) there exist a constant \( 1/2 < r < 1 \) and an integer \( N_0 \) independent of \( p \) and \( M \) such that for \( N \geq N_0, \)
\[ s_p(\phi_N) \geq N - \frac{pM \ln |Q_N(2\pi/3)| + \ln(2 + 2M^N r^p)}{pM \ln 2}. \]
Also
\[ s_{\infty}(\phi_N) \geq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}. \]

Obviously our main theorem follows from Propositions 1 and 2 by choosing the above \( M \) as the integral part of \( -pN \ln r / \ln 2 \). We need some lemmas to prove the proposition. The main estimate is Lemma 6, based on the accurate estimates of \( Q_N(\xi) \) on \([0, \frac{2\pi}{3}]\) and \( Q_N(\xi)Q_N(2\xi) \) on \([\frac{2\pi}{3}, \pi]\). First we introduce an auxiliary function
\[ g(\xi) = \begin{cases} 
(\cos \frac{\xi}{2})^{-2}, & |\xi| \leq \frac{\pi}{2}, \\
4(\sin \frac{\xi}{2})^2, & \frac{\pi}{2} \leq |\xi| \leq \pi,
\end{cases} \]
(6)

**Lemma 3.** There exists a constant \( C \) independent of \( N \) and \( \xi \) such that
\[ C^{-1}N^{-C} g(\xi)^N \leq |Q_N(\xi)|^2 \leq g(\xi)^N. \]

**Proof.** The right inequality was proved by Cohen and Séré [5, Lemma 2.3]. It remains to prove the left inequality. Write
\[ a_k(\xi) = \left( N - 1 + \frac{k}{2} \right) (\sin \frac{\xi}{2})^{2k}, \quad 0 \leq k \leq N - 1. \]
Then
\[ \frac{a_k(\xi)}{a_{k-1}(\xi)} = \frac{N + k - 1}{k} \sin^2 \frac{\xi}{2}, \]

Let \( k_0 \) be the integral part of \((N - 1) \tan^2 \frac{\xi}{2}\). Then by observing that
\[ \frac{a_k(\xi)}{a_{k-1}(\xi)} \geq 1 \quad \text{if and only if} \quad k \leq (N - 1) \tan^2 \frac{\xi}{2} \]
and that \(|\tan \frac{\xi}{2}| \leq 1\) for \(|\xi| \leq \frac{\pi}{2}\), we have
\[ \max_{1 \leq k \leq N-1} a_k(\xi) = a_{k_0}(\xi), \quad |\xi| \leq \pi/2. \]

By using the Stirling formula
\[ k! = k^k e^{-k} \sqrt{2\pi k}(1 + o(1)), \]
we have for \(|\xi| \leq \pi/2,
\[ a_{k_0}(\xi) = \frac{(N + k_0 - 1)!}{k_0!(N - 1)!} \left( \sin \frac{\xi}{2} \right)^{2k_0} = \frac{(N + k_0 - 1)^{N+k_0-1}}{k_0!(N - 1)^{N-1}} \left( \sin \frac{\xi}{2} \right)^{2k_0} B_N \]
where \( C^{-1}N^{-C} \leq B_N \leq CN^{-C} \). By substituting \(-1 < k_0 - (N - 1) \tan^2 \frac{\xi}{2} \leq 0\) into the above expression and simplifying, we have
\[ a_{k_0}(\xi) = \hat{B}_N \left( \cos \frac{\xi}{2} \right)^{-2N} = \hat{B}_N g(\xi)^N, \quad |\xi| \leq \pi/2, \]
where \((C')^{-1}N^{-C'} \leq \hat{B}_N \leq C'N^{-C'}\). This yields the left inequality of (7) for \(|\xi| \leq \pi/2.
For \( \frac{\pi}{2} \leq |\xi| \leq \pi\), \( \tan^2 \frac{\xi}{2} \geq 1\) implies that
\[ a_0(\xi) \leq a_1(\xi) \leq \cdots \leq a_{N-1}(\xi). \]

By using the Stirling formula again and making a similar estimation, we have
\[ C^{-1}N^{-C} g(\xi)^N = C^{-1}N^{-C} \left( 2 \sin \frac{\xi}{2} \right)^{2N} \leq a_N(\xi) \leq |Q_N(\xi)|^2, \quad \pi/2 \leq |\xi| \leq \pi, \]
which completes the proof. \(\square\)

**Lemma 4.** Let \( g(\xi) \) be defined by (6). Then
\[ 0 \leq g(\xi) g(2\xi) \leq |g(\frac{2\pi}{3})|^2, \quad |\xi| \in \left[ \frac{2\pi}{3}, \pi \right], \]
and for \( 0 < \delta < \frac{\pi}{6} \) there exists \( 0 < r_1 < 1 \) such that
\[ 0 \leq g(\xi) g(2\xi) \leq r_1 |g(\frac{2\pi}{3})|^2, \quad |\xi| \in \left[ \frac{2\pi}{3} + \delta, \pi \right]. \]

**Proof.** Recall that \( g(\xi) \) is an even periodic function, hence it suffices to prove (8) for \( \xi \in [0, \pi] \). Note that
\[ g(\xi) g(2\xi) = \begin{cases} 16 \sin^2 \frac{\xi}{2} \sin^2 \xi, & \xi \in \left[ \frac{2\pi}{3}, \frac{3\pi}{2} \right], \\ 4 \sin^2 \frac{\xi}{2} \cos^2 \xi, & \xi \in \left[ \frac{3\pi}{2}, \pi \right]. \end{cases} \]
It is easy to check that the product is strictly decreasing on \( \left[ \frac{2\pi}{3}, \pi \right] \). Hence
\[ 0 \leq g(\xi) g(2\xi) \leq g(\frac{2\pi}{3}) g(\frac{4\pi}{3}) = |g(\frac{2\pi}{3})|^2. \]
The second part follows from the strictly decreasing property. \(\square\)
Lemma 5. For any integer $N \geq 1$,

\begin{align}
|Q_N(\xi)| &\leq |Q_N(\frac{2\pi}{3})|, \quad |\xi| \in [0, \frac{2\pi}{3}), \\
|Q_N(\xi)Q_N(2\xi)| &\leq |Q_N(\frac{2\pi}{3})|^2, \quad |\xi| \in [\frac{2\pi}{3}, \pi). 
\end{align}

Furthermore for any $0 < \delta < \pi/6$, there exists $0 < r_2 < 1$ and an integer $N_1$ such that for $N > N_1$,

\begin{align}
|Q_N(\xi)| &\leq r_2^N|Q_N(\frac{2\pi}{3})|, \quad |\xi| \in [0, \frac{2\pi}{3} - \delta), \\
|Q_N(\xi)Q_N(2\xi)| &\leq r_2^N|Q_N(\frac{2\pi}{3})|^2, \quad |\xi| \in [\frac{2\pi}{3} + \delta, \pi].
\end{align}

Proof. The first two inequalities were proved in \cite[p. 222]{1}. We use Lemma 3 to prove (12): for $|\xi| \in [0, \frac{2\pi}{3} - \delta)$, there exists $0 < r < 1$ such that

\[ |Q_N(\xi)|^2 \leq g(\xi)^N \leq r^N g(\frac{2\pi}{3})^N \leq CNr^N|Q_N(\frac{2\pi}{3})|^2. \]

We pick $r_2$ so that $0 < r < r_2 < 1$. Hence (12) holds for $N$ large enough. The proof of (13) is similar by using Lemma 4.

Figure 1.

Lemma 6. Let $N_1$ be as in Lemma 5. Then there exists a constant $C_N$ and a constant $0 < r_3 < 1$ depending on $0 < \delta < \pi/6$ only, such that for $k > 2$ and $N \geq N_1$,

\begin{align}
\prod_{j=1}^{k}|Q_N(2^j\xi)| \leq C_N r_3^{N_k(\xi,\delta)}|Q_N(\frac{2\pi}{3})|^k.
\end{align}

Proof. We use $r_2(\delta)$ to denote the $r_2$ in Lemma 5, and choose $r_3(\delta)$ so that $r_2(\delta), r_2(\delta/2) < r_3(\delta) < 1$. It is easy to see that by letting $C_N$ be large enough, the lemma holds for $k = 1$ and $k = 2$. We assume that (14) holds for $k < l$ with $l \geq 3$.

For $k = l$, we divide the proof into four cases:
Let $\beta > 0$.

(i) If $2\xi \in [-\frac{2\pi}{3}, -\frac{2\pi}{3} + \delta) + 2m\pi$, then $i_k(\xi, \delta) = i_{k-1}(2\xi, \delta) + 1$. We can write

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| = |Q_N(2\xi)| \prod_{j=1}^{k-1} |Q_N(2^j(2\xi))|,$$

and (14) follows from (12) with $r_2(\delta) < r_3(\delta) < 1$ and the induction hypothesis.

(ii) If $2\xi \in \left(\left(-\frac{2\pi}{3} - \frac{\delta}{2}, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{2\pi}{3} + \delta\right)\right) + 2m\pi$, then $i_k(\xi, \delta) = i_{k-1}(2\xi, \delta)$ and the same induction hypothesis together with (10) implies (14).

(iii) If $2\xi \in \left(\left[-\frac{2\pi}{3} - \frac{\delta}{2}, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{2\pi}{3} + \delta\right)\right) + 2m\pi$, then it follows that $2\xi, 4\xi \notin \bigcup_{m \in \mathbb{Z}} \left(\left[-\frac{2\pi}{3} + \delta, \frac{2\pi}{3} - \delta\right] + 2m\pi\right)$, hence $i_k(\xi, \delta) = i_{k-2}(4\xi, \delta)$. Write

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| = |Q_N(2\xi)Q_N(4\xi)\prod_{j=1}^{k-2} |Q_N(2^j(4\xi))|,$$

and (14) follows from (11).

(iv) If $2\xi \in \left(\left(-\frac{2\pi}{3} - \frac{\delta}{2}, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{2\pi}{3} + \delta\right)\right) + 2m\pi$, then $i_k(\xi, \delta) \leq i_{k-2}(4\xi, \delta) + 1$.

By using the above product, $r_2(\delta/2) < r_3(\delta) < 1$ and (13), we have

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| \leq r_2(\delta/2)^N C_N N_3^{-i_k(4\xi, \delta)} Q_N(\frac{2\pi}{3})^{-k} \leq C_N N_3^{-i_k(\xi, \delta)} Q_N(\frac{2\pi}{3})^{-k}.$$

The induction step follows from these four cases. $\square$

For any integer $M \geq 2$, $k \geq 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_{kM})$ with $\epsilon_i = 0$ or 1, let $\alpha_{kM}(\epsilon)$ be the cardinality of the set

$$A_{kM}(\epsilon) = \{l : 1 \leq l \leq k, (\epsilon_{l(1)M+1}, \cdots, \epsilon_{lM}) \text{ has two consecutive } 0 \text{ or } 1\}.$$

Then $\alpha_{kM}(\epsilon) = \sum_{\ell=0}^{k-1} \alpha_M(\ell')$ where $\ell' = (\ell_{(M+1)}, \cdots, \ell_{(i+1)M})$ and

$$\sum_{\epsilon=(\epsilon_1,\cdots,\epsilon_{kM}) \in \{0,1\}^{kM}} r^{\epsilon_{kM}(\epsilon)} = \sum_{l=0}^{k-1} \sum_{\epsilon'=(\epsilon_{M+1},\cdots,\epsilon_{(i+1)M}) \in \{0,1\}^M} \prod_{j=0}^{k-1} r^{\alpha_M(\epsilon')}$$

$$= \left( \sum_{\epsilon=(\epsilon_1,\cdots,\epsilon_{kM}) \in \{0,1\}^{kM}} r^{\alpha_M(\epsilon)} \right)^k = (2 + (2^M - 2)r)^k,$$

where $r > 0$ and the last equality follows from the fact that $\alpha_M(\epsilon) = 1$ for any $\epsilon \in \{0,1\}^M$ except $\epsilon = (0,1,0,1,\cdots) \in \{0,1\}^M$ or $(1,0,1,0,\cdots) \in \{0,1\}^M$.

**Lemma 7.** Let $0 < \delta < \pi/6$. For $\xi \in [\pi, 2\pi)$, write $\xi = 2\pi \left(\sum_{j=1}^{kM} \epsilon_j 2^{-j} + \eta\right)$ with $0 \leq \eta < 2^{-kM}$. Then

$$\alpha_{kM}(\epsilon) - 1 \leq i_{kM}(\xi, \delta).$$

**Proof.** Suppose $l \in A_{kM}(\epsilon)$ and $l \geq 2$. Then there exists an index $j \geq 2$ such that $(l-1)M + 1 \leq j \leq lM - 1$ and $\epsilon_j = \epsilon_{j+1}$. Hence

$$2^{j-1}\xi = 2m\pi + 2\pi \left(\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta'\right)$$

for some integer $m$ and $0 \leq \eta' < 1/4$. For $\epsilon_j = \epsilon_{j+1} = 0$,

$$2\pi \left(\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta'\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
and for \( \epsilon_j = \epsilon_{j+1} = 1, \)
\[
2\pi (\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta' - 1) \in [-\frac{\pi}{2}, \frac{\pi}{2}].
\]
Hence \( 2^j - \xi \in \bigcup_{m \in \mathbb{Z}} [-2\pi/3 + \delta, 2\pi/3 - \delta] + 2m\pi, \) i.e., \( j - 1 \in I_{kM}(\xi, \delta). \) What we have just shown is that each \( l \in A_{kM}(\epsilon) \) corresponds to at least one distinct \( j \in I_{kM}(\xi, \delta) \) provided that \( l \geq 2. \) The lemma follows from this assertion. \( \square \)

**Proof of Proposition 2.** Recall that
\[
\hat{\phi}_N(\xi) = \left( \frac{1 - e^{-i\xi}}{i\xi} \right)^N \prod_{j=1}^{\infty} Q_N(\xi/2^j).
\]
Let \( r = r_3(\pi/6). \) Then for \( \xi \in [2^{kM-1}\pi, 2^{kM}\pi] \) and \( N \geq N_1, \) Lemma 6 implies that
\[
|\hat{\phi}_N(\xi)| \leq C 2^{-kM} \prod_{j=1}^{kM-1} |Q_N(2^{j-kM}\xi)|
\]
\[
\leq C' 2^{-kM} \prod_{j=1}^{kM-1} |Q_N(2^{j-kM}\xi, \pi/6)| |Q_N(\pi/3)|^{kM},
\]
where \( C' \) depends on \( N \) only. It now follows from (3), (16) and (15) that
\[
\int_{2^{kM-1}\pi}^{2^{(k+1)M-1}\pi} |\hat{\phi}_N(\xi)|^p d\xi = \sum_{l=0}^{M-1} \int_{2^{kM-1+l}\pi}^{2^{kM+l}\pi} |\hat{\phi}_N(\xi)|^p d\xi \leq 2^M \int_{2^{kM-1}\pi}^{2^{kM}\pi} |\hat{\phi}_N(\xi)|^p d\xi
\]
\[
\leq C' 2^{-NkM} |Q_N(\pi/3)|^{kM} \int_{2^{kM-1}\pi}^{2^{kM}\pi} \prod_{j=1}^{kM-1} |Q_N(2^{j-kM}\xi, \pi/6)| d\xi
\]
\[
\leq C' 2^{-NkM} |Q_N(\pi/3)|^{kM} \sum_{\epsilon_j \in \{0,1\}, 1 \leq j \leq kM} \prod_{j=1}^{kM} |Q_N(2^{j-kM}\xi, \pi/6)| d\xi
\]
\[
\leq C' 2^{-NkM} |Q_N(\pi/3)|^{kM} (2 + 2^M \prod_{j=1}^{kM} |Q_N(2^{j-kM}\xi, \pi/6)|).\]

This completes the proof. \( \square \)

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