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ON A CHARACTERIZATION OF THE MAXIMAL IDEAL SPACES OF COMMUTATIVE C^*-ALGEBRAS IN WHICH EVERY ELEMENT IS THE SQUARE OF ANOTHER

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Dedicated to Professor Junzo Wada on his seventieth birthday

Abstract. A topological condition is given for a locally connected compact Hausdorff space on which every complex-valued continuous function is the square of another. The condition need not be necessary nor sufficient unless the space is locally connected.

1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X. Suppose that X is locally connected and A is a uniform algebra on X. Čirka [2] proved that if the following condition $(*)$ is satisfied for A, then $A = C(X)$.

$(*)$ For every $f \in A$ there exists a $g \in A$ with $g^2 = f$.

On the other hand $C(X)$ does not always satisfy the condition $(*)$: there is no continuous function on the unit circle S^1 whose square is the identity function on S^1. In this paper we give a necessary and sufficient condition for a locally connected compact Hausdorff space X on which $(*)$ for $C(X)$ is satisfied. We also show that the condition is neither necessary nor sufficient for $(*)$ unless X is locally connected.

2. Main results

Definition 2.1. Let X be a compact Hausdorff space. We say that the condition $(*)$ holds for $C(X)$ if for each $f \in C(X)$ there exists a $g \in C(X)$ such that $f = g^2$.

Proposition 2.1. *If X is a totally disconnected compact Hausdorff space, then the condition $(*)$ holds for $C(X)$.*

A proof of the proposition is elementary and is omitted.

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Definition 2.2. Let X be a normal space. The covering dimension $\dim X$ of X is less than or equal to n if for every finite covering $\mathfrak{A}$ there exists a refinement $\mathfrak{B}$ of $\mathfrak{A}$ such that each $x \in X$ is in at most $n+1$ elements in $\mathfrak{B}$.

Note that the following is well-known (cf. [5]): $\dim X \leq n$ if and only if for every closed subset F of X and every S^n-valued continuous function f defined on F, there exists S^n-valued continuous function $\tilde{f}$ defined on X such that $\tilde{f}|_F = f$, where S^n denotes the n-sphere.

Theorem 2.2. *Let X be a locally connected compact Hausdorff space. Then the following are equivalent.*
$(*)$ *For every $f \in C(X)$ there exists a $g \in C(X)$ such that $f = g^2$.*
$(**)$ *The first Čech cohomology group of X with integer coefficients $\check{H}^1(X, \mathbb{Z})$ is trivial and* $\dim X \leq 1$.

Note that $\check{H}^1(X, \mathbb{Z})$ is isomorphic to $C(X)^{-1}/\exp C(X)$ by a theorem of Arens and Royden (cf. [4, Theorem 7.2]). We require two lemmas before proving Theorem 2.2. We assume in Lemma 2.3 and Lemma 2.4 that X is a locally connected compact Hausdorff space.

Lemma 2.3. *Suppose that the condition $(*)$ holds for $C(X)$. Then the equality $C(X)^{-1} = \exp C(X)$ holds.*

Proof. We show that each $f \in C(X)^{-1}$ with $f(X) \subset S^1$ belongs to $\exp C(X)$. It will follow by simple calculation that the conclusion holds. There exists a family $\{G_j\}_{j=1}^n$ of connected open subsets of X such that $X = \bigcup_{j=1}^n G_j$ and $\overline{f(G_j)}$ is a proper subset of S^1 for $j = 1, 2, \cdots, n$, where $\bar{\cdot}$ denotes the closure in $\mathbb{C}$. There exist $a_j, b_j \in \mathbb{R}$ such that $f(G_j) \subset \{e^{i\theta} : a_j \leq \theta \leq b_j\}$ and $b_j - a_j < 2\pi$. By the condition $(*)$ for $C(X)$, for each $l \in \mathbb{N}$ there exists a $g_l \in C(X)$ such that $f = {g_l}^{2^l}$. Since $g_l(G_j)$ is connected, there correspond $a_{j_l}, b_{j_l} \in \mathbb{R}$ such that $g_l(G_j) \subset \{e^{i\theta} : a_{j_l} \leq \theta \leq b_{j_l}\}$ and $b_{j_l} - a_{j_l} = (b_j - a_j)/2^l$. For sufficiently large $l \in \mathbb{N}$, we have

$$\sum_{j=1}^{n} (b_{j_l} - a_{j_l}) < \frac{n\pi}{2^{l-1}} < 2\pi.$$

Hence $g_l(X)$ is a proper subset of S^1. Therefore there exists an $h \in C(X)$ such that $g_l = e^h$. Then

$$f = {g_l}^{2^l} = e^{2^l h} \in \exp C(X).$$

We have completed the proof. □

Lemma 2.4. *Suppose that the equalities*

$$C(X)^{-1} = \exp C(X), \ \overline{C(X)^{-1}} = C(X)$$

hold, where $\bar{\cdot}$ denotes the closure in $C(X)$. Then the condition $(*)$ holds for $C(X)$.

Proof. For each $f \in C(X)^{-1}$ there exists an $h \in C(X)$ such that $f = e^h$, by the hypothesis. Put $g = e^{\frac{h}{2}}$. Then $g \in C(X)^{-1}$ and $f = g^2$. In a way similar to the proof of [1, Corollary 5.9] we see that $(*)$ holds for $C(X)$. □

Lemma 2.5. *Suppose that* $\dim X \leq 1$. *Then the equality $\overline{C(X)^{-1}} = C(X)$ holds.*

Proof. We show that for each $f \in C(X)$ there corresponds $\{f_n\} \subset C(X)^{-1}$ such that $f_n \to f$ uniformly on X as $n \to \infty$. For each $n \in \mathbb{N}$, put $X_n = \{x \in X : |f(x)| \leq \frac{1}{n}\}$ and $E_n = \overline{{X_n}^c}$, the closure of ${X_n}^c$ in X. By Tietze's extension theorem there exists a $v_n \in C(X)$ such that $v_n = |f|$ on E_n and $v_n(X) \subset [1/n, \|f\|_\infty]$. Further we may assume that v_n satisfies the inequalities $1/n \leq v_n \leq 1/(n-1)$ on X_n. Since $\dim X \leq 1$, there exists $w_n \in C(X)$ such that

$$w_n|_{E_n} = \frac{f|_{E_n}}{|f|_{E_n}|} \text{ and } w_n(X) \subset S^1.$$

Put $f_n = v_n w_n$; then $f_n \in C(X)^{-1}$. Since

$$|f(x) - f_n(x)| \leq \frac{1}{n} + \frac{1}{n-1}$$

holds for every $x \in X$, f_n converges to f uniformly on X. □

Proof of Theorem 2.2. Assume that $(**)$ holds. By a theorem of Arens and Royden [4], we have $C(X)^{-1} = \exp C(X)$. Therefore $(*)$ holds by Lemma 2.4 and Lemma 2.5.

Conversely, assume that $(*)$ holds. Then by Lemma 2.3 $C(X)^{-1} = \exp C(X)$, so that $\check{H}^1(X, \mathbb{Z})$ is trivial. It remains to show that $(*)$ implies $\dim X \leq 1$. Assume that $\dim X \geq 2$. Then we see the following: there exist a closed subset F of X and an $f \in C(F)$ with $f(F) \subset S^1$ such that $\tilde{f}(X) \setminus S^1 \neq \emptyset$ for any $\tilde{f} \in C(X)$ with $\tilde{f}|_F = f$. In this case the following condition $(\#)$ holds. Let D be the open unit disk and $\bar{D}$ its closure in $\mathbb{C}$.

> $(\#)$ For every $j \in \mathbb{N}$, the range $g(X)$ of any function $g \in C(X)$ with $g^{2^j}|_F = f$ contains $\bar{D}$.

Suppose not. Then $g_j(X) \not\supset \bar{D}$ for some $j \in \mathbb{N}$ and $g_j \in C(X)$ with ${g_j}^{2^j}|_F = f$. Then there exists a $z_0 \in D$ such that $z_0 \notin g_j(X)$. Choose a homeomorphism φ from $\mathbb{C}$ onto itself such that $\varphi(z_0) = 0$ and that $\varphi|_{S^1}$ is identity. Put $\tilde{\varphi} = (\varphi \circ g_j/|\varphi \circ g_j|)^{2^j}$. Then $\tilde{\varphi} \in C(X)$ with $\tilde{\varphi}|_F = f$ and $\tilde{\varphi}(X) \subset S^1$, which is a contradiction. We have proved that $(\#)$ holds.

With $(\#)$ we arrive at a contradiction. Fix a $\psi \in C(X)$ such that $\psi|_F = f$. For each $x \in X$ there corresponds a connected open neighborhood of x which satisfies the following: if $\psi(x) = 0$, then O_x is a connected open neighborhood of x such that $|\psi(y)| < \frac{1}{2}$ for every $y \in O_x$; if $\psi(x) \neq 0$, then O_x is a connected open neighborhood of x such that

$$\psi(O_x) \subset \{re^{i\theta} : 0 < r < 1 + \|\psi\|_\infty, |\theta - \theta_x| < \pi\},$$

where $\theta_x \in [0, 2\pi)$ is an argument of $\psi(x)$. Then by the compactness of X, $X = \bigcup_{k=1}^n O_{x_k}$ for some $\{O_{x_k}\}_{k=1}^n \subset \{O_x\}_{x \in X}$. Without loss of generality we may assume that $\psi(x_k) \neq 0$ for $k = 1, 2, \cdots, m$, where m is some constant such that $1 \leq m \leq n-1$. Choose a sequence $\{\eta_j\}$ of functions in $C(X)$ such that ${\eta_j}^{2^j} = \psi$. As in the proof of Lemma 2.3, for each $j \in \mathbb{N}$ and for $k = 1, 2, \cdots, m$ there exists a $\theta_{k_j} \in \mathbb{R}$ such that

$$\eta_j(O_{x_k}) \subset \{re^{i\theta} : 0 < r < (1 + \|\psi\|_\infty)^{\frac{1}{2^j}}, |\theta - \theta_{k_j}| < \frac{\pi}{2^j}\}.$$

For a sufficiently large $j \in \mathbb{N}$ we have

$$\sum_{k=1}^{m} \left\{ (\theta_{k_j} + \frac{\pi}{2^j}) - (\theta_{k_j} - \frac{\pi}{2^j}) \right\} = \frac{m\pi}{2^{j-1}} < 2\pi.$$

Hence $S^1 \setminus \eta_j(X) \neq \emptyset$, which is a contradiction since $\eta_j(X) \supset \bar{D} \supset S^1$. □

Note that the hypothesis in Theorem 2.2 that X is locally connected is essential.

(i) Let X be the Stone-Čech compactification of $[0,1] \times \mathbb{N}$. Then we see that $(*)$ holds for $C(X)$ and $\dim X = 1$ [3, Theorem 12 in Chapter 7]. But $\check{H}^1(X, \mathbb{Z})$ is not trivial. In fact, put $f_n(t) = e^{int}$ $(t \in [0,1])$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ is extended to a function $\tilde{f}$ in $C(X)^{-1}$. Suppose that g_n is a function in $C([0,1])$ with $f_n = e^{g_n}$. Then we see that $\sup_{n \in \mathbb{N}} \|g_n\|_\infty = +\infty$, so $\tilde{f} \notin \exp C(X)$. Hence $\check{H}^1(X, \mathbb{Z})$ is not trivial, by a theorem of Arens and Royden [4].

(ii) Let $X = \bigcup_{n=0}^{\infty} I_n$, where $I_0 = [-1,1] \times \{0\}$ and $I_n = [-1,1] \times \{\frac{1}{n}\}$. We show that $\check{H}^1(X, \mathbb{Z})$ is trivial and $\dim X = 1$ while $(*)$ does not hold for $C(X)$. First we show that $\check{H}^1(X, \mathbb{Z})$ is trivial. Let $f \in C(X)^{-1}$. For each $(t, \alpha) \in X$, put $\tilde{f}(t, \alpha) = f(t, \alpha)/f(t, 0)$. Then $\tilde{f} = 1$ on I_0, so there exists an $n_0 \in \mathbb{N}$ such that $\tilde{f}(I_n) \subset \{z \in \mathbb{C} : |z - 1| < 1\}$ for any $n \geq n_0$. Put $J = I_0 \cup (\bigcup_{k=n_0}^{\infty} I_k)$. Since $\tilde{f}(J) \subset \{z \in \mathbb{C} : |z - 1| < 1\}$ there exists a $g \in C(J)$ such that $\tilde{f}|_J = e^g$. For each $n \leq n_0 - 1$, choose a $g_n \in C(I_n)$ such that $\tilde{f}|_{I_n} = e^{g_n}$. Let h_1 be as follows:

$$h_1(t, \alpha) = \begin{cases} g(t, \alpha), & (t, \alpha) \in J, \\ g_n(t, \alpha), & (t, \alpha) \in I_n,\ n \leq n_0 - 1. \end{cases}$$

Then $h_1 \in C(X)$ and $\tilde{f} = e^{h_1}$. Thus $f(t, \alpha) = f(t, 0)e^{h_1(t, \alpha)}$. Choose $h \in C(I_0)$ such that $f|_{I_0} = e^h$ and put $h_2(t, \alpha) = h(t)$ for every $(t, \alpha) \in X$. Then $f = e^{h_1 + h_2} \in \exp C(X)$.

Next we show that $\dim X = 1$. Let F be a closed subset of X. A proof for the case where $F \cap I_0 = \emptyset$ is simple and is omitted. We consider the case where $F \cap I_0 \neq \emptyset$. Suppose that $f \in C(F)$ and $f(F) \subset S^1$. We show that a function f is extended to a function $\tilde{f} \in C(X)$ such that $\tilde{f}(X) \subset S^1$. First we consider the case where $f = 1$ on $F \cap I_0$. There exist a $k_0 \in \mathbb{N}$ and a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of real numbers tending to zero as $j \to \infty$ such that $f(F \cap I_l) \subset \{e^{i\theta} : |\theta| < \varepsilon_l\}$ for each $l \geq k_0$. We may assume that $\varepsilon_j < \pi$. Then for each $l \geq k_0$ with $F \cap I_l \neq \emptyset$ there exists an $\dot{f}_l \in C(I_l)$ such that $\dot{f}_l = f$ on $F \cap I_l$ and $\dot{f}_l(I_l) \subset \{e^{i\theta} : |\theta| < \varepsilon_l\}$. For each $l < k_0$ with $I_l \cap F \neq \emptyset$ there exists an $\ddot{f}_l \in C(I_l)$ such that $\ddot{f}_l = f$ on $F \cap I_l$ and $\ddot{f}(I_l) \subset S^1$ since $\dim I_l = 1$. Put

$$\tilde{f}(t, \alpha) = \begin{cases} 1, & (t, \alpha) \in I_0 \text{ or } (t, \alpha) \in I_n \text{ with } F \cap I_n = \emptyset, \\ \dot{f}_l(t, \alpha), & (t, \alpha) \in I_l,\ l \geq k_0 \text{ with } F \cap I_l \neq \emptyset, \\ \ddot{f}_l(t, \alpha), & (t, \alpha) \in I_l,\ l < k_0 \text{ with } F \cap I_l \neq \emptyset. \end{cases}$$

Then $\tilde{f} \in C(X)$, $\tilde{f}(X) \subset S^1$ and $\tilde{f}|_F = f$. Next we consider the general case. There exists an $f_0 \in C(I_0)$ such that $f_0 = f$ on $F \cap I_0$ and $f_0(I_0) \subset S^1$ since $\dim I_0 = 1$. For each $(t, \alpha) \in F$, put $f_1(t, \alpha) = f(t, \alpha)/f_0(t)$. Then $f_1 \in C(F), f_1(F) \subset S^1$ and $f_1|_{F \cap I_0} = 1$. Then by the first part, there exists an $f_2 \in C(X)$ such that $f_2|_F = f_1$ and $f_2(X) \subset S^1$. For each $(t, \alpha) \in X$, put $\tilde{f}(t, \alpha) = f_2(t, \alpha) f_0(t)$. Then $\tilde{f}$ is a desired function.

Finally we show that the condition $(*)$ for $C(X)$ is not satisfied. Let f be as follows:

$$f(t,0) = \begin{cases} 0, & t = 0, \\ |t|e^{\frac{2\pi i}{t}}, & t \neq 0; \end{cases}$$

if n is an even number

$$f(t,n) = \begin{cases} \frac{1}{n}, & |t| \leq \frac{1}{n}, \\ |t|e^{\frac{2\pi i}{t}}, & |t| > \frac{1}{n}; \end{cases}$$

if n is an odd number

$$f(t,n) = \begin{cases} \frac{1}{n}e^{(nt+1)\pi i}, & |t| \leq \frac{1}{n}, \\ |t|e^{\frac{2\pi i}{t}}, & |t| > \frac{1}{n}. \end{cases}$$

It is easy to see that $f \in C(X)$ and there is no $g \in C(X)$ with $f = g^2$.

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