ON A CHARACTERIZATION OF THE MAXIMAL IDEAL SPACES OF COMMUTATIVE $C^*$-ALGEBRAS IN WHICH EVERY ELEMENT IS THE SQUARE OF ANOTHER

OSAMU HATORI AND TAKESHI MIURA

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Dedicated to Professor Junzo Wada on his seventieth birthday

Abstract. A topological condition is given for a locally connected compact Hausdorff space on which every complex-valued continuous function is the square of another. The condition need not be necessary nor sufficient unless the space is locally connected.

1. Introduction

Let $X$ be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on $X$. Suppose that $X$ is locally connected and $A$ is a uniform algebra on $X$. Čirka \cite{2} proved that if the following condition $(\ast)$ is satisfied for $A$, then $A = C(X)$.

\begin{equation}
\text{For every } f \in A \text{ there exists a } g \in A \text{ with } g^2 = f.
\end{equation}

On the other hand $C(X)$ does not always satisfy the condition $(\ast)$: there is no continuous function on the unit circle $S^1$ whose square is the identity function on $S^1$. In this paper we give a necessary and sufficient condition for a locally connected compact Hausdorff space $X$ on which $(\ast)$ for $C(X)$ is satisfied. We also show that the condition is neither necessary nor sufficient for $(\ast)$ unless $X$ is locally connected.

2. Main results

Definition 2.1. Let $X$ be a compact Hausdorff space. We say that the condition $(\ast)$ holds for $C(X)$ if for each $f \in C(X)$ there exists a $g \in C(X)$ such that $f = g^2$.

Proposition 2.1. If $X$ is a totally disconnected compact Hausdorff space, then the condition $(\ast)$ holds for $C(X)$.

A proof of the proposition is elementary and is omitted.

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Definition 2.2. Let $X$ be a normal space. The covering dimension $\dim X$ of $X$ is less than or equal to $n$ if for every finite covering $\mathcal{A}$ there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ such that each $x \in X$ is in at most $n + 1$ elements in $\mathcal{B}$.

Note that the following is well-known (cf. [4]): $\dim X \leq n$ if and only if for every closed subset $F$ of $X$ and every $S^n$-valued continuous function $f$ defined on $F$, there exists $S^n$-valued continuous function $f$ defined on $X$ such that $f|_F = f$, where $S^n$ denotes the $n$-sphere.

Theorem 2.2. Let $X$ be a locally connected compact Hausdorff space. Then the following are equivalent.

(*) For every $f \in C(X)$ there exists a $g \in C(X)$ such that $f = g^2$.

(**) The first Čech cohomology group of $X$ with integer coefficients $\check{H}^1(X, \mathbb{Z})$ is trivial and $\dim X \leq 1$.

Note that $\check{H}^1(X, \mathbb{Z})$ is isomorphic to $C(X)^{-1}/\exp C(X)$ by a theorem of Arens and Royden (cf. [4] Theorem 7.2). We require two lemmas before proving Theorem 2.2. We assume in Lemma 2.3 and Lemma 2.4 that $X$ is a locally connected compact Hausdorff space.

Lemma 2.3. Suppose that the condition (*) holds for $C(X)$. Then the equality $C(X)^{-1} = \exp C(X)$ holds.

Proof. We show that each $f \in C(X)^{-1}$ with $f(X) \subseteq S^1$ belongs to $\exp C(X)$. It will follow by simple calculation that the conclusion holds. There exists a family $\{G_j\}_{j=1}^n$ of connected open subsets of $X$ such that $X = \bigcup_{j=1}^n G_j$ and $\overline{f(G_j)}$ is a proper subset of $S^1$ for $j = 1, 2, \ldots, n$, where $\overline{\cdot}$ denotes the closure in $\mathbb{C}$. There exist $a_j, b_j \in \mathbb{R}$ such that $f(G_j) \subseteq \{e^{i\theta} : a_j \leq \theta \leq b_j\}$ and $b_j - a_j < 2\pi$. By the condition (*) for $C(X)$, for each $l \in \mathbb{N}$ there exists a $g_l \in C(X)$ such that $f = g_l^{2\pi}$. Since $g_l(G_j)$ is connected, there correspond $a_{j_l}, b_{j_l} \in \mathbb{R}$ such that $g_l(G_j) \subseteq \{e^{i\theta} : a_{j_l} \leq \theta \leq b_{j_l}\}$ and $b_{j_l} - a_{j_l} = (b_j - a_j)/2\pi$. For sufficiently large $l \in \mathbb{N}$, we have

$$\sum_{j=1}^n (b_{j_l} - a_{j_l}) < \frac{n\pi}{2\pi} < 2\pi.$$ 

Hence $g_l(X)$ is a proper subset of $S^1$. Therefore there exists an $h \in C(X)$ such that $g_l = e^h$. Then

$$f = g_l^{2\pi} = e^{2\pi h} \in \exp C(X).$$

We have completed the proof.

Lemma 2.4. Suppose that the equalities

$$C(X)^{-1} = \exp C(X), \quad \overline{C(X)^{-1}} = C(X)$$

hold, where $\overline{\cdot}$ denotes the closure in $C(X)$. Then the condition (*) holds for $C(X)$.

Proof. For each $f \in C(X)^{-1}$ there exists an $h \in C(X)$ such that $f = e^h$, by the hypothesis. Put $g = e^{2\pi h}$. Then $g \in C(X)^{-1}$ and $f = g^2$. In a way similar to the proof of [4] Corollary 5.9] we see that (*) holds for $C(X)$.

Lemma 2.5. Suppose that $\dim X \leq 1$. Then the equality $\overline{C(X)^{-1}} = C(X)$ holds.
Proof. We show that for each $f \in C(X)$ there corresponds \{${f_n}$\} $\subset C(X)^{-1}$ such that $f_n \rightarrow f$ uniformly on $X$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, put $X_n = \{x \in X : |f(x)| \leq \frac{1}{n}\}$ and $E_n = \overline{X_n}^c$, the closure of $X_n^c$ in $X$. By Tietze’s extension theorem there exists a $v_n \in C(X)$ such that $v_n = |f|$ on $E_n$ and $v_n(X) \subset [1/n, ||f||]$. Further we may assume that $v_n$ satisfies the inequalities $1/n \leq v_n \leq 1/(n-1)$ on $X_n$. Since $\dim X \leq 1$, there exists $w_n \in C(X)$ such that

$$w_n|_{E_n} = \frac{f|_{E_n}}{|f|_{E_n}}$$ and $w_n(X) \subset S^1$.

Put $f_n = v_nw_n$; then $f_n \in C(X)^{-1}$. Since

$$|f(x) - f_n(x)| \leq \frac{1}{n} + \frac{1}{n-1}$$

holds for every $x \in X$, $f_n$ converges to $f$ uniformly on $X$. \hfill \Box

Proof of Theorem 2.2. Assume that (**) holds. By a theorem of Arens and Royden [3], we have $C(X)^{-1} = \exp C(X)$. Therefore (**) holds by Lemma 2.4 and Lemma 2.5.

Conversely, assume that (*) holds. Then by Lemma 2.3 $C(X)^{-1} = \exp C(X)$, so that $\hat{H}^1(X, \mathbb{Z})$ is trivial. It remains to show that (*) implies $\dim X \leq 1$. Assume that $\dim X \geq 2$. Then we see the following: there exist a closed subset $F$ of $X$ and an $f \in C(F)$ with $f(F) \subset S^1$ such that $f(x) \neq 0$ for any $f \in C(X)$ with $f_F = f$. In this case the following condition (#) holds. Let $D$ be the open unit disk and $\tilde{D}$ its closure in $\mathbb{C}$.

(#) For every $j \in \mathbb{N}$, the range $g(X)$ of any function $g \in C(X)$ with $g^2 |_{F}$ contains $\tilde{D}$.

Suppose not. Then $g_j(X) \notin \tilde{D}$ for some $j \in \mathbb{N}$ and $g_j \in C(X)$ with $g_j^2 |_{F} = f$. Then there exists a $z_0 \in D$ such that $z_0 \notin g_j(X)$. Choose a homeomorphism $\varphi$ from $\mathbb{C}$ onto itself such that $\varphi(z_0) = 0$ and that $\varphi|_{S^1}$ is identity. Put $\bar{\varphi} = (\varphi \circ g_j)/|\varphi \circ g_j|)^2$. Then $\bar{\varphi} \in C(X)$ with $\bar{\varphi}|_{F} = f$ and $\bar{\varphi}(X) \subset S^1$, which is a contradiction. We have proved that (#) holds.

With (#) we arrive at a contradiction. Fix a $\psi \in C(X)$ such that $\psi|_{F} = f$. For each $x \in X$ there corresponds a connected open neighborhood of $x$ which satisfies the following: if $\psi(x) = 0$, then $O_x$ is a connected open neighborhood of $x$ such that $|\psi(y)| < \frac{1}{2}$ for every $y \in O_x$; if $\psi(x) \neq 0$, then $O_x$ is a connected open neighborhood of $x$ such that

$$\psi(O_x) \subset \{re^{i\theta} : 0 < r < 1 + \|\psi\|_{\infty}, |\theta - \theta_x| < \pi\},$$

where $\theta_x \in [0, 2\pi)$ is an argument of $\psi(x)$. Then by the compactness of $X$, $X = \bigcup_{k=1}^{m} O_{x_k}$ for some $\{O_{x_k}\}_{k=1}^{m} \subset \{O_x\}_{x \in X}$. Without loss of generality we may assume that $\psi(x_k) \neq 0$ for $k = 1, 2, \cdots, m$, where $m$ is some constant such that $1 \leq m \leq n - 1$. Choose a sequence $\{\eta_j\}$ of functions in $C(X)$ such that $\eta_j^2 = \psi$. As in the proof of Lemma 2.3, for each $j \in \mathbb{N}$ and for $k = 1, 2, \cdots, m$ there exists a $\theta_{k_j} \in \mathbb{R}$ such that

$$\eta_j(O_{x_k}) \subset \{re^{i\theta} : 0 < r < (1 + \|\psi\|_{\infty})^{\frac{1}{2}}, |\theta - \theta_{k_j}| < \frac{\pi}{2}\}. $$
For a sufficiently large $j \in \mathbb{N}$ we have

$$\sum_{k=1}^{m} \left\{ (\theta_{k_j} + \frac{\pi}{2^j}) - (\theta_{k_j} - \frac{\pi}{2^j}) \right\} = \frac{m\pi}{2^{j-1}} < 2\pi.$$ 

Hence $S^1 \setminus \eta_j(X) \neq \emptyset$, which is a contradiction since $\eta_j(X) \supset \tilde{D} \supset S^1$. □

Note that the hypothesis in Theorem 2.2 that $X$ is locally connected is essential.

(i) Let $X$ be the Stone-Čech compactification of $[0, 1] \times \mathbb{N}$. Then we see that (*) holds for $C(X)$ and $\dim X = 1$ [3 Theorem 12 in Chapter 7]. But $\tilde{H}^1(X, \mathbb{Z})$ is not trivial. In fact, put $f_n(t) = e^{int}$ ($t \in [0, 1]$). Then the sequence $\{f_n\}_{n=1}^\infty$ is extended to a function $\tilde{f}$ in $C(X)^{-1}$. Suppose that $g_n$ is a function in $C([0, 1])$ with $f_n = e^{g_n}$. Then we see that $\sup \|g_n\|_\infty = +\infty$, so $\tilde{f} \not\in \exp C(X)$. Hence $\tilde{H}^1(X, \mathbb{Z})$ is not trivial, by a theorem of Arens and Royden [4].

(ii) Let $X = \bigcup_{n=0}^\infty I_n$, where $I_0 = [-1, 1] \times \{0\}$ and $I_n = [-1, 1] \times \{\frac{1}{n}\}$. We show that $\tilde{H}^1(X, \mathbb{Z})$ is trivial and $\dim X = 1$ while (*) does not hold for $C(X)$. First we show that $\tilde{H}^1(X, \mathbb{Z})$ is trivial. Let $f \in C(X)^{-1}$. For each $(t, \alpha) \in X$, put $\tilde{f}(t, \alpha) = f(t, \alpha)/f(t, 0)$. Then $\tilde{f} = 1$ on $I_0$, so there exists an $n_0 \in \mathbb{N}$ such that $f(I_n) \subset \{z \in \mathbb{C} : |z - 1| < 1\}$ for any $n \geq n_0$. Put $J = I_0 \cup (\bigcup_{k=n_0}^\infty I_k)$. Since $f(J) \subset \{z \in \mathbb{C} : |z - 1| < 1\}$ there exists a $g \in C(J)$ such that $\tilde{f} = e^g$. For each $n \leq n_0 - 1$, choose a $g_n \in C(I_n)$ such that $\tilde{f}|_{I_n} = e^{g_n}$. Let $h_1$ be as follows:

$$h_1(t, \alpha) = \begin{cases} g(t, \alpha), & (t, \alpha) \in J, \\ g_n(t, \alpha), & (t, \alpha) \in I_n, n \leq n_0 - 1. \end{cases}$$

Then $h_1 \in C(X)$ and $\tilde{f} = e^{h_1}$. Thus $f(t, \alpha) = f(t, 0)e^{h_1(t, \alpha)}$. Choose $h \in C(I_0)$ such that $f|_{I_0} = e^h$ and put $h_2(t, \alpha) = h(t)$ for every $(t, \alpha) \in X$. Then $f = e^{h_1 + h_2} \in \exp C(X)$.

Next we show that $\dim X = 1$. Let $F$ be a closed subset of $X$. A proof for the case where $F \cap I_0 = \emptyset$ is simple and is omitted. We consider the case where $F \cap I_0 \neq \emptyset$. Suppose that $f \in C(F)$ and $f(F) \subset S^1$. We show that a function $f$ is extended to a function $\tilde{f} \in C(X)$ such that $\tilde{f}(X) \subset S^1$. First we consider the case where $f = 1$ on $F \cap I_0$. There exist a $k_0 \in \mathbb{N}$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty$ of real numbers tending to zero as $j \to \infty$ such that $f(F \cap I_l) \subset \{e^{i\theta} : |\theta| < \varepsilon_l\}$ for each $l \geq k_0$. We may assume that $\varepsilon_j < \pi$. Then for each $l \geq k_0$ with $F \cap I_l \neq \emptyset$ there exists an $f_l \in C(I_l)$ such that $\tilde{f}_l = f$ on $F \cap I_l$ and $\tilde{f}_l(I_l) \subset \{e^{i\theta} : |\theta| < \varepsilon_l\}$. For each $l < k_0$ with $I_l \cap F \neq \emptyset$ there exists an $\tilde{f}_l \in C(I_l)$ such that $f_l = f$ on $F \cap I_l$ and $\tilde{f}_l(I_l) \subset S^1$ since $\dim I_l = 1$. Put

$$\tilde{f}(t, \alpha) = \begin{cases} 1, & (t, \alpha) \in I_0 \text{ or } (t, \alpha) \in I_n \text{ with } F \cap I_n = \emptyset, \\ \tilde{f}_l(t, \alpha), & (t, \alpha) \in I_l, l \geq k_0 \text{ with } F \cap I_l \neq \emptyset, \\ \tilde{f}_l(t, \alpha), & (t, \alpha) \in I_l, l < k_0 \text{ with } F \cap I_l \neq \emptyset. \end{cases}$$

Then $\tilde{f} \in C(X)$, $\tilde{f}(X) \subset S^1$ and $\tilde{f}|_F = f$. Next we consider the general case. There exists an $f_0 \in C(I_0)$ such that $f_0 = f$ on $F \cap I_0$ and $f_0(I_0) \subset S^1$ since $\dim I_0 = 1$. For each $(t, \alpha) \in F$, put $f_1(t, \alpha) = f(t, \alpha)/f_0(t)$. Then $f_1 \in C(F)$, $f_1(F) \subset S^1$ and $f_1|_{F \cap I_0} = 1$. Then by the first part, there exists an $f_2 \in C(X)$ such that $f_2|_F = f_1$ and $f_2(X) \subset S^1$. For each $(t, \alpha) \in X$, put $\tilde{f}(t, \alpha) = f_2(t, \alpha)f_0(t)$. Then $\tilde{f}$ is a desired function.
Finally we show that the condition (\#) for \(C(X)\) is not satisfied. Let \(f\) be as follows:

\[
f(t, 0) = \begin{cases} 
0, & t = 0, \\
|t|e^{\frac{2\pi i}{n}}, & t \neq 0;
\end{cases}
\]

if \(n\) is an even number

\[
f(t, n) = \begin{cases} 
\frac{1}{n}, & |t| \leq \frac{1}{n}, \\
|t|e^{\frac{2\pi i}{n}}, & |t| > \frac{1}{n};
\end{cases}
\]

if \(n\) is an odd number

\[
f(t, n) = \begin{cases} 
\frac{1}{n}e^{(nt+1)\pi i}, & |t| \leq \frac{1}{n}, \\
|t|e^{\frac{2\pi i}{n}}, & |t| > \frac{1}{n};
\end{cases}
\]

It is easy to see that \(f \in C(X)\) and there is no \(g \in C(X)\) with \(f = g^2\).

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Department of Mathematical Sciences, Graduate School of Science and Technology, Niigata University, 8050 Ikarashi 2-no-chou, Niigata 950-21, Japan