A MODIFICATION OF LOUVEAU AND VELIČKOVIĆ’S CONSTRUCTION FOR $F_\sigma$-IDEALS

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Abstract. We show that the construction of Louveau and Veličković can be modified to obtain an embedding of $(\omega^\omega, \subset^*)$ into the preorder $(F_\sigma$-ideals, $\leq)$ where $\leq$ is the relation of Borel reducibility.

The notion of reducibility appeared in [1]. Generally, reducibility is a preorder on all Borel equivalence relations in Polish spaces. We will be interested here only in one Polish space $P(\omega)$ (or $P(A)$, where $A$ is a countable set). We equip this space with the Tychono topology transferred from the Cantor cube $2^\omega$. For $s \in 2^{<\omega}$ = $\{0, 1\}^n$ let $s = \{x \in \omega : x \cap \text{dom}(s) = s^{-1}\{1\}\}$. Thus $\{s : s \in 2^{<\omega}\}$ is a basis of the topology defined above. By $[A]^\omega$ we will define a set of all infinite, and by $[A]^{<\omega}$ a set of all finite subsets of a set $A$. Let $I$ be the Borel ideal in the space $P(\omega)$. Additionally we will restrict our attention only to equivalences which are of the form $=_I$ (i.e. congruences modulo ideal $I$).

Thus we will define reducibility (in symbols $\leq$) and continuous reducibility (in symbols $\leq_c$) only for ideals. The definitions look as follows:

(1) $(I \leq J) \equiv \exists F : P(\omega) \xrightarrow{\text{Borel}} P(\omega) \forall x, y \in P(\omega)[(x \Delta y \in I) \Leftrightarrow (F(x) \Delta F(y) \in J)]$

and the definition of $\leq_c$ is almost the same with “Borel” replaced by “continuous”.

By submeasure on a set $X$ we mean a function $\mu : P(X) \to [0, \infty]$ with the following properties:

$\forall A, B \subseteq X [\mu(A) \leq \mu(A \cup B) \leq \mu(A) + \mu(B)]$

$\mu(\varnothing) = 0, \mu(X) > 0, \forall x \in X [\mu(\{x\}) < \infty]$

$\mu(A) = \sup_{a \in [A]^{<\omega}} \mu(a)$.

Let us define preorder $\subset^*$ on the set $P(\omega)$ by the formula:

$S \subset^* T \Leftrightarrow S \setminus T \in [\omega]^{<\omega}$.
Louveau and Velicković found a family of $F_{\sigma\delta}$-ideals $(I_S^\omega)_{S \in [\omega]^\omega}$ satisfying:

$$(3) \quad \forall S, T \in [\omega]^\omega [S \subset T \iff I_S \leq I_T].$$

We will show that there exists a family of $F_{\sigma}$-ideals $I_S$ for which the same is true.

Let us recall some basic facts about the original construction of Louveau and Velicković [2]. They start by partitioning $\omega$ into finite pieces $(P_n)_n$ and constructing a sequence of submeasures $((\|\|)_n)_n$ such that for every $n$, $\|\|_n$ is originally defined on $P_n$ and $\forall n(\|P_n\|_n \geq 1)$. These submeasures extend naturally to $\mathcal{P}(\omega)$ by the formula:

$$\|Y\|_n \overset{df}{=} \|Y \cap P_n\|_n.$$ 

Then they define their $F_{\sigma\delta}$-ideals $(I_S^\omega)_{S \in [\omega]^\omega}$ by the formulas:

$$Y \in I_S^\omega \iff \lim_{n \in S} \|Y\|_n = 0.$$ 

Our $F_{\sigma}$-ideals $(I_S)_{S \in [\omega]^\omega}$ will be defined by the formulas:

$$I_S = \{Y \subseteq \omega : \sup_{n \in S} \|Y\|_n < \infty\}.$$ 

Of course we must require something like: $\forall n(\|P_n\|_n \geq n)$ in order to obtain proper ideals.

Now we will give our (very close to the original) definition of $P_n$’s and $\|\|_n$’s. Let us create two increasing sequences of natural numbers $(a_n)_n$ and $(b_n)_n$. Put $a_0 = b_0 = 2, a_{n+1} = 2^{n+1}(a_n + b_n + 2), b_{n+1} = 2^{(n+1)(a_{n+1} + b_{n+1})}$. Let additionally $m_{n} = \sum_{k<n} b_{k}$, $P_n = \lfloor m_{n}, m_{n+1}\rfloor$. Then we have of course $|P_n| = b_{n}$ and we will define a submeasure $\|\|_n$ supported by $P_n$ by the formulas:

$$\|Y\|_n = \frac{\log_2(|Y \cap P_n| + 1)}{a_{n}}.$$ 

Notice that the above definitions imply that $\forall n(\|P_n\|_n \geq n + 1)$.

Let us begin the proof of the equivalence (3) for ideals $(I_S)_{S \in [\omega]^\omega}$.

The proof of the implication “$\Rightarrow$” is the same as in [2]. If we define $\omega_S = \bigcup_{n \in S} P_n$, then the appropriate reducing function is $F(Y) = Y \cap \omega_S$.

The proof of “$\Leftarrow$”: Let us take a pair $S, T$ of infinite subsets of $\omega$ such that $I_S \leq I_T$. Assume towards a contradiction that $S \not\subseteq T$. As in [2] again (Lemma 2) we can observe that if there is a Borel reduction, then there exists a continuous one (possibly for smaller $S$) and that we can assume $S, T \subseteq [\omega]^\omega$ are disjoint. Let us define submeasures $\varphi_S, \varphi_T$ on $\omega$ connected with the ideals $I_S, I_T$, respectively:

$$\varphi_S(Y) = \sup_{n \in S} \|Y\|_n$$

and $\varphi_T$ in the similar way. We will prove:

**Lemma 1.** Assume that $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous and reduces $I_S$ to $I_T$. Then we can find $K \in \omega$, $S' \in [S]^\omega$, $F': \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ continuously reducing $I_{S'}$ to $I_T$ such that:

$$(*) \quad \forall X, Y \subseteq \omega[(\varphi_{S'}(X \Delta Y) \leq 1) \Rightarrow (\varphi_T(F'(X) \Delta F'(Y)) \leq K)].$$

**Proof.** The proof will be split into two facts:

**Fact 2.** Assume that $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ continuously reduces $I_S$ to $I_T$. Then there exist an $S' \in [S]^\omega$ and $F'$ reducing $I_{S'}$ to $I_T$ satisfying:

$$(**) \quad \forall n \in \omega \exists m_n \in \omega \forall X, Y \subseteq \omega[(X \Delta Y) \subseteq n) \Rightarrow (\varphi_T(F'(X) \Delta F'(Y)) \leq m_n)].$$
Fact 3. Assume that $F$ is a continuous function reducing $I_S'$ to $I_T$ and satisfying (**) Then $F$ also satisfies (*).

Proof of Fact 2. We will define first a suitable dense $G_δ$-set and then we will proceed as in [2], Lemma 2. For $m, n ∈ ω$ let:

$$C^n_m = \{ x : ∀ y (\langle x \rangle n = y) \Rightarrow (ϕ_T(F(x)ΔF(y)) ≤ m) \}. $$

For any $n ∈ ω$, $(C^n_m)_m$ is an increasing family of closed sets and $∪_{m∈ω}C^n_m = P(ω)$. Hence, by the Baire category theorem, the set $∪_{m∈ω}int(C^n_m)$ is open dense for any $n ∈ ω$. Our $G$ will be of the form $\bigcap_{m∈ω}G_n$ where $G_n = ∪_{n∈ω}int(C^n_m)$. Proceeding as in [2], Lemma 2, take $S' ∈ [S]'$ and $Z ⊂ ω \setminus ω_S$ such that $∀ A ∈ ω_S (A ∪ Z ∈ G)$. The set $\{ A ∪ Z : A ⊂ ω_S \}$ is compact and contained in every $G_n$. Hence:

$$∀ n \exists m_n[\{ A ∪ Z : A ⊂ ω_S \} ⊂ int(C^n_m)],$$

i.e.,

$$∀ n ∀ x, y ∈ \{ A ∪ Z : A ⊂ ω_S \} [x ∆ y ⊂ n \Rightarrow (ϕ_T(F(x)ΔF(y)) ≤ m_n)].$$

Define $F' : P(ω) → P(ω)$ by the formula: $F'(X) = F((X \cap ω_S) ∪ Z)$. Then $F'$ is as required.

Proof of Fact 3. We know that: $\{ F(X)ΔF(Y) : ϕ_S(XΔY) ≤ 1 \}$ is a compact set covered by the countable union of closed sets: $∪_{n∈ω}A : ϕ_T(A) ≤ n$. Hence, by the Baire category theorem applied to this space there exist a $u ∈ 2^{<ω}$ and $m_1 ∈ ω$ such that $\emptyset \neq \{ F(X)ΔF(Y) : ϕ_S(XΔY) ≤ 1 \} \cap \hat{u} ⊂ \{ A : ϕ_T(A) ≤ m_1 \}$. By the continuity of $F$ we can also find $s_1, t_1 ∈ 2^{<ω}$ such that $lh(s_1) = lh(t_1)$, $ϕ_S(s_1^{-1}{1} Δ t_1^{-1}{1}) ≤ 1$ and

$$\{ F(X)ΔF(Y) : ϕ_S(XΔY) ≤ 1, X ∈ s_1, Y ∈ t_1 \} ⊂ \{ A : ϕ_T(A) ≤ m_1 \}.$$

Take any $X, Y ⊂ ω$ such that $ϕ_S(XΔY) ≤ 1$. Let $C = ∪_{k∈ω}P_k\cap dom(ϕ_S) ≠ \emptyset P_k$, $X_1 = s_1^{-1}{1} \cup (X \setminus C)$, $Y_1 = t_1^{-1}{1} \cup (Y \setminus C)$. Let $n = sup(C)$. Using Fact 2 we can find $m_2$ such that: $∀ Z_1, Z_2 (\{ Z_1ΔZ_2 ⊂ n \} \Rightarrow (ϕ_T(F(Z_1)ΔF(Z_2)) ≤ m_2))$. Now we have:

$$ϕ_T(F(X)ΔF(X_1)) ≤ m_2,$$

$$ϕ_T(F(X_1)ΔF(Y_1)) ≤ m_1,$$

$$ϕ_T(F(Y_1)ΔF(Y)) ≤ m_2.$$

From the above we infer that: $ϕ_T(F(X)ΔF(Y)) ≤ m_1 + 2m_2$, which concludes the proof of Fact 3 and the lemma.

Next we will prove two interesting properties of the sequence $(P_n, \| n \_n)$. 

Lemma 4. Let $n < m$ and let $(A_k)_{k<l≤b_n}$ be a family of subsets of $P_m$. Then

$$\left\| \bigcup_{k<l} A_k \right\|_m ≤ sup_{k<l} \| A_k \|_m + \frac{1}{2n+1}.$$ 

Proof.

$$log_2 \left( \left\| \bigcup_{k<l} A_k \right\| + 1 \right) ≤ log_2(l) + sup_{k<l} log_2 \left( |A_k| + \frac{1}{7} \right) \leq log_2(b_n) + sup_{k<l} [log_2(|A_k| + 1)].$$
Dividing \( \log_2(b_n) + \sup_{k < l} [\log_2(|A_k| + 1)] \) by \( a_m \) for \( m > n \), and noting that
\[
\frac{\log_2(b_n)}{a_m} \leq \frac{1}{2n+1},
\]
we infer the lemma.

**Lemma 5.** Let \( n < m \) and assume that \( f : \mathcal{P}(P_n) \to \mathcal{P}(P_m) \) satisfies:
\[
\forall A, B \subseteq P_n [\|A \Delta B\|_n \leq 1 \Rightarrow \|f(A) \Delta f(B)\|_m \leq K].
\]
Then
\[
\forall A, B \subseteq P_n \left( \|f(A) \Delta f(B)\|_m \leq K + \frac{1}{2n+1} \right).
\]

**Proof.** Enumerate \( A \Delta B = \{t_k : k < l\} \) where \( l \leq b_n \). For \( p \leq l \) let \( U_p = A \Delta \{t_k : k < p\} \). We have \( U_0 = A, U_l = B \). For every \( p < l \), \( |U_p \Delta U_{p+1}| \leq 1 \), hence \( \|U_p \Delta U_{p+1}\|_n \leq 1 \). Thus from our assumptions on \( f \) we have \( \|f(U_p) \Delta f(U_{p+1})\|_m \leq K \). Using Lemma 4 we can calculate:
\[
\|f(A) \Delta f(B)\|_m = \left\| \bigcup_{p=0}^{l-1} (f(U_p) \Delta f(U_{p+1})) \right\|_m \leq K + \frac{1}{2n+1}.
\]

Now we want to construct a sequence of natural numbers \((i_n)\) and two sequences \((A_n)\) and \((B_n)\) of subsets of \( \omega \) such that
\begin{enumerate}
  \item \( A_n, B_n \subseteq m_{i_n}; A_{n+1} \cap m_{i_n} = A_n; B_{n+1} \cap m_{i_n} = B_n \),
  \item \( \|A_{n+1} \Delta B_{n+1}\|_{i_n} \geq n \),
  \item \( \forall X \subseteq \omega \backslash m_{i_n} \left[ \varphi_T[F(A_n \cup X) \Delta F(B_n \cup X) \leq K + 1 - \frac{1}{2n+1}].\right. \)
\end{enumerate}
We start the construction by taking \( i_1 = 1 \) as the first element of \( S \) and \( A_1, B_1 \subseteq m_{i_1} \) such that \( \varphi_S(A_1 \Delta B_1) \leq 1 \). Now we describe how to do the inductive step. Let us find a family \( F \subseteq \mathcal{P}(P_{i_n}) \) such that \( |F| \geq 2^{m_{i_n}+1} \), consisting of disjoint sets, each of cardinality \( 2^{a_{i_n}} \). This is possible because:
\[
b_{i_n} = |P_{i_n}| = 2^{i_n(b_{i_n-1}+a_{i_n}+1)} = 2^{i_n+1}b_{i_n-1}2^{i_n}a_{i_n} \geq (2^{m_{i_n}+1})2^{a_{i_n}}.
\]
From the pigeon-hole principle it follows that we can find \( A, B \in F \) such that:
\[
F(A_n \cup A) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.
\]
Let us choose \( i_{n+1} > i_n, i_{n+1} \in S \), such that for any \( X \subseteq \omega \backslash i_{n+1} \)
\[
F(A_n \cup A \cup X) \cap m_{i_n} = F(A_n \cup A) \cap m_{i_n},
\]
\[
F(A_n \cup B \cup X) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.
\]
Finally we put:
\[
A_{n+1} = A_n \cup A,
\]
\[
B_{n+1} = B_n \cup B.
\]
By the properties of the family \( F, A \) and \( B \) are disjoint and the submeasure \( ||_{i_n} \) is \( \geq n \) on both of them. Therefore \( \|A_{n+1} \Delta B_{n+1}\|_{i_n} \geq \|A \Delta B\|_{i_n} \geq n \). Now we want to check if iii) holds for \( n + 1 \). Take any \( X \subseteq \omega \backslash i_{n+1} \) and \( m \in T \). We want to show that:
\[
\|F(A_{n+1} \cup X) \Delta F(B_{n+1} \cup X)\|_m \leq K + 1 - \frac{1}{2n+1}.
\]
We can partition this symmetric difference introducing the intermediate factor $F(A_n \cup B \cup X)$. We obtain:

$$F(A_{n+1} \cup X) \triangle F(B_{n+1} \cup X) = [F(A_{n+1} \cup X) \triangle F(A_n \cup B \cup X)] \triangle [F(A_n \cup B \cup X) \triangle F(B_{n+1} \cup X)]_{(I)}$$

For our $m \in T$ there are two possibilities: $m < i_n$ and $m > i_n$. Recall that because $i_n \in S$ and $S$ and $T$ are disjoint, the case $m = i_n$ is impossible. When $m < i_n$, then $\|((I)|_m = 0$ and $\|((II)|_m$ is small by our inductive assumption iii). When $m > i_n$, then, if we take $f : \mathcal{P}(P_n) \to \mathcal{P}(P_m)$ defined by $f(C) = F(A_n \cup C \times X) \cap P_m$, then from Lemma 5 we have $\|((I))|_m \leq K + \frac{1}{n} \leq K + \frac{1}{2n+1}$ and from our inductive assumption $\|((II)|_m \leq K + 1 - \frac{1}{2n+1}$. From Lemma 4 we obtain:

$$\|((I) \triangle (II)|_m \leq \|((I) \cup (II)|_m \leq \sup(\|((I)|_m, \|((II)|_m) + \frac{1}{2n+1} \leq K + 1 - \frac{1}{2n+1}$$.

Finally, if we put $\tilde{A} = \bigcup_{n\in \omega} A_n, \tilde{B} = \bigcup_{n\in \omega} B_n$, then $\tilde{A} \triangle \tilde{B} \notin I_S$ but $\varphi_T[F(\tilde{A}) \triangle F(\tilde{B})] \leq K + 1$. Thus the pair $\tilde{A}, \tilde{B}$ is an example showing that $F$ does not reduce $I_S$ to $I_T$.

Let us recall the definitions of two important Borel ideals:

$$Fin \times \emptyset = \{ x \in \omega^2 : \exists n \in n \times \omega \}$$,

$$\emptyset \times Fin = \{ x \in \omega^2 : \forall m \forall n \forall k \geq n \langle m, k \rangle \notin x \}$$.

It is not difficult to prove that the original Louveau and Veličković family of ideals $(I_S^{|\omega|})_{S \in [\omega]^{\omega}}$ satisfies:

$$\forall S \in [\omega]^{\omega} |I_S^{|\omega|} \geq \emptyset \times Fin|$$.

Similarly the family $(I_S)_{S \in [\omega]^{\omega}}$ constructed above satisfies:

$$\forall S \in [\omega]^{\omega} |I_S \geq Fin \times \emptyset|$$.

Thus, in connection with the results of Solecki (see [3], Theorems 2.1 and 3.3), stating that every ideal not greater in the sense of reducibility from either $\emptyset \times Fin$ or $Fin \times \emptyset$ is a $p$-ideal of the class $F_\sigma$, it is interesting to ask the following

**Question 6. Does there exist a family of $p$-ideals of the class $F_\sigma$, $(I_S^{|\omega|})_{S \in [\omega]^{\omega}}$, satisfying the formula analogous to (3)?**

**References**


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