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THE HOCHSCHILD COHOMOLOGY RING OF A CYCLIC BLOCK

STEPHEN F. SIEGEL AND SARAH J. WITHERSPOON

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ABSTRACT. Suppose B is a block of a group algebra kG with cyclic defect group. We calculate the Hochschild cohomology ring of B, giving a complete set of generators and relations. We then show that if B is the principal block, the canonical map from $H^*(G,k)$ to the Hochschild cohomology ring of B induces an isomorphism modulo radicals.

1. INTRODUCTION

The representation theory of cyclic blocks (that is, blocks with cyclic defect groups) plays an important role in the representation theory of finite groups. The principal result states that a cyclic block is a Brauer tree algebra (see Alperin's book [1] for this and other results from the cyclic theory). These algebras are complicated enough to be interesting but simple enough so that many aspects of the theory afford an elegant combinatorial description. For this reason, the cyclic theory has provided an important testing ground for new developments in representation theory.

In this note we calculate the Hochschild cohomology ring of a cyclic block. Specifically, let G be a finite group, p a prime, and k an algebraically closed field of characteristic p. Suppose kG has a cyclic block B. Then Theorem 1 below gives a complete set of generators and relations for $H^*(B,B)$ as a commutative graded k-algebra.

Earlier, T. Holm [6] calculated the even subring of $H^*(B,B)$. His approach, which we share, used the fact that B is derived equivalent to kT, where T is a split extension of a cyclic p-group by a cyclic p'-group. This follows from a theorem of J. Rickard [7, Thm. 4.2], which states that two Brauer tree algebras are derived equivalent if, and only if, the trees have the same number of edges and the same multiplicity. Because algebras which are derived equivalent have isomorphic Hochschild cohomology rings ([8, Prop. 2.5]), it suffices to calculate $H^*(kT,kT)$.

At this point we part with the methods of [6] (which involve extensive calculations in the category of kT-bimodules) and instead exploit the isomorphism of $H^*(kT,kT)$ with $H^*(T,kT)$ ([10, Prop. 3.1]). The latter ring denotes the ordinary cohomology of T with coefficients in kT *considered as a kT-module by conjugation.* The ring structure is provided by the composition of the cup product with the map

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on cohomology induced by multiplication $kT \otimes kT \to kT$ ($\otimes = \otimes_k$). It turns out that this ring is quite easy to describe, using some elementary results from group cohomology.

We should point out that this result is a special case of more recent work by K. Erdmann and T. Holm [4], in which they calculate the Hochschild cohomology of a more general class of algebras, the self-injective Nakayama algebras. For the cyclic block case, however, the approach here does have the advantage that the proof, and the relations arrived at, are particularly simple.

In Section 3, we further prove that in case the principal block B_0 is cyclic, the canonical map from $H^*(G,k)$ to $H^*(B_0,B_0)$ induces an isomorphism modulo radicals. This question was raised for principal blocks in general in [10] and answered positively in the cases where G is a p-group, G is Abelian, and in a few other specific cases. This result for cyclic blocks provides further evidence for a positive answer to this question in general, and so we have elevated the question to the status of "conjecture."

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2. Generators and relations

Suppose kG has a block B with defect group cyclic of order p^n and inertial index e. Let $m = (p^n - 1)/e$.

Theorem 1. *If $e > 1$, then $H^*(B,B)$ is generated as a commutative k-algebra by elements z, $p_1, \dots, p_{e-1}$ of degree 0, and elements y_1, y_2, y_{2e-1}, y_{2e}, where $\deg(y_r) = r$, subject to the relations*

$$z^{m+1} = z^m y_1 = z^m y_2 = y_1 y_{2e-1} = y_1^2 = y_{2e-1}^2 = 0,$$
$$y_1 y_2^{e-1} = z y_{2e-1}, \quad y_2^e = z y_{2e}, \quad y_2 y_{2e-1} = y_1 y_{2e},$$
$$z p_i = y_1 p_i = y_2 p_i = y_{2e-1} p_i = y_{2e} p_i = 0 \ (1 \le i < e),$$
$$p_i p_j = 0 \ (1 \le i, j < e).$$

If $e = 1$ and $p^n \neq 2$, then

$$H^*(B,B) = k[z, y_1, y_2 \mid \deg(z) = 0, \deg(y_r) = r, z^{p^n} = 0 = y_1^2],$$

while if $e = 1$ and $p^n = 2$,

$$H^*(B,B) = k[z, y_1 \mid \deg(z) = 0, \deg(y_1) = 1, z^2 = 0].$$

The rest of this section is devoted to the proof of Theorem 1. First we note that by [3, Prop. 62.35], e divides $p - 1$. We consider the group algebra kT, where

$$T = \langle a, b \mid a^{p^n} = 1 = b^e, bab^{-1} = a^s \rangle,$$

and $s + p^n\mathbb{Z}$ is an element of order e in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, the group of units of $\mathbb{Z}/p^n\mathbb{Z}$. By [3, Lemma 60.9], kT has only one block, and so by [1, p. 123], kT is a Brauer tree algebra for the star with e edges and exceptional multiplicity m. As we have seen, we may conclude from this that $H^*(B,B)$ is isomorphic as a graded algebra to $H^*(T,kT)$, where kT is considered a kT-module under conjugation. It is this algebra to which we now turn.

If $e = 1$, then T is cyclic of order p^n. Since T is Abelian, $H^*(T, kT) \cong kT \otimes H^*(T, k)$ ([2, Thm. 2.1] or [10, Prop. 3.2]), and $H^*(T, k)$ is well known (see [5, §3.2]). This finishes the proof for $e = 1$. From this point on, we will assume $e > 1$.

First we give the conjugacy classes of T. Let $H = \langle a \rangle$ and $K = \langle b \rangle$. Then T is a Frobenius group with Frobenius kernel H and complement K; that is, K acts freely on $H \setminus \{1\}$ by conjugation. Therefore H is the union of the class $\{1\}$ and m classes each of size e. We claim that the remaining conjugacy classes of T are the $e - 1$ classes Hb^j $(1 \leq j < e)$. Indeed, for $h \in H$ we have $h^{-1}b^j h = h^{-1}(b^j h b^{-j})b^j$, an element of Hb^j as H is normal in T. As h runs over the elements of H, $h^{-1}(b^j h b^{-j})$ also runs over the elements of H, since K acts freely on $H \setminus \{1\}$.

From the above remarks, we have a direct sum decomposition of kT-modules (under the conjugation action)

$$kT = kH \oplus \bigoplus_{j=1}^{e-1} kHb^j.$$

Hence as graded k-modules (but not necessarily as rings), we have

$$H^*(T, kT) = H^*(T, kH) \oplus \bigoplus_{j=1}^{e-1} H^*(T, kHb^j). \tag{1}$$

Note that the restriction of kHb^j to H is a free kH-module, and H is a Sylow p-subgroup of T. Thus kHb^j is a projective kT-module [1, Cor. 9.3], so $H^*(T, kHb^j) = H^0(T, kHb^j) \cong k$. Using the identification of $H^0(T, kT)$ with the center $Z(kT)$, we set

$$p_j = \sum_{i=1}^{p^n} a^i b^j \quad (1 \leq j < e),$$

so that $H^0(T, kHb^j)$ is spanned by p_j. It is clear that $p_i p_j = 0$. Furthermore, if $\alpha \in H^r(T, kT)$ with $r > 0$, then $\alpha p_j = 0$. This is because the p_j form a basis of the projective ideal in $Z(kT)$, that is, all elements of the form $\sum_{x \in G} xyx^{-1}$, $y \in kT$. This ideal annihilates Hochschild cohomology in degree nonzero, as left multiplication by the element $\sum_{x \in G} xyx^{-1}$ in kT is a homomorphism that factors through a projective $kT \otimes kT^{op}$-module.

We now turn to the kT-module structure of kH. As K acts freely on $H \setminus \{1\}$, we have $kH \cong k \oplus (kK)^m$ as kK-modules. As K is a cyclic p'-group, $kK \cong S_0 \oplus S_1 \oplus \cdots \oplus S_{e-1}$ as kK-modules, where each S_j is one dimensional and b acts on S_j as multiplication by the eth root of unity s^j. The kK-module kK is the restriction of the induced kT-module $k \uparrow_H^T = kT \otimes_{kH} k$, on which H acts trivially. By abuse of notation, we will identify this kT-module with $S_0 \oplus S_1 \oplus \cdots \oplus S_{e-1}$, where S_j becomes a kT-module by specifying that H acts trivially. For any integer j, let V_j be the eigenspace in the kT-module kH, with eigenvalue s^j, for the linear transformation induced by conjugation by b. We have shown that as a kT-module (under the conjugation action),

$$kH \cong k \oplus (k \uparrow_H^T)^m \cong V_0 \oplus V_1 \oplus \cdots \oplus V_{e-1},$$

where $\dim(V_0) = m + 1$ and $\dim(V_j) = m$ for $1 \leq j < e$.

As H is the Sylow p-subgroup of G, the restriction res_H^T from $H^*(T, kH)$ to $H^*(H, kH)$ is injective [5, Prop. 4.2.2]. By [5, Cor. 4.2.7], the image is $H^*(H, kH)^K$, and therefore res^T_H provides an isomorphism of graded algebras

$$H^*(T, kH) \cong H^*(H, kH)^K.$$

Since H is Abelian, [10, Prop. 3.2] provides an isomorphism of graded algebras

$$H^*(H, kH) \xrightarrow{\cong} kH \otimes H^*(H, k).$$

This is also a map of kK-modules, where K acts diagonally on the tensor product. Composing these two isomorphisms, we have the following isomorphism of graded algebras.

Lemma 2. *$H^*(T, kH) \cong (kH \otimes H^*(H, k))^K$, where K acts by conjugation on kH.*

We claim that the action of K on

$$H^*(H, k) = k[x, y \mid \deg(x) = 1, \deg(y) = 2, x^2 = 0]$$

in the lemma is given by $bx = s^{-1}x$, $by = s^{-1}y$. To see this, let $\mathbf{P} \xrightarrow{\epsilon} k$ be the standard (minimal) kH-resolution, i.e., each P_n is a free kH-module on one generator and the differentials alternate between multiplication by $a - 1$ and multiplication by $\sum_{h \in H} h$. Let t be the integer between 1 and $p^n - 1$ satisfying $[t] = [s]^{-1}$ in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We may define a chain map θ from $\mathbf{P}$ to itself, commuting with ϵ and satisfying $\theta(hx) = b^{-1}hb\theta(x)$ $(h \in H,\ x \in \mathbf{P})$, such that in degrees 1 and 2, θ multiplies the generator by $1 + a + \cdots + a^{t-1}$. On the level of cocycles, the action of b is given by precomposing with θ, so in degrees 1 and 2 this just multiplies the cocycle by t.

It follows from Lemma 2 that $H^*(T, kH)$ is isomorphic to the subring

$$V_0 \otimes 1 \quad \oplus \quad \bigoplus_{i=1}^{\infty} V_i \otimes \langle xy^{i-1}, y^i \rangle.$$

We now turn to the problem of finding generators and relations for this subring. Let

$$w = \sum_{j=0}^{e-1} s^{-j} a^{s^j}.$$

Clearly $w \in V_1$. Shortly we will show that $w \in J \setminus J^2$, where J is the radical of the algebra kH, that is, the ideal generated by $1 - a$. It will follow that $w = u(1 - a)$, where u is an invertible element of kH. Hence $w^r \in J^r \setminus J^{r+1}$ for $0 \le r < p^n$. In particular the w^r form a basis for kH. In fact, since $V_q V_r \subseteq V_{q+r}$ for any integers q and r, we see that the w^{ei} $(0 \le i \le m)$ form a basis for V_0, and the w^{ei+j} $(0 \le i < m)$ form a basis for V_j $(1 \le j < e)$.

To prove our claim about w, we look at its image in kH/J^2. First note that for any $r \ge 0$,

$$a^r \equiv (1 + (a - 1))^r \equiv 1 + r(a - 1) \equiv ra + 1 - r \mod J^2.$$

Also note that $\sum_{j=0}^{e-1} s^{-j} = 0$ (multiply by $1 - s^{-1}$). Hence

$$w \equiv \sum_{j=0}^{e-1} s^{-j}(s^j a + 1 - s^j) = \sum_{j=0}^{e-1} (a - 1 + s^{-j}) \equiv e(a - 1) \mod J^2,$$

which establishes the claim that $w \in J \setminus J^2$.

Now define the remaining generators as follows:

$$z = w^e \otimes 1,\ \ y_1 = w \otimes x,\ \ y_2 = w \otimes y,\ \ y_{2e-1} = 1 \otimes xy^{e-1},\ \ y_{2e} = 1 \otimes y^e.$$

It is easy to see these generate $H^*(T, kH)$. First, the z^i $(0 \le i \le m)$ form a basis for $V_0 \otimes 1$; in particular z corresponds to the element w^e generating the radical of $(kH)^K$. Second, if $1 \le j < e$, then the $z^i y_1 y_2^{j-1}$ $(0 \le i < m)$ form a basis for $V_j \otimes xy^{j-1}$, and the $z^i y_2^j$ form a basis for $V_j \otimes y^j$. Third, the $z^i y_{2e-1}$ $(0 \le i \le m)$ form a basis for $V_e \otimes xy^{e-1}$. Finally, multiplication by y_{2e} is an isomorphism from $H^r(T, kH)$ to $H^{r+2e}(T, kH)$ for all $r \ge 0$, and using this we obtain an explicit basis for each homogeneous component of $H^*(T, kH)$.

Since $w = u(1-a)$, it is clear that $zp_j = 0$. This may also be seen by noting that the projective ideal of kT (having basis p_j $(1 \le j < e)$) is contained in the socle of $Z(kT)$, and so annihilates the element z of the radical of $Z(kT)$. The remaining relations are equally easy to check. We conclude that there is a surjective map of graded algebras from A^* to $H^*(T, kT)$, where A^* is the algebra defined abstractly by the generators and relations of the Theorem. We wish to show that this map is an isomorphism, and to do this it suffices to show $\dim(A^r) \le \dim(H^r(T, kT))$ for all $r \ge 0$.

Since the product of p_j $(1 \le j < e)$ with each generator of A^* is 0, we have $A^* = A_1 \oplus A_2$, where A_2 is spanned by the p_j and A_1 is the subalgebra generated by the remaining generators. Direct inspection of the relations shows that A_1 is spanned by the same elements described above which form a basis for $H^*(T, kH)$. Thus the dimension inequality is satisfied, and the proof is complete.

3. The principal block case

Suppose for now that G is any finite group and let

$$kG = B_0 + \cdots + B_s$$

be the block decomposition of kG, with B_0 the principal block. Considering kG as a module under conjugation, this yields an isomorphism of graded k-algebras

$$H^*(G, kG) \cong H^*(G, B_0) \oplus \cdots \oplus H^*(G, B_s).$$

It is not hard to see that $H^*(G, B_i) \cong H^*(B_i, B_i)$ $(0 \le i \le s)$. Now there are maps of kG-modules

$$k \longrightarrow B_0 \longrightarrow k$$

whose composite is the identity: the first map sends 1 to the principal block idempotent, the second is the restriction of the augmentation map to B_0. (If B_0 is replaced by B_i for $i > 0$, then the restriction of the augmentation map is 0.) Both of these are also maps of k-algebras and so applying the functor $H^*(G, -)$ we obtain maps of graded algebras

$$H^*(G, k) \xrightarrow{f} H^*(G, B_0) \longrightarrow H^*(G, k),$$

whose composite is the identity. In particular f is injective, so the induced map

$$\frac{H^*(G, k)}{\operatorname{rad}(H^*(G, k))} \xrightarrow{\bar{f}} \frac{H^*(G, B_0)}{\operatorname{rad}(H^*(G, B_0))}$$

is also injective.

Conjecture 1. *Let G be a finite group and k a field of characteristic p. Then the map $\bar{f}$ is an isomorphism.*

In [10, §§10–11], we showed that this conjecture holds when G is a p-group, when G is Abelian, when $G = A_4$ and $p = 2$, and when $G = S_3$ and p is 2 or 3. We now show that it holds whenever B_0 is cyclic:

Theorem 3. *Suppose G has cyclic Sylow p-subgroups. Then Conjecture 1 holds.*

Proof. First note that once it is shown that $H^*(G,k)$ and $H^*(G,B_0)$ are isomorphic after modding out by their radicals, then $\bar{f}$ must itself be an isomorphism, being an injection between graded algebras having the same dimension in each degree.

Next note that $H^*(G,k) \cong \mathrm{Ext}^*_{B_0}(k,k)$ as k lies in the block B_0. Using the same notation as in §2, we further see that $\mathrm{Ext}^*_{B_0}(k,k)$ is isomorphic to $H^*(T,k)$, as B_0 and kT are derived equivalent under an equivalence preserving the trivial module k [9, Thm. 10]. On the other hand, $H^*(G,B_0) \cong H^*(B_0,B_0)$ is isomorphic to $H^*(kT,kT)$ as B_0 and kT are derived equivalent [8, Prop. 2.5]. But $H^*(kT,kT) \cong H^*(T,kT)$, where kT is a module under conjugation. If we show that $H^*(T,k)$ and $H^*(T,kT)$, are isomorphic modulo radicals, it will follow from the above isomorphisms that $H^*(G,k)$ and $H^*(G,B_0)$ are also isomorphic modulo radicals. Thus we have reduced to the case $G = T$.

Considering (1), we note that $\bigoplus_{j=1}^{e-1} H^*(T,kHb^j)$ is spanned by the nilpotent elements p_j, so modulo radicals $H^*(T,kT)$ and $H^*(T,kH)$ are isomorphic. The quotient of $H^*(T,kH)$ by the nilpotent subalgebra corresponding to $(J(kH) \otimes H^*(H,k))^K$, under the isomorphism of Lemma 2, is isomorphic to $(H^*(H,k))^K$, since kH is local. As H is the Sylow p-subgroup of T, $H^*(T,k) \cong (H^*(H,k))^K$ by [5, Prop. 4.2.2 and Cor. 4.2.7], with the isomorphism provided by the restriction map from T to H. Therefore $H^*(T,kT)$ and $H^*(T,k)$ are isomorphic after modding out by their radicals. □

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Department of Computer Science, University of Massachusetts, Amherst, Massachusetts 01003-4610

E-mail address: `siegel@cs.umass.edu`

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

E-mail address: `withersp@math.wisc.edu`