A MODIFICATION OF LOUVÉAU AND VELIČKOVIĆ’S CONSTRUCTION FOR \( F \)-IDEALS

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Abstract. We show that the construction of Louveau and Veličković can be modified to obtain an embedding of \((\omega^\omega, \subset^\ast)\) into the preorder \((F_\ast \text{-ideals}, \leq)\) where \(\leq\) is the relation of Borel reducibility.

The notion of reducibility appeared in [1]. Generally, reducibility is a preorder on all Borel equivalence relations in Polish spaces. We will be interested here only in one Polish space \(P(\omega)\) (or \(P(A)\), where \(A\) is a countable set). We equip this space with the Tychonoff topology transferred from the Cantor cube \(2^\omega\). For \(s \in 2^{<\omega} = \{0, 1\}^n\) let 
\[s = \{ X \subseteq \omega : x \in \text{dom}(s) = s^{-1}\{1\}\}. \]
Thus \(\{s : s \in 2^{<\omega}\}\) is a basis of the topology defined above. By \([A]^{<\omega}\) we will define a set of all infinite, and by \([A]^{<\omega}\) a set of all finite subsets of a set \(A\). Let \(I\) be the Borel ideal in the space \(P(\omega)\). Additionally we will restrict our attention only to equivalences which are of the form \(=_I\) (i.e. congruences modulo ideal \(I\)).

Thus we will define reducibility (in symbols \(\leq\)) and continuous reducibility (in symbols \(\leq_c\)) only for ideals. The definitions look as follows:

\[(I \leq J) \equiv \exists F : P(\omega) \xrightarrow{\text{Borel}} P(\omega) \forall x, y \in P(\omega)[(x \Delta y \in I) \Leftrightarrow (F(x) \Delta F(y) \in J)]\]

and the definition of \(\leq_c\) is almost the same with “Borel” replaced by “continuous”.

By submeasure on a set \(X\) we mean a function \(\mu : P(X) \to [0, \infty]\) with the following properties:

\[\forall A, B \subseteq X [\mu(A) \leq \mu(A \cup B) \leq \mu(A) + \mu(B)]\]
\[\mu(\emptyset) = 0, \mu(X) > 0, \forall x \in X [\mu(\{x\}) < \infty]\]
\[\mu(A) = \sup_{a \in [A]^{<\omega}} \mu(a).\]

Let us define preorder \(\subset^\ast\) on the set \(P(\omega)\) by the formula:

\[S \subset^\ast T \iff S \setminus T \in [\omega]^{<\omega}.\]
Louveau and Velicković found a family of $F_{σδ}$-ideals $(I_S)_S ∈ [ω]^ω$ satisfying:

(3) \[ ∀S, T ∈ [ω]^ω [S ⊆ T ⇔ I_S ≤ I_T]. \]

We will show that there exists a family of $F_{σδ}$-ideals $I_S$ for which the same is true.

Let us recall some basic facts about the original construction of Louveau and Velicković [2]. They start by partitioning $ω$ into finite pieces $(P_n)_n$ and constructing a sequence of submeasures $(||)_n$ such that for every $n$, $||_n$ is originally defined on $P_n$ and $∀n(||P_n||_n ≥ 1)$. These submeasures extend naturally to $P(ω)$ by the formula:

\[ ||Y||_n \overset{df}{=} ||Y \cap P_n||_n. \]

Then they define their $F_{σδ}$-ideals $(I_S)_S ∈ [ω]^ω$ by the formulas:

\[ Y ∈ I_S ⇔ \lim_{n→S} ||Y||_n = 0. \]

Our $F_{σδ}$-ideals $(I_S)_S ∈ [ω]^ω$ will be defined by the formulas:

\[ I_S = \{Y ⊆ ω: \sup_{n ∈ S} ||Y||_n < ∞\}. \]

Of course we must require something like: $∀n(||P_n||_n ≥ n)$ in order to obtain proper ideals.

Now we will give our (very close to the original) definition of $P_n$’s and $||_n$’s. Let us create two increasing sequences of natural numbers $(a_n)_n$ and $(b_n)_n$. Put $a_0 = b_0 = 2$, $a_{n+1} = 2^{n+1}(a_n + b_n + 2)$, $b_{n+1} = 2^{(n+1)(a_{n+1} + b_{n+1})}$. Let additionally $m_n = \sum_{k<n} b_k$, $P_n = [m_n, m_{n+1})$. Then we have of course $|P_n| = b_n$ and we will define a submeasure $||_n$ supported by $P_n$ by the formulas:

\[ ||Y||_n = \log_2((Y \cap P_n) + 1). \]

Notice that the above definitions imply that $∀n(||P_n||_n ≥ n + 1)$.

Let us begin the proof of the equivalence (3) for ideals $(I_S)_S ∈ [ω]^ω$.

The proof of the implication “⇒” is the same as in [2]. If we define $ω_S = \bigcup_{n ∈ S} P_n$, then the appropriate reducing function is $F(Y) = Y \cap ω_S$.

The proof of “⇐”: Let us take a pair $S, T$ of infinite sets of $ω$ such that $I_S ≤ I_T$. Assume towards a contradiction that $S ⊈ T$. As in [2] again (Lemma 2) we can observe that if there is a Borel reduction, then there exists a continuous one (possibly for smaller $S$) and that we can assume $S, T ∈ [ω]^ω$ are disjoint. Let us define submeasures $φ_S, φ_T$ on $ω$ connected with the ideals $I_S, I_T$, respectively:

\[ φ_S(Y) = \sup_{n ∈ S} ||Y||_n \]

and $φ_T$ in the similar way. We will prove:

**Lemma 1.** Assume that $F: P(ω) → P(ω)$ is continuous and reduces $I_S$ to $I_T$. Then we can find $K ∈ ω$, $S' ∈ [S]^ω$, $F': P(ω) → P(ω)$ continuously reducing $I_{S'}$ to $I_T$ such that:

\[ ∀X, Y ⊆ ω [(φ_{S'}(XΔY) ≤ 1) ⇒ (φ_T(F'(X)ΔF'(Y)) ≤ K)]. \]

**Proof.** The proof will be split into two facts:

**Fact 2.** Assume that $F: P(ω) → P(ω)$ continuously reduces $I_S$ to $I_T$. Then there exist an $S' ∈ [S]^ω$ and $F'$ reducing $I_{S'}$ to $I_T$ satisfying:

\[ ∀n ∈ ω \exists m_n ∈ ω ∀X, Y ⊆ ω [(XΔY) ≤ n) ⇒ (φ_T(F'(X)ΔF'(Y)) ≤ m_n)]. \]
Fact 3. Assume that $F$ is a continuous function reducing $I_{S'}$ to $I_T$ and satisfying (**) Then $F$ also satisfies (*).

Proof of Fact 2. We will define first a suitable dense $G_δ$-set and then we will proceed as in [2], Lemma 2. For $m, n ∈ ω$ let:

$$C^m_n = \{ x : ∀y[(x \setminus n = y) \cap n) ⇒ (ϕ_T(F(x) ∆ F(y)) ≤ m)\} .$$

For any $n ∈ ω$, $(C^m_n)_m$ is an increasing family of closed sets and $∃_{m ∈ ω} C^m_n = P(ω)$. Hence, by the Baire category theorem, the set $\bigcup_{m ∈ ω} \text{int}(C^m_n)$ is open dense for any $n ∈ ω$. Our $G$ will be of the form $\bigcap_{n ∈ ω} G_n$ where $G_n = ∃_{n ∈ ω} \text{int}(C^m_n)$. Proceeding as in [2], Lemma 2, take $S' ∈ [S']^ω$ and $Z ⊆ ω\setminus ω_{S'}$ such that $∀A ∈ ω_{S'} (A \cup Z ∈ G)$. The set $\{ A \cup Z : A ∈ ω_{S'} \}$ is compact and contained in every $G_n$. Hence:

$$∀n \exists m_n [\{ A \cup Z : A ∈ ω_{S'} \} ⊆ \text{int}(C^m_n) ,$$

i.e.,

$$∀n, x, y ∈ \{ A \cup Z : A ∈ ω_{S'} \} ((x ∆ y ∈ n) ⇒ (ϕ_T(F(x) ∆ F(y)) ≤ m_n)) .$$

Define $F' : P(ω) → P(ω)$ by the formula: $F'(X) = F((X ∩ ω_{S'}) ∪ Z)$. Then $F'$ is as required. □

Proof of Fact 3. We know that: $\{ F(X) ∆ F(Y) \}_{X, Y} : ϕ_S(X ∆ Y) ≤ 1 \}$ is a compact set covered by the countable union of closed sets: $∪_{A ∈ ω} \{ A : ϕ_T(A) ≤ n \}$. Hence, by the Baire category theorem applied to this space there exist a $u ∈ 2^{<ω}$ and $m_1 ∈ ω$ such that $∅ ≠ \{ F(X) ∆ F(Y) : ϕ_S(X ∆ Y) ≤ 1 \} \cap u ⊆ \{ A : ϕ_T(A) ≤ m_1 \}$. By the continuity of $F$ we can also find $s_1, t_1 ∈ 2^{<ω}$ such that $lh(s_1) = lh(t_1)$, $ϕ_S(s_1^{-1}\{ 1 \} ∆ t_1^{-1}\{ 1 \}) ≤ 1$ and

$$\{ F(X) ∆ F(Y) : ϕ_S(X ∆ Y) ≤ 1, X ∈ s_1, Y ∈ t_1 \} ⊆ \{ A : ϕ_T(A) ≤ m_1 \} .$$

Take any $X, Y ⊆ ω$ such that $ϕ_S(X ∆ Y) ≤ 1$. Let $C = ∪_{k ∈ ω} \{ P_k : ϕ_S(X) \neq ∅ \}$. $X_1 = s_1^{-1}\{ 1 \} ∪ (X \setminus C)$, $Y_1 = t_1^{-1}\{ 1 \} ∪ (Y \setminus C)$. Let $n = sup(C)$. Using Fact 2 we can find $m_2$ such that: $∀Z_1, Z_2 [([Z_1 ∆ Z_2] \subseteq n) ⇒ (ϕ_T(F(Z_1) ∆ F(Z_2)) ≤ m_2)]$. Now we have:

$$ϕ_T(F(X) ∆ F(X_1)) ≤ m_2 ,$$

$$ϕ_T(F(Y_1) ∆ F(Y)) ≤ m_2 .$$

From the above we infer that: $ϕ_T(F(X) ∆ F(Y)) ≤ m_1 + 2m_2$, which concludes the proof of Fact 3 and the lemma. □

Next we will prove two interesting properties of the sequence $(P_n, ||n||_n)$.

Lemma 4. Let $n < m$ and let $(A_k)_{k < l ≤ b_n}$ be a family of subsets of $P_m$. Then

$$\left\| \bigcup_{k < l} A_k \right\|_m ≤ sup_{k < l} ||A_k||_m + \frac{1}{2^{n+1}} .$$

Proof. \begin{align*}
\log_2 \left( \left\| \bigcup_{k < l} A_k \right\| + 1 \right) & ≤ \log_2 (l) + sup_{k < l} \log_2 (||A_k|| + 1) \\
& ≤ \log_2 (b_n) + sup_{k < l} [\log_2 (||A_k|| + 1)].
\end{align*}
From the pigeon-hole principle it follows that we can find sequences $(\ldots)$ we want to check if iii) holds for $n$, and noting that $rac{\log_2(b_n)}{a_{n+1}} \leq \frac{1}{2^{n+1}}$, we infer the lemma.

**Lemma 5.** Let $n < m$ and assume that $f \colon \mathcal{P}(P_n) \to \mathcal{P}(P_m)$ satisfies:
\[
\forall A, B \subseteq P_n (\|A \Delta B\|_n \leq 1) \Rightarrow (\|f(A) \Delta f(B)\|_m \leq K).
\]
Then
\[
\forall A, B \subseteq P_n \left( \|f(A) \Delta f(B)\|_m \leq K + \frac{1}{2^{n+1}} \right).
\]

**Proof.** Enumerate $A \Delta B = \{t_k \colon k < l\}$ where $l \leq b_n$. For $p \leq l$ let $U_p = A \Delta \{t_k \colon k < p\}$. We have $U_0 = A, U_1 = B$. For every $p < l$, $|U_p \Delta U_{p+1}| = 1$, hence $\|U_p \Delta U_{p+1}\|_n \leq 1$. Thus from our assumptions on $f$ we have $\|f(U_p) \Delta f(U_{p+1})\|_m \leq K$. Using Lemma 4 we can calculate:
\[
\|f(A) \Delta f(B)\|_m = \left\| \bigcup_{p=0}^{l-1} (f(U_p) \Delta f(U_{p+1})) \right\|_m 
\leq K + \frac{1}{2^{n+1}}.
\]

Now we want to construct a sequence of natural numbers $(i_n)_n \subseteq S$ and two sequences $(A_n)_n$ and $(B_n)_n$ of subsets of $\omega$ such that
i) $A_n, B_n \subseteq m_{i_n}; A_{n+1} \cap m_{i_n} = A_n; B_{n+1} \cap m_{i_n} = B_n$;
ii) $\|A_{n+1} \Delta B_{n+1}\|_{i_n} \geq n$;
iii) $\forall X \subseteq \omega \setminus m_{i_n} [\varphi_T [F(A_n \cup X) \Delta F(B_n \cup X)] \leq K + 1 - \frac{1}{2^{n+1}}].$

We start the construction by taking $i_1$ as the first element of $S$ and $A_1, B_1 \subseteq m_{i_1}$ such that $\varphi_S(A_1 \Delta B_1) \leq 1$. Now we describe how to do the inductive step. Let us find a family $\mathcal{F} \subseteq \mathcal{P}(P_{i_n})$ such that $|\mathcal{F}| \geq 2^{a_{i_n}+1}$, consisting of disjoint sets, each of cardinality $2^{a_{i_n}}$. This is possible because:
\[
b_{i_n} = |P_{i_n}| = 2^{i_n(b_{i_n-1}+a_{i_n}+1)} = 2^{i_n+1}2^{i_n}2^{2^{a_{i_n}}} \geq (2^{m_{i_n}+1})2^{a_{i_n}}.
\]
From the pigeon-hole principle it follows that we can find $A, B \in \mathcal{F}$ such that:
\[
F(A_n \cup A) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.
\]
Let us choose $i_{n+1} > i_n, i_{n+1} \in S$, such that for any $X \subseteq \omega \setminus i_{n+1}$
\[
F(A_n \cup A \cup X) \cap m_{i_n} = F(A_n \cup A) \cap m_{i_n},
\]
\[
F(A_n \cup B \cup X) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.
\]
Finally we put:
\[
A_{n+1} = A_n \cup A,
\]
\[
B_{n+1} = B_n \cup B.
\]
By the properties of the family $\mathcal{F}$, $A$ and $B$ are disjoint and the submeasure $\|\_i_n$ is $\geq n$ on both of them. Therefore $\|A_{n+1} \Delta B_{n+1}\|_{i_n} \geq \|A \Delta B\|_{i_n} \geq n$. Now we want to check if iii) holds for $n + 1$. Take any $X \subseteq \omega \setminus i_{n+1}$ and $m \in T$. We want to show that:
\[
\|F(A_{n+1} \cup X) \Delta F(B_{n+1} \cup X)\|_m \leq K + 1 - \frac{1}{2^{n+1}}.
\]
We can partition this symmetric difference introducing the intermediate factor \( F(A_{n+1} \cup X) \). We obtain:

\[
F(A_{n+1} \cup X) \Delta F(B_{n+1} \cup X) = [F(A_{n+1} \cup X) \Delta F(A_u \cup B \cup X)] \triangle [F(A_u \cup B \cup X) \triangle F(B_{n+1} \cup X)].
\]

For our \( m \in T \) there are two possibilities: \( m < i_n \) and \( m > i_n \). Recall that because \( i_n \in S \) and \( S \) and \( T \) are disjoint, the case \( m = i_n \) is impossible. When \( m < i_n \), then \( \| (I) \|_m = 0 \) and \( \| (II) \|_m \) is small by our inductive assumption iii). When \( m > i_n \), then, if we take \( f : \mathcal{P}(P_n) \to \mathcal{P}(P_m) \) defined by \( f(C) = F(A_u \cup C \cup X) \cap P_m \), then from Lemma 5 we have \( \| (I) \|_m \leq K + \frac{1}{2n-1} \leq K + \frac{1}{2n} \) and from our inductive assumption \( \| (II) \|_m \leq K + 1 - \frac{1}{2n} \). From Lemma 4 we obtain:

\[
\| (I) \|_m \leq \| (I) \cup (II) \|_m \leq \sup(\| (I) \|_m, \| (II) \|_m) + \frac{1}{2n+1} \leq K + 1 - \frac{1}{2n+1}.
\]

Finally, if we put \( \hat{A} = \bigcup_{n \in \omega} A_n, \hat{B} = \bigcup_{n \in \omega} B_n \), then \( \hat{A} \Delta \hat{B} \not\subset I_S \) but \( \varphi_T[F(\hat{A}) \triangle F(\hat{B})] \leq K + 1 \). Thus the pair \( \hat{A}, \hat{B} \) is an example showing that \( F \) does not reduce \( I_S \) to \( I_T \).

Let us recall the definitions of two important Borel ideals:

\[
\text{Fin} \times \emptyset = \{ x \subset \omega^2 : \exists n \subset n \times \omega \},
\]

\[
\emptyset \times \text{Fin} = \{ x \subset \omega^2 : \forall m \exists n \forall k \geq n (m, k) \notin x \}.
\]

It is not difficult to prove that the original Louveau and Veličković family of ideals \((I_S)_{S \in [\omega]^{\omega}}\) satisfies:

\[
\forall S \in [\omega]^{\omega}[I_S \supset \emptyset \times \text{Fin}].
\]

Similarly the family \((I_S)_{S \in [\omega]^{\omega}}\) constructed above satisfies:

\[
\forall S \in [\omega]^{\omega}[I_S \supset \text{Fin} \times \emptyset].
\]

Thus, in connection with the results of Solecki (see [3], Theorems 2.1 and 3.3), stating that every ideal not greater in the sense of reducibility from either \( \emptyset \times \text{Fin} \) or \( \text{Fin} \times \emptyset \) is a p-ideal of the class \( F_\sigma \), it is interesting to ask the following

**Question 6.** Does there exist a family of p-ideals of the class \( F_\sigma \), \((I_S)_{S \in [\omega]^{\omega}}\), satisfying the formula analogous to (3)?

**References**


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