THE RESIDUES OF THE RESOLVENT ON DAMEK-RICCI SPACES

R. J. MIATELLO AND C. E. WILL

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Abstract. We determine the poles and residues of the resolvent kernel of the Laplacian on a Damek-Ricci space $S$. We show that all poles are simple and the residues define convolution operators of finite rank. This generalizes a result of Guillope-Zworski for the real hyperbolic $n$-space. If $S$ corresponds to a symmetric space of negative curvature $G=K$, the image of each residue is a $g_c$-module with a specific highest weight. We compute the dimension by the Weyl dimension formula.

1. Preliminaries

In this section we will recall some basic notions on $H$-type groups and their canonical solvable extensions, following mainly [2] (see also [1]).

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. If $\mathfrak{n}$ is abelian, we shall use the convention that $\mathfrak{v} = 0$ and $\mathfrak{n} = \mathfrak{z}$.

Define a linear mapping $J : \mathfrak{z} \to \text{End}(\mathfrak{v})$ by
\[
\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle
\]
(note that $J_Z$ is skew-symmetric). Now $\mathfrak{n}$ is said to be an $H$-type algebra if for any $Z_1, Z_2 \in \mathfrak{z}$,
\[
J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -2 \langle Z_1, Z_2 \rangle.
\]

The corresponding $H$-type group is the simply connected Lie group $N$ with Lie algebra $\mathfrak{n}$, endowed with the left-invariant riemannian metric induced by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$.

Consider the solvable extension, $S = AN$, the semidirect product of $A = \mathbb{R}^+$ and $N$, where each $t \in A$ acts on $N$ by $(x, z) \mapsto (tx, tz)$.

Let $\mathfrak{s}, \mathfrak{a}$, denote respectively the Lie algebras of $S, A$. Then $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{a} = \mathbb{R}H$, where $ad H$ is the derivation of $\mathfrak{n}$ such that $ad H|_\mathfrak{a} = \frac{1}{2}I$ and $ad H|_\mathfrak{z} = I$. Also, $\mathfrak{s}$ carries the inner product extending the one on $\mathfrak{n}$ such that $\|H\| = 1$, $\langle H, \mathfrak{n} \rangle = 0$; $S$ carries the induced left-invariant riemannian structure. Furthermore, let $q = \dim \mathfrak{z}$, $p = \dim \mathfrak{v}$, $n = \dim \mathfrak{s} = p + q + 1$ and $Q = \frac{1}{2}(p + 2q)$.

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Using coordinates from $v \oplus z \oplus R^+$, the product on $S$ is expressed as

$$(X, Z, a)(X', Z', a') = (X + a^\perp X', Z + aZ' + \frac{1}{2} a^\perp [X, X'], a a').$$

The volume element of the induced left-invariant riemannian metric on $S$ is the left Haar measure

$$d \mu = a^{-Q-1} dX dZ d\alpha.$$

We will use the fact that $S$ can be realized as the unit ball in $s$:

$$B(s) = \{(X, Z, u) : |X|^2 + |Z|^2 + u^2 = 1\}$$

via a Cayley type transform $\tilde{C} : S \rightarrow B(s)$ (see [2], Section 4).

In $B(s)$ the geodesics through the origin are the diameters and the geodesic distance to the origin $r = d(\tilde{p}, 0) = \log \frac{1 + |\tilde{p}|}{1 - |\tilde{p}|}$, thus $|\tilde{p}| = \tanh(r/2)$, with $\tilde{p} = \tilde{C}(p)$, if $p \in S$. Furthermore $\cosh (k^{-1}) = \frac{1 + a + b}{(1 + a - b)(1 - a + b)}$ and the image of the left Haar measure on $S$ via $\tilde{C}^{-1}$ is $d\mu = J(r) d\sigma dr$, where $r, \sigma$ are the radial coordinates on $B$, $r^2 = |X|^2 + |Z|^2 + u^2$ and $J(r) = 2^p \sinh(r/2)^p \sinh(r)^q$ (see [2], Section 4).

The symmetric spaces of negative curvature are a main subclass of the Damek-Ricci spaces. Let $G$ be a connected, noncompact, semisimple Lie group of real rank one. Let $K$ be a maximal compact subgroup of $G$ and let $g$ and $k$ be the corresponding Lie algebras. If $G = NA \ltimes K$ is an Iwasawa decomposition of $G$, then $N$ is an $H$-type group and $S = NA \approx G/K$ is a solvable Lie group in the class introduced above. Indeed, if $a$ and $n$ denote the Lie algebras of $A$ and $N$ respectively, $n$ splits $n = g_{\alpha/2} \oplus g_{\alpha}$, where $g_{\alpha}$, $j = 1/2, 1$, denote the $j$-root spaces of $a$. In the notation above we have $n_0 = z$, $n_{\alpha/2} = v$, $a = RH_o$, with $H_o \in a$ such that $\alpha(H_o) = 1$. If on $S = NA$ we use the $G$-invariant metric induced by $2(p + 4q)^{-1}B$ (the Killing form of $g$), then $S$ is isometric to a Damek-Ricci space. We note that, because of our convention, if $n$ is abelian, then $p = 0$, $q = \dim n$.

2. THE RESOLVENT OF THE LAPLACIAN ON $S$

Damek-Ricci spaces have strong similarities with symmetric spaces of negative curvature, in particular they are harmonic spaces. On $S$ there is a radialization operator $\pi$ which corresponds to the standard operator in the case of the ball model of $S$ (see [2], p. 230). If $f \in C^\infty_c(S)$, $p \in S$ and $\tilde{p} = \tilde{C}(p)$ then

$$\pi f(p) := \int_{S^{p+q}} \check{f}(||\tilde{p}||\sigma) d\sigma,$$

where $\check{f} := f \circ \tilde{C}^{-1}$. In the symmetric case, if $f \in C^\infty(NA)$, then $\pi f(x) = \int_K \check{f}(kx) dk$, where $\check{f}$ denotes the right $K$-invariant extension of $f$.

If $\{Z_i\}, \{V_j\}$ are orthonormal bases of $z$ and $v$ respectively, the Laplace-Beltrami operator is given by $L = \sum_i Z_i^2 + \sum_j V_j^2 + H^2 - QH$; $L$ generates the algebra of left-invariant differential operators on $S$ which commute with $\pi$ (see [2], Theorem 5.2).

If $f$ is a smooth radial function on $S - \{e\}$, we will often abuse notation by writing $f(r) = f(x)$, where $r = d(x, e)$. The action of $L$ on radial functions is given by

$$Lf(r) = \frac{d^2}{dr^2} f(r) + \frac{1}{2} \left( p \coth(r/2) + 2q \coth(r) \right) \frac{d}{dr} f(r).$$
In the symmetric case, if \( \mathfrak{n} \) is not abelian and we set \( r = 2t \), then \( Lf(t) \) corresponds to \( 4Cf(a_t) \), \( C \) the Casimir element; [6], Section 1 (1). If \( \mathfrak{n} \) is abelian, then \( L \) corresponds to \( C \).

A spherical function \( \psi \) on \( S \) is a radial eigenfunction of \( L \) such that \( \psi(e) = 1 \). This generalizes the corresponding notion in the symmetric case and one has the following characterization (2).

**Proposition 2.1.** Let \( \nu \in \mathbb{C} \). The function \( \phi_\nu = \pi(a^{\nu+Q/2}) \) is a spherical function with eigenvalue \( \lambda(\nu) = \nu^2 - Q^2/4 \). Any spherical function on \( S \) is of this form.

As in the symmetric case, we can express \( \phi_\nu \) by a hypergeometric function as follows. By letting \( z = -\sinh(r/2) \), the equation

\[
\frac{d^2}{dr^2} + \frac{1}{2} (p \coth(r/2) + 2q \coth(r)) \frac{d}{dr} - \lambda(\nu) \}
\]

transforms into the hypergeometric equation with parameters \( a = Q/2 - \nu, b = Q/2 + \nu, c = n/2 \). Since \( \phi_\nu(e) = 1 \), it follows that

\[
\phi_\nu(r) = F \left( \nu + Q/2, \nu + Q/2, \frac{n}{2} - \sinh(r/2)^2 \right).
\]

Furthermore, if \( \Re \nu > 0 \), the asymptotic behavior of \( \phi_\nu(r) \), as \( r \to \infty \), is given by (see [2], p. 239)

\[
\phi_\nu(r) \sim c(\nu) e^{r(\nu+Q/2)}, \quad \text{where} \quad c(\nu) = \frac{2^{-2\nu+Q} \Gamma(n/2) \Gamma(2\nu)}{\Gamma(n+Q/2) \Gamma(\nu+Q/2)}.
\]

Here \( c(\nu) \) coincides with Harish Chandra’s \( c \)-function in the symmetric case. The Plancherel measure, \( \mu(\nu) = (c(\nu)c(-\nu))^{-1} \), can be written \( \mu(\nu) = c_0 p(\nu) D(\nu), c_0 \) a constant and \( p(\nu) \) the polynomial given by

\[
\prod_{j=0}^{q-1} (-\nu^2 + ((2j + 1)^2/4)) \prod_{j=0}^{q-1} (-\nu^2 + (j^2/4)), \quad q \text{, \( \frac{q}{2} \) even,}
\]

\[
-\prod_{j=1}^{q/2} (\nu^2 + j^2)^2 \nu^3, \quad q = 1, \frac{q}{2} \text{ odd,}
\]

\[
-\prod_{j=0}^{q-1} (-\nu^2 + ((2j + 1)^2/4)) \prod_{j=0}^{q-1} (-\nu^2 + ((2j + 1)^2/4)) \nu, \quad q \text{ odd, \( \frac{q}{2} \) even,}
\]

and \( D(\nu) \) equals respectively 1, \( \cot(\pi\nu) \), and \( \tan(\pi\nu) \) (1).

**Remark.** We note that \( p \) is always even, since \( \mathfrak{g} \) is a module over the Clifford algebra of \( \mathfrak{g} \). If \( \nu = 0 \), then \( X \approx H^{q+1} \), \( G \approx \text{SO}(q+1, 1) \) and in this case \( D(\nu) \) equals 1 or \( \tan(\pi\nu) \) depending on whether \( q \) is even or odd.

In [6], the resolvent of the Laplacian \( R(\lambda(\nu)) \) was studied on symmetric (and locally symmetric spaces) of negative curvature. In the symmetric case, it is given for \( \Re \nu > p \) by convolution with a smooth radial function \( Q_\nu \) on \( S - \{ e \} \) which is an eigenfunction of \( L \) with eigenvalue \( \lambda(\nu) \), and which has a meromorphic continuation to \( C \). As we shall now see, these properties remain valid for any \( S \) as above. Many arguments in [6] can be adapted, so we shall omit several proofs. On the other hand, we shall show how to obtain \( Q_\nu \) by using a series solution. We thank N. Wallach for useful discussions on this point, which helped us to simplify the original argument.
If \( b \in \mathbb{R} \) and \( \delta > 0 \), let \( S_{b,\delta} = \{ \nu : \Re \nu > b, |\nu + j| > \delta \ \forall j \in \mathbb{N} : b \leq j \} \). That is, \( S_{b,\delta} = \{ \nu : \Re \nu > b \} \), if \( b \geq 0 \), and \( S_{b,\delta} \) is a half plane with finitely many discs removed, centered at \(-1, -2, \ldots, -k\), with \(-k \geq b\), if \( b < 0 \).

**Theorem 2.2.** If \( \nu \in \mathbb{C} \), \( 2\nu \notin -\mathbb{N} \), then there exists a radial function \( Q_\nu \in C^\infty(S - \{ e \}) \) with the following properties:

(a) \((L - \lambda(\nu))Q_\nu = 0\). For each \( x \in S \), \( Q_\nu(x) \) is holomorphic for \( \nu \notin -\frac{1}{2}\mathbb{N} \) and in \( \nu \in \frac{1}{2}\mathbb{N} \), \( Q_\nu(s) \) has at most a simple pole. Furthermore, for any \( b \in \mathbb{R} \), \( \delta, r_o > 0 \), there exists \( K = K(b, \delta, r_o) \) such that \( |Q_\nu(r)| \leq K \) for any \( r \geq r_o, \nu \in S_{b,\delta} \).

(b) Where defined, \( \phi_\nu = c(-\nu)Q_\nu + c(\nu)Q_{-\nu} \).

(c) As \( r \to 0 \), \( Q_\nu(r) \sim d(\nu)r^{-p-q+1}|\log r|^{b+q+1} \), for some meromorphic function \( d(\nu) \) on \( \mathbb{C} \), holomorphic if \( 2\nu \notin -\mathbb{N} \).

(d) \( \lim_{r \to 0^+} J(r) \frac{d}{dx} Q_\nu(r) = -2\nu c(\nu) \).

(e) If \( f \in C^\infty_c(S) \) and \( 2\nu \notin -\mathbb{N} \), then

\[
\int_S Q_\nu(x^{-1}y)(L - \lambda(\nu))f(y)dy = -2\nu c(\nu)f(x).
\]

**Proof.** We look for a solution of (11) of the form \( q_\nu(r) = \sum_{j=0}^\infty a_j(\nu)e^{-((\nu+j)/2)r} \).

Substituting in (11) and using \( \coth(r) = \frac{1+e^{-2r}}{1-e^{-2r}} \), we get that

\[
\sum_{j \geq 0} (Q + j)(2\nu + Q + j)a_j(\nu)e^{-jr} + p \sum_{j \geq 1} (\nu + Q/2 + j + 1)a_{j+1}(\nu)e^{-jr} + \sum_{j \geq 2} (j + 2)(2\nu + j + 2)a_{j+2}(\nu)e^{-jr} = 0.
\]

Thus, the coefficients \( a_j(\nu) \) must satisfy the recurrence relations

\[
a_1(\nu) = a_0(\nu)f_{-1}(\nu), \quad a_{j+2}(\nu) = a_{j+1}(\nu)f_j(\nu) + a_j(\nu)g_j(\nu),
\]

where \( f_j(\nu) = p\frac{\nu+Q/2+j+1}{Q+j+1} \) and \( g_j(\nu) = \frac{(Q+j)(2\nu+Q+j)}{(Q+j+1)(2\nu+j+2)} \), for \( j \geq 0 \).

We thus set \( q_\nu(r) = e^{-((\nu+j)/2)r} \sum_{j=0}^\infty a_j(\nu)e^{-jr} \), where \( a_0 = 1 \), and if \( 2\nu \notin -\mathbb{N} \), then the \( a_j(\nu) \) are given by (8).

If \( b \in \mathbb{R} \), \( \delta > 0 \) and \( \nu \in S_{b,\delta} \), we have

\[
|f_j(\nu)| \leq \frac{p}{2j+4} \left( 1 + \frac{Q+j}{2\nu+j+2} \right) \leq \frac{p}{2j+4} \left( 1 + \frac{Q+j}{j+2-2k} \right),
\]

\[
|g_j(\nu)| \leq \frac{Q+j}{2j+2} \left( 1 + \frac{|Q-2j|}{2\nu+j+2} \right) \leq \frac{Q+j}{j+2} \left( 1 + \frac{|Q-2|}{j+2-2k} \right)
\]

for \( j + 2 > 2|k| \), where \( k \) is the first integer such that \( k \leq b \). These estimates clearly imply that given \( \varepsilon > 0 \) there exist \( j_0 \) and \( M = M(\varepsilon) \) such that \( |f_j(\nu)| \leq \varepsilon, |g_j(\nu)| \leq 1 + \varepsilon, j \geq j_0, |f_j(\nu)| \leq M, |g_j(\nu)| \leq M, j < j_0 \), uniformly for \( \nu \in S_{b,\delta} \). Using these estimates we see that if \( \nu \in S_{b,\delta} \), if \( M' = M'(\varepsilon) = j_0M_{j_0} \), then

\[
|a_j(\nu)| \leq \left\{ \begin{array}{ll}
\frac{jM^j}{M'(1+2\varepsilon)^{j-j_0}+1}, & j \leq j_0, \\
\frac{jM^j}{M'(1+2\varepsilon)^{j-j_0+1}}, & j \geq j_0.
\end{array} \right.
\]

Now, by (9) \( |q_\nu(r)| \leq e^{-(\Re \nu + 1/2)r}M' \left( j_0 + \sum_{l \geq 0} (1+2\varepsilon)^{l+1}e^{-(l+j_0)r} \right) \); hence the series defining \( q_\nu \) converges absolutely and uniformly for \( \nu \in S_{b,\delta} \) and \( r > r_o \), for
Theorem 3.1. If $q(x) = q_{e}(x)$ with $r = d(x,e)$, for $x \in S$, then $Q_{\nu \nu} \in C^{\infty}(S - \{e\})$ is a radial eigenfunction of $L$ of eigenvalue $\lambda(\nu)$ and has the properties stated in (a).

From now on we shall write $Q_{\nu}(r) = q_{e}(r)$, for simplicity. By the asymptotic behavior as $r \to +\infty$, it follows that if $2\nu \notin \mathbb{Z}$, $Q_{\nu}(r), Q_{-\nu}(r)$ form a fundamental system of solutions of (4). Writing $\phi_{\nu}$ in terms of $Q_{\nu}$ and $Q_{-\nu}$, the functional equation in (b) follows as in the symmetric case (see [6], p. 671).

The proof of (d) is similar to that of [6], Lemma 1.3, and will be omitted.

We now prove (c). Equation (4) has a regular singular point at $r = 0$ and the corresponding indicial equation is $s(s - 1) + (p + q)s = 0$, with roots $s = 0, s = 1 - p - q$. The solution $\phi_{\nu}(r)$ is associated to the root $s = 0$ and is continuous at $r = 0$. If $2\nu \notin \mathbb{N}$, and if $p + q > 1$, $Q_{\nu}$ is a second linearly independent solution; hence $\lim_{r \to +0} Q_{\nu}(r)r^{p+q-1} := d(\nu)$ exists and the meromorphy of $Q_{\nu}$ implies that of $d(\nu)$. Similarly, if $p + q = 1$, $Q_{\nu}(r) \sim d(\nu) \log r$ as $r \to 0^{\pm}$. Thus (c) follows.

The proof of (d) is similar to that of [6], Lemma 1.3, and will be omitted.

To see (e) we may assume that $x = e$. We have, for any $f \in C_{c}^{\infty}(S)$,

$$
\int_{S} Q_{\nu}(y)(L - \lambda(\nu)I)f(y) \, d\mu(y) = \int_{\partial B_{0}} \int_{0}^{\infty} \tilde{Q}_{\nu}(r\sigma)(L - \lambda(\nu)I)\tilde{f}(r\sigma)J(r)drd\sigma
$$

$$
= \int_{0}^{\infty} Q_{\nu}(r)J(r)(L - \lambda(\nu)I)f(r) \, dr.
$$

Now we observe that for a radial function $h$ on $S$, $J(r)^{1/2}Lh(r) = \frac{d^{2}}{dr^{2}}J^{1/2}(r)h(r) + J(r)^{1/2}\eta(r)\eta(r)h(r)$, where $\eta = \frac{J^{1/2}(r) - 2J^{1/2}(r)\eta}{J^{1/2}(r)}$. Hence we see that the above equals

$$
\int_{0}^{\infty} \frac{d^{2}}{dr^{2}} \left( J(r)^{1/2}f(r) \right) J(r)^{1/2}Q_{\nu}(r) - J(r)^{1/2}\pi f(r) \frac{d^{2}}{dr^{2}} \left( J(r)^{1/2}Q_{\nu}(r) \right) dr
$$

$$
= \int_{0}^{\infty} \frac{d}{dr} \left[ \frac{d}{dr} (J(r)^{1/2}\pi f(r)) J(r)^{1/2}Q_{\nu}(r) - J(r)^{1/2}\pi f(r) \frac{d}{dr} (J(r)^{1/2}Q_{\nu}(r)) \right] dr
$$

$$
= -2\nu c(\nu)f(e)
$$

using (c) and (d). This gives (e); hence the theorem follows.

3. The Residues of the Resolvent

Let $\hat{R}(\lambda(\nu))$ denote the kernel operator with kernel $K_{\nu}(x,y) = -\frac{Q_{\nu}(x^{-1}y)}{2\nu c(\nu)}$. If $\text{Re} \, \nu > 0$, then $\hat{R}(\lambda(\nu)) = R(\lambda(\nu))$.

Theorem 3.1. If $p, q$ are both even, then $\hat{R}(\lambda(\nu))$ is everywhere holomorphic. Otherwise, it has simple poles lying at $\nu_{k} = -Q/2 - k$ with $k \in \mathbb{N} \cup \{0\}$. If $\nu = \nu_{k}$, set $T_{\nu_{k}}(f) := \text{Res}_{\nu=\nu_{k}} \hat{R}(\lambda(\nu))(f)$. Then $T_{\nu_{k}}(f) = (2\pi \nu_{k})^{-1} p(\nu_{k}) f * \phi_{\nu}$ and $T_{\nu_{k}}$ is a finite rank operator, for each value of $k$.

Proof. The possible poles of $K_{\nu}(x,y)$ lie at $-\frac{1}{2}\mathbb{N}$ or at the zeros of $c(\nu)$. By using formula [3] one sees that $c(\nu)$ has no zeros in $\mathbb{C}$, if $p$ and $q$ are both even. Otherwise, $q$ is odd and $c(\nu)$ has simple zeros at $\nu_{k} = -Q/2 - k$, for any $k \in \mathbb{N} \cup \{0\}$, and possibly simple poles at $\nu \in -\frac{1}{2}\mathbb{N}$.
Since \( \nu = 0 \) is a simple pole of \( c(\nu) \), and \( Q_\nu \) is holomorphic at 0, \( \frac{Q_\nu}{2\nu c(\nu)} \) is
holomorphic at \( \nu = 0 \).

On the other hand \( c(-\nu) \) and \( Q_{-\nu} \) are holomorphic and nonvanishing on \( \mathbb{R}^{<0} \),
\( \phi_\nu \) is everywhere holomorphic and \( \phi_\nu(0) = 1 \). So Theorem 2.2 (b) implies that a
pole of \( Q_\nu \) must be compensated by a pole of \( c(\nu) \) and a zero of \( c(\nu) \) cannot be a
zero of \( Q_\nu \).

Therefore, \( \frac{Q_\nu}{2\nu c(\nu)} \) has a pole at \( \nu \) if and only if \( \nu \) is a zero of \( c(\nu) \), that is,
\( \nu = \nu_k = -Q/2 - k \), \( k \in \mathbb{N} \cup \{0\} \). On the other hand, \( \frac{Q_\nu}{2\nu c(\nu)} \) is analytic at \( \nu = \nu_k \).
Thus, if \( f \in C^\infty_c(S) \) and using that \( -\frac{Q_\nu}{2\nu c(\nu)} = \frac{Q_{-\nu}}{2\nu c(-\nu)} - \frac{\mu(\nu)\phi_\nu}{2\nu} \), we have
\[
(10) \quad T_{\nu_k}(f) = \text{Res}_{\nu = \nu_k} \tilde{R}(\lambda(\nu))(f) = \frac{p(\nu_k)}{2\pi \nu_k} f * \tilde{\phi}_{\nu_k}.
\]

From (5) and the expression for \( \cosh(\sqrt{2}) \) in the Preliminaries, we have that
\[
(11) \quad \phi_\nu(X, Z, a) = \sum_{i \geq 0} \frac{(Q/2 - \nu)_i}{i!} \frac{(Q/2 + \nu)_i}{(n/2)_i} \left[ \frac{(a + \frac{1}{2}|X|^2 + |Z|^2)^2}{4a} \right]^{i/2},
\]
where \( (u)_i = \prod_{l=0}^{i-1} u + l \) for \( u \in \mathbb{C} \). Hence we see that the coefficients in the expansion
of \( \phi_\nu \) are zero for \( i \geq k+1 \), for the special values \( \nu_k = -Q/2 - k \). Fix \( \{V_i\} \) and \( \{W_j\} \),
othornormal bases of \( v \) and \( z \) respectively, and write \( X = \sum_{i=1}^p x_i V_i \) and \( Z = \sum_{j=1}^q z_j W_j \).

If \( I = (i_1, \ldots, i_p) \), \( J = (j_1, \ldots, j_q) \), set \( X^I = \prod x_{i_j}^k \), \( Z^J = \prod z_{j_i}^k \), \( |I| = \sum_{j} i_j \) and
similarly for \( |J| \). Let \( \mathcal{F}_k \) be the linear span of the functions \( a^i X^{2I} Z^{J} : i \in \mathbb{Z} \),
\( |i| \leq k \), \( |I|, |J| \leq 2k \). Clearly \( \phi_{\nu_k} \in \mathcal{F}_k \). If \( t = (Y, U, b) \) with \( Y = \sum_{i=1}^p y_i V_i \in v \),
\( U = \sum_{i=1}^q u_i W_i \in z \), \( b \in A \) and \( s = (X, Z, a) \in S \), then
\[
t^{-1}s = \left( b^{-\frac{1}{2}}(X - Y), b^{-1}(Z - U + \frac{1}{2}[X, Y]), b^{-1}a \right)
= \left( b^{-\frac{1}{2}} \sum_{i} (x_i - y_i) V_i, b^{-1} \sum_{j} (z_j - u_j) W_j + \frac{1}{2} \sum_{i,j} x_i y_j a_{i,j} W_i, b^{-1}a \right),
\]
where \( [X, Y] = \sum_{i,j} x_i y_j a_{i,j} W_i \). Hence, by (11) \( \phi_{\nu_k}(t^{-1}s) \) is a linear combination
of functions of the form \( a^i b^{j_2} X^{2I_j} Y^{2J_k} Z^{J_1} U^{J_2} \) with \( j_i \in \mathbb{Z} \), \( |j_i| \leq k \), \( i = 1, 2 \), and
\( |I_i|, |J_i| \leq 2k \) for \( i = 1, 2 \).

Therefore, if \( f \in C^\infty_c(S) \), it follows that \( f * \tilde{\phi}_{\nu_k} (t) = \int_S f(s) \phi_{\nu_k}(t^{-1}s)ds \) is a
linear combination of expressions of the form
\[
t \mapsto b^{j_2} Y^{2J_2} U^{J_2} \int_{\mathcal{F}_k} \int_{A} \int_{\mathbb{Z}} f(X, Z, a) a^{j_1} X^{2I_1} Z^{J_1} a^{-Q-1} da dX dZ.
\]
Therefore, \( f * \tilde{\phi}_{\nu_k} \) belongs to \( \mathcal{F}_k \), a finite dimensional space, as asserted.
4. The symmetric case

In the case when $S$ is of symmetric type one can get more precise information on the operators $T_{\nu_k}$ by using representation theory.

The group of isometries $G$ of $S$ is a noncompact semisimple Lie group of real rank one. Let $\mathfrak{g}$, $\mathfrak{t}$, $N$, and $A$ be as in Section 4 let $M$ be the centralizer of $A$ in $K$, let $P = MAN$ and let $\mathfrak{p}$ be the Lie algebra of $P$. Extend $\mathfrak{a}$ in the usual way to a Cartan subalgebra $\mathfrak{h}_c = \mathfrak{a}_c + \mathfrak{h}_c^-$ of $\mathfrak{g}$, where $\mathfrak{h}_c^-$ is a maximal abelian subalgebra of $\mathfrak{m}$, and introduce compatible orderings in the dual spaces of $\mathfrak{a}$ and $\mathfrak{a} + \sqrt{-1}\mathfrak{h}_c^-$.

Let $\Sigma^+(\Delta^+)$ denote the corresponding set of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ (respectively $(\mathfrak{g}_c, \mathfrak{h}_c)$). Since $\mathfrak{g}$ has real rank one, there is only one real root $\alpha \in \Delta^+$. It satisfies $\delta_{\mathfrak{h}_c^-} \alpha = 0$ and $\delta_{\mathfrak{a}} \alpha = \alpha$.

For $\nu \in \mathcal{C}$, let $(\pi_\nu, H^\nu)$ be the spherical principal series representation of $G$ (see [7], Section 3.6). The zonal spherical function $\phi_\nu$ is given by $\phi_\nu(g) = \langle \pi_\nu(g)1_{\nu}, 1_{\nu} \rangle$, where $1_{\nu} \in H^\nu$ is such that $1_{\nu}(nak) = a^{(\nu+\rho)\alpha}$, $n \in N, a \in A, k \in K$, and $\langle , \rangle$ is the standard inner product on $H^\nu$.

**Theorem 4.1.** Let $S = G/K$ be a noncompact symmetric space of real rank one and let $\nu_k = -\rho - k$ with $k \in \mathbf{N} \cup \{0\}$. Then $\text{Im}(T_{\nu_k})$ is an irreducible $\mathfrak{g}_c$-module of highest weight $\kappa$.

**Proof.** By a result of Helgason (see [4], Ch. V, Theorem 4.1), the $K$-spherical finite dimensional representations of $G$ can be characterized as the representations of $\mathfrak{g}_c$ of highest weight $\Lambda \in \mathfrak{h}_c^\ast$ such that: $\Lambda_{\mathfrak{h}_c^-} = 0$ and $\langle \Lambda, \lambda \rangle / \langle \lambda, \lambda \rangle \in \mathbf{Z}_{\geq 0}$, for any $\lambda \in \Sigma^+$. Since in our case $\Sigma^+ = \{\alpha, \alpha/2\}$ or $\{\alpha\}$, this is equivalent to $\Lambda_{\mathfrak{a}} = k\kappa$, with $k \in \mathbf{Z}_{\geq 0}$, and $\kappa$ the real root. We shall denote by $V_{\kappa\kappa}$ the $\mathfrak{g}_c$-module with highest weight $k\kappa$.

Our claim is that $1_{\nu_k}$ generates a finite dimensional $(\mathfrak{g}_c, K)$-submodule $V_{\nu_k}$ of $H^\nu$, isomorphic to $V_{\kappa\kappa}$.

In the notation of Lemma 3.8.2 in [7], we have that

$$\text{Hom}_{\mathfrak{g}_c,K}(V_{\kappa\kappa}, H^\nu) \simeq \text{Hom}_{\mathfrak{p}, M}(V_{\kappa\kappa}/nV_{\kappa\kappa}, C_{\nu_k}),$$

where $C_{\nu_k}$ denotes the MAN-module $C$, with $MN$ acting trivially and $a \in A$ acting by multiplication by $a^{(\nu+\rho)\alpha}$. To prove our claim it will thus be sufficient to show that there exists a nontrivial $(\mathfrak{p}, M)$-morphism $f : V_{\kappa\kappa}/nV_{\kappa\kappa} \to C_{\nu_k}$. We denote by $\Lambda_\alpha$ the lowest weight of $V_{\kappa\kappa}$ and by $\nu_0$ the corresponding lowest weight vector. Then $\Lambda_\alpha = s_\alpha \Lambda$, $s_\alpha$ the long element of the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Since $\Lambda' = -s_\alpha \Lambda$ is the highest weight of the dual representation of $V_{\kappa\kappa}$, which is also $K$-spherical, $\Lambda'$ satisfies Helgason’s conditions. This implies that $s_\alpha \Lambda_{\mathfrak{h}_c^-} = 0$ and $s_\alpha \Lambda_{\mathfrak{a}} = -\kappa$. Arguing as in the proof of Theorem 4.1, Ch. V in [4], one shows that $\pi_\Lambda(M)v_0 = v_0$.

Since $s_\alpha \Lambda_{\mathfrak{h}_c^-} = 0$, it follows that $V = C_{\nu_0} \oplus (n + m)V$. Now we can define a $(\mathfrak{p}, M)$-morphism $f : V/nV \to C_{\nu}$ such that $f : [v_0] \mapsto 1$, where $[v_0]$ is the class of $v_0$ and $f = 0$ on $mV$. Hence, by (12), there is a nonzero $G$-map of $V_{\kappa\kappa}$ onto a subspace $V_{\nu_k}$ of $H^\nu$, which must contain $1_{\nu_k}$.

Now we prove the statement in the theorem. If $f \in C^\infty_c(G/K)$, and $x \in G$, we have by (3.1)

$$T_{\nu_k}(f) = p_k f * \hat{\phi}_{\nu_k}(x) = p_k \langle \pi(x^{-1}) \pi(f)1_{\nu_k}, 1_{\nu_k} \rangle,$$

where $p_k = -\frac{p(\nu_k)}{\pi(\nu_k)} \neq 0$, for all $k$ (see the formula of $p(\nu)$ in Section 2).
By irreducibility, as \( f \) varies, \( \pi(f)_{V_{r_k}} \) fills \( V_{r_k} \simeq V_{r_0} \). Hence, the image of 
\( T_{r_k} \) coincides with the image of the \( G \)-morphism \( T_k : V_{r_k} \mapsto \mathcal{C}^\infty(G/K) \) given by \( T_k(v)(x) = \langle \pi_{r_k}(x^{-1}) v, 1_v \rangle \), for \( v \in V_{r_k} \). This proves the theorem.

Remark 4.2. We will now use the Weyl dimension formula to calculate the dimension of the \( \mathfrak{g}_\ast \)-module \( V_{k\tilde{a}} \) in each case. The real roots \( \tilde{\alpha} \) can be read from the Satake diagram of \( \mathfrak{g} \). They are listed, for each rank one group, in [5] (for instance). We shall thus use the notation in [5].

(i) \( \mathfrak{g} = \mathfrak{so}(n, 1) \) (\( n \) even). In this case, the real root is \( \tilde{\alpha} = \epsilon_1 \), the first fundamental weight. The corresponding \( \mathfrak{g}_\ast \)-module \( V_{k\tilde{a}} \) is isomorphic to the representation of \( G \) on \( \mathcal{H}_k \), the space of homogeneous harmonic polynomials of degree \( k \) in \( n + 1 \) variables, which has dimension \( \frac{(k+n-2)!(2k+n-1)}{k!(n-1)!} \). This can easily be computed by the Weyl dimension formula.

(ii) \( \mathfrak{g} = \mathfrak{su}(n, 1) \). Here, the real root is \( \tilde{\alpha} = \epsilon_1 - \epsilon_{n+1} \), and the positive roots are \( \epsilon_i - \epsilon_j, \ i < j \) and \( 2\rho = \sum_{j=1}^{n+1} (n - 2j + 2) \epsilon_j \). Thus

\[
\dim(V_{k\tilde{a}}) = \prod_{1 \leq i < j \leq n+1} \frac{\langle k(\epsilon_1 - \epsilon_{n+1}) + \rho, \epsilon_i - \epsilon_j \rangle}{\langle \rho, \epsilon_i - \epsilon_j \rangle} = \prod_{2 \leq j \leq n} \frac{k + j - 1}{j - 1} \prod_{2 \leq i \leq n} \frac{k + n + 1 - i}{n + 1 - i} \frac{2k + n}{k} = \left( \frac{k + n - 1}{n} \right)^2 \left( \frac{2k + n}{k} \right) = \left( \frac{2n + k - 1}{k} \right)^2 \frac{2n + k}{(2n + k)(2n + k - 1)}.
\]

(iii) \( \mathfrak{g} = \mathfrak{sp}(n, 1) \). In this case, \( \tilde{\alpha} = \epsilon_1 + \epsilon_2 \), and the positive roots are \( \epsilon_i \pm \epsilon_j, \ 1 \leq i < j \leq n + 1 \), and \( 2\epsilon_i, \ 1 \leq i \leq n + 1 \). Also, \( \rho = \sum_{j=1}^{n+1} (n + 2 - j) \epsilon_j \). Hence

\[
\dim(V_{k\tilde{a}}) = \left( \frac{2n + k}{k} \right)^2 \frac{2n + k}{(2n + k)(2n + k - 1)} = \left( \frac{2n + k}{k} \right)^2 \frac{2n + k + 1}{(2n + k)(2n + k - 1)} \left( \frac{2n + k}{k} \right)^2 \frac{2n + k}{(2n + k)(2n + k - 1)}.
\]

(iv) \( \mathfrak{g} = \mathfrak{f}_4 \). The real root is \( \tilde{\alpha} = \lambda_4 \ (= \epsilon_1) \), the fourth fundamental weight. The positive roots are \( \epsilon_i, \ \epsilon_i \pm \epsilon_j, \ 1 \leq i < j \leq 4 \), \( \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \) and \( 2\rho = 11\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + \epsilon_4 \). Using the Weyl dimension formula we obtain in this case

\[
\dim(V_{k\tilde{a}}) = \frac{2k + 11}{11} \prod_{j=1}^{10} \frac{k + j}{j} \prod_{j=4}^{7} \frac{k + j}{j}.
\]

4.1. The real hyperbolic \( n \)-space. If \( S \cong H^n \), one can make the results in Theorem 2.2 more precise. In this case one can solve the recurrence in \( \mathfrak{S} \), obtaining an explicit series expression for \( Q_\nu \). Indeed, since \( p = 0 \), by \( \mathfrak{S} \) we see that \( a_{2j+1} = 0 \) for \( j \geq 0 \); hence \( a_{2j} = a_2 - \frac{(j-1+p)(j-1+p+\nu)}{j(j+\nu)} \), \( j \geq 0 \). Thus, if we set \( c_j := a_2 \), for \( j \geq 0 \), and if \( c_0 := 1 \), we obtain for \( j \geq 1 \)

\[
c_j(\nu) = \frac{\rho_j (\nu + \rho_j)}{j! (\nu + 1)_j}.
\]
Furthermore, \( c(\nu) = \frac{2^{n-1} \Gamma(n/2)}{\pi^{1/2}} \Gamma(\nu) \) hence, using (13) and the duplication formula for the Gamma function, we obtain

\[
Q_\nu(r) = \frac{2^{-2n+3}}{(n-2)!} e^{-(\nu+i\rho)r} \sum_{j=0}^{\infty} \frac{\Gamma(\nu+j)}{\Gamma(\nu+j+1)} e^{-2jr}.
\]

Now, if \( S_{b,\delta} \) is as in Theorem 2.2 one sees, by using Stirling’s estimates, that there exists a constant \( K = K(b, \delta) \) such that the coefficients in (14) are bounded by \( K j^{\rho-1} |\nu+j|^\rho-1 \), uniformly for \( \nu \) in \( S_{b,\delta} \). This gives an alternative proof of the convergence, as stated in Theorem 2.2 (a).

Regarding the poles, we see that if \( \rho = \frac{n-1}{2} \in \mathbb{N} \), the coefficients in (14) are polynomial functions in \( \nu \); hence \( \hat{R}(\lambda(\nu)) \) is everywhere holomorphic in this case.

If \( n \) is even, (14) implies that the kernel is meromorphic with poles at \( \nu_k = -\rho - k, \ k \in \mathbb{N} \cup \{0\} \). Since \( \Gamma(\nu+j) \) is holomorphic at \( \nu = \nu_k \) for \( j > k \), we get

\[
\text{Res}_{\nu=\nu_k} Q_\nu(r) = \sum_{j=0}^{\infty} \frac{\Gamma(\nu+j)(-1)^{k-j}}{j!(k-j)\Gamma(-k-\rho+j+1)} e^{-2j-k}.
\]

since \( \frac{\Gamma(\nu+j)(-1)^k}{\Gamma(-k-\rho+j+1)} = \frac{\Gamma(\rho-k-j)}{\Gamma(-\rho+j+1)} \), for \( 0 \leq j \leq k \).

**Remark 4.3.** We note that in [3], Section 2, Guillopé-Zworski consider the resolvent kernel for the real hyperbolic \( n \)-space, giving the location of the poles and showing that the residues define operators of finite rank.

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**References**


Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina

E-mail address: miatello@mate.uncor.edu

E-mail address: cwill@mate.uncor.edu