DECIDABILITY OF THE REPRESENTATION EXTENSION
PROPERTY FOR FINITE SEMIGROUPS

KUNITAKA SHOJI

(Communicated by Ronald M. Solomon)

Abstract. We prove that the decision problem of whether or not a finite
semigroup has the representation extension property is decidable.

1. The main theorem and preliminaries

It is an immediate consequence of the normal form theorem for amalgamated free
products of groups that every amalgam of groups embeds in some group. However,
this result fails for semigroup amalgams: an early result of Kimura [6] shows that
amalgams of semigroups do not necessarily embed in any semigroup (see also [4],
Vol. II, page 138). More recently, Sapir [7] has shown that it is fact undecidable
whether an amalgam of (finite) semigroups embeds in any (finite) semigroup. A
semigroup $S$ is called an amalgamation base for semigroups if every amalgam of
semigroups containing $S$ as a subsemigroup embeds in some semigroup. It is natural
to ask if it is decidable whether or not a finite semigroup is an amalgamation base.

According to [3], we say that a semigroup $S$ has the representation extension
property if for any right $S$-set $X$ and any left $S$-set $M$ containing $S^1$ as a left $S$-
subset, the canonical map: $X \to X \otimes_S M (x \mapsto x \otimes 1)$ is injective. Hall [5]
proved that any semigroup which is an amalgamation base in the class of all semigroups
has the representation extension property. In this paper we prove

The Main Theorem. It is decidable whether or not a finite semigroup has the
representation extension property.

Let $S$ be a semigroup. Let $M$ be a nonempty set with a unitary and associative
operation of $S : S^1 \times M \to M ((s, w) \mapsto sw)$, where $S^1$ is the monoid obtained
from $S$ by adjoining a new identity $1$. Then $M$ is called a left $S$-set. Dually, a right
$S$-set is defined. If a left $S$-set [resp. right $S$-set] $M$ contains elements $m_1, \ldots, m_n$
such that $M = S^1 m_1 \cup \cdots \cup S^1 m_n$ [resp. $M = m_1 S^1 \cup \cdots \cup m_n S^1$], then we say
that $m_1, \ldots, m_n$ are generators of $M$.

A relation $\rho$ on a left [resp. right] $S$-set $M$ is called an $S$-congruence if $(m, m') \in \rho$
and $s \in S$ implies $(sm, sm') \in \rho$ [resp. $(ms, m's) \in \rho$]. Let $M, N$ be left [resp.
right] $S$-sets. Then a map $\phi : M \rightarrow N$ is called an $S$-map if $\phi(sm) = s\phi(m)$ [resp. $\phi(ms) = \phi(m)s$] for any $m \in M$ and $s \in S$.

**Result** ([3 Proposition 1.5]). Let $S$ be a semigroup and $A, B, C, D$ left [resp. right] $S$-sets such that $C$ is a left [resp. right] $S$-subset of $A$ and $D$ is a left [resp. right] $S$-subset of $B$. Let $\alpha$ be a bijective $S$-map: $C \rightarrow D$. Then there exist a left [resp. right] $S$-set $W$ and injective $S$-maps $\beta : A \rightarrow W$, $\lambda : B \rightarrow W$ such that $\alpha \lambda = \beta$ on $C$, $W = A\beta \cup B\lambda$, $A\beta \cap B\lambda = C\beta$.

In this case, we say that the left [resp. right] $S$-set $W$ is the left [resp. right] $S$-set obtained by gluing $A$ and $B$ with $\alpha$ and write $W = A\#_\alpha B$.

If $C$, $D$ are generated by $x$ and $y$ respectively and $\alpha(x) = y$, then we write $A \#_{x=y} B$ instead of $A\#_\alpha B$.

Let $X$ be a right $S$-set. Consider a set $\Sigma$ of equations $x_i t_i = x_{i+1} s_{i+1}$ $(1 \leq i \leq n-1)$ on $X$, where $t_i , s_{i+1} \in S$, $x_i \in X$. Then for any $1 \leq i \leq n$, we define the $S$-congruence $\rho_i$ on the right $S$-set $S^1$ as follows:

\( (s, t) \in \rho_i \) if and only if $x_i s = x_i t$ in $X$ for all $s , t \in S^1$.

Let $X_i = S^1/\rho_i$ and $\pi_i = 1_{\rho_i}$. Then we can obtain a right $S$-set $\overline{X}$ such that

\[ \overline{X} = \overline{X}_1 \#_{\pi_1, t_1 = \pi_2, s_2} \overline{X}_2 \cdots \#_{\pi_{n-1}, t_{n-1} = \pi_n, s_n} \overline{X}_n. \]

Then we have the set of equations $\pi_i t_i = \pi_i+1 s_{i+1}$ $(1 \leq i \leq n-1)$ in $\overline{X}$. We call $\overline{X}$ the relatively free right $S$-set associated to $X$ with respect to $\Sigma$.

2. **Decision problem for the representation extension property**

Let $S$ be a finite semigroup and let $T(S^1)$ denote the set of all mappings of $S^1$. Then we define an operation of $S$ on $T(S^1)$: $tf$ is defined by $(st)f = (st)f$ for all $s, t \in S^1$ and $f \in T(S^1)$. Then $T(S^1)$ is a left $S$-set.

Next let $\rho : S^1 \rightarrow T(S^1)$ denote the right regular representation of $S^1$. Then by [1 Theorem 6], the left $S$-set $T(S^1)$ is injective. As a consequence, we have (see Remark after Corollary 2.3 of [3]).

**Lemma.** Let $S$ be a finite semigroup and $T(S^1)$ as above. Then $S$ has the representation extension property if and only if for any right $S$-set $X_S$, the canonical map: $X \rightarrow X \otimes_S T(S^1)$ $(x \mapsto x \otimes 1)$ is injective.

**Proof.** This follows from Theorem 2.1 of [8] by substituting $T(S^1)$ for the left $S$-set $W$.

Then we shall prove that for any finite semigroup $S$, there is a positive integer $k$ depending only on the order $|S|$ of $S$ such that if there exist a right $S$-set $X$ and a scheme of length greater than $k$ which shows $x \otimes 1 = x' \otimes 1$ in $X \otimes_S T(S^1)$ but $x \neq x'$ in $X$, then there exist a right $S$-set $Y$ with fewer generators than $k$ and $y, y' \in Y$ such that $y \otimes 1 = y' \otimes 1$ in $Y \otimes_S T(S^1)$ but $y \neq y'$ in $Y$ (see [2] for undefined terms).

**Proof of the main theorem.** Let $S$ be a finite semigroup. By the lemma, suppose that there exist a right $S$-set $X$ and $x, x' \in X$ such that $x \neq x'$ in $X$ and $x \otimes 1 = x' \otimes 1$ in $X \otimes_S T(S^1)$. Then by Lemma 1.2 of [2], there exist $x_1, \cdots, x_n \in X$,
Let $x_1, \ldots, x_n \in \mathcal{T}(S^1)$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that

\[
\begin{align*}
x &= x_1s_1, & s_1 &= t_1y_1 \\
x_1t_1 &= x_2s_2, & s_2y_2 &= t_2y_3 \\
\vdots & \quad \vdots \\
x_{n-1}t_{n-1} &= x_ns_n, & s_ny_n &= t_n \\
x_n & = x'.
\end{align*}
\]

Then we can assume that $X = \mathcal{X}$, the relatively free right $S$-set of the right $S$-set associated to $X$ with respect to the set of equations (1). Thus we have

\[
X = X_1#x_1t_1=x_2s_2X_2# \cdots #X_i#x_{i-1}t_{i-1}=x_is_i+1X_{i+1}# \cdots #X_{n-1}#x_{n-1}t_{n-1}=x_ns_nX_n.
\]

Next we let $C_r(S)$ be the set of $S$-congruences on the right $S$-set $S^1$. In particular, let $\rho_i$ denote the $S$-congruence on the right $S$-set $S^1$ given by

\[
(s, t) \in \rho_i \quad \text{if and only if} \quad x_i s = x_i t \quad \text{for all} \quad s, t \in S.
\]

Let $m = |S \times C_r(S) \times S \times C_r(S) \times \mathcal{T}(S^1)|$ and $k = 2m$. If $n > k$, then there exist $1 \leq i_1 < i_2 < i_3 \leq n$ such that

\[
(t_{i_1}, \rho_{i_1}, s_{i_1+1}, \rho_{i_1+1}, y_{i_1+1}) = (t_{i_2}, \rho_{i_2}, s_{i_2+1}, \rho_{i_2+1}, y_{i_2+1}) = (t_{i_3}, \rho_{i_3}, s_{i_3+1}, \rho_{i_3+1}, y_{i_3+1}),
\]

where $y_{n+1} = 1$.

By the construction of $X$, we know that $X_i$ is $S$-isomorphic to $S^1/\rho_i$ as a right $S$-set with an $S$-map mapping $x_i$ on $1\rho_i$. Then, for $(p, q) = (1, 2), (1, 3), (2, 3)$, there are bijective $S$-maps:

\[
\begin{align*}
(x_{i_p}t_{i_p}S & \to x_{i_q}t_{i_q}S) \quad (x_{i_p}t_{i_p} \to x_{i_q}t_{i_q}), \\
(x_{i_p+1}s_{i_p+1}S & \to x_{i_q+1}s_{i_q+1}S) \quad (x_{i_p+1}s_{i_p+1} \to x_{i_q+1}s_{i_q+1})
\end{align*}
\]

and

\[
(x_{i_p}t_{i_p}S \to x_{i_q+1}s_{i_q+1}S) \quad (x_{i_p}t_{i_p} \to x_{i_q+1}s_{i_q+1}).
\]

By (5), for $(p, q) = (1, 2), (1, 3), (2, 3)$, we obtain three right $S$-sets

\[
X_{i_p,i_q} = X_1#x_{i_1}t_{i_1}=x_2s_2\mathcal{T}_2# \cdots #X_{i_p-1}#x_{i_p}t_{i_p}=x_{i_p+1}s_{i_p+1}X_{i_q+1}# \cdots #x_n#x_{n-1}=x_ns_nX_n.
\]

Then it follows from (1) and

\[
(t_{i_p}, \rho_{i_p}, s_{i_p+1}, \rho_{i_p+1}, y_{i_p+1}) = (t_{i_q}, \rho_{i_q}, s_{i_q+1}, \rho_{i_q+1}, y_{i_q+1})
\]

that $x \otimes 1 = x' \otimes 1$ in $X_{i_p,i_q} \otimes_S \mathcal{T}(S^1)$ for all $1 \leq p < q \leq 3$. On the other hand, there must exist at least one pair $(p, q)$ such that $x \neq x'$ in $X_{i_p,i_q}$. Suppose, on the contrary, that $x = x'$ holds in $X_{i_1,i_2}$, $X_{i_2,i_3}$ and $X_{i_1,i_3}$. Then it follows from the first equation $x = x'$ in $X_{i_1,i_2}$ and the way of construction of $X_{i_1,i_2}$ that

\[
x \in \bigcap_{i=1}^{i_1}x_it_iS, \quad x' \in \bigcap_{j=i_2+1}^{n}x_js_jS
\]

and there exists $u \in S$ such that

\[
x = x_it_iu \quad \text{in} \quad X_1#x_1t_1=x_2s_2X_2# \cdots #X_i.
\]
and

\[ x' = x_{i_2+1}s_{i_2+1}u \text{ in } X_{i_2+1} \#_{x_{i_2+1}t_{i_2+1}u = x_{i_2+2}s_{i_2+2}} X_{i_2+2} \cdots \# X_n. \]

Similarly, it follows from the other equations in \( X_{i_2,i_2} \) or \( X_{i_3,i_3} \) that there exist \( v, w \in S \) such that

\[ x = x_{i_2}t_{i_2}v \text{ in } X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_2}, \]

\[ x' = x_{i_3+1}s_{i_3+1}v \text{ in } X_{i_3+1} \#_{x_{i_3+1}t_{i_3+1}v = x_{i_3+2}s_{i_3+2}} X_{i_3+2} \cdots \# X_n \]

and

\[ x = x_{i_1}t_{i_1}w \text{ in } X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_1}, \]

\[ x' = x_{i_3+1}s_{i_3+1}w \text{ in } X_{i_3+1} \#_{x_{i_3+1}t_{i_3+1}w = x_{i_3+2}s_{i_3+2}} X_{i_3+2} \cdots \# X_n. \]

Of course, it can be assumed that

\[ X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_1} \subseteq X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_2} \]

and

\[ X_{i_3+1} \#_{x_{i_3+1}t_{i_3+1} = x_{i_3+2}s_{i_3+2}} X_{i_3+2} \cdots \# X_n \subseteq X_{i_2+1} \#_{x_{i_2+1}t_{i_2+1} = x_{i_2+2}s_{i_2+2}} X_{i_2+2} \cdots \# X_n. \]

Consequently, we have

\[ x = x_{i_1}t_{i_1}u = x_{i_2}t_{i_2}v = x_{i_1}t_{i_1}w \text{ in } X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_2} \]

and

\[ x' = x_{i_2+1}s_{i_2+1}u = x_{i_3+1}s_{i_3+1}v = x_{i_3+1}s_{i_3+1}w \text{ in } X_{i_2+1} \#_{x_{i_2+1}t_{i_2+1}u = x_{i_2+2}s_{i_2+2}} X_{i_2+2} \cdots \# X_n. \]

Since \( x_{i_1}t_{i_1}u = x_{i_1}t_{i_1}w \), by (3) we have \( x_{i_2}t_{i_2}u = x_{i_2}t_{i_2}w \). Also, by (5) we have \( x_{i_1}t_{i_1}w = x_{i_1}t_{i_1}u \). Similarly, \( x_{i_2}t_{i_2}v = x_{i_2}t_{i_2}u \) and \( x_{i_3}t_{i_3}v = x_{i_3}t_{i_3}u \). Therefore \( x' = x_{i_2+1}s_{i_2+1}u = x_{i_2+1}s_{i_2+1}w \), by (4) we have \( x_{i_2+1}s_{i_2+1}v = x_{i_2+1}s_{i_2+1}u \). Therefore \( x' = x_{i_2+1}s_{i_2+1}v = x_{i_2+1}s_{i_2+1}u = x_{i_2+1}s_{i_2+1}w = x_{i_2+1}s_{i_2+1}u \) in \( X_{i_2+1} \#_{x_{i_2+1}t_{i_2+1}u = x_{i_2+2}s_{i_2+2}} X_{i_2+2} \cdots \# X_n \), which implies \( x = x_{i_2}t_{i_2}u = x_{i_2}t_{i_2}v = x_{i_2}t_{i_2}w \) in \( X_1 \#_{x_1t_1 = x_2s_2} X_2 \cdots \# X_{i_1} \). Hence, by (2) \( x = x_{i_1}t_{i_1}u = x_{i_2+1}s_{i_2+1}u = x' \) in \( X \). This is a contradiction. Consequently, in order to decide whether or not the lemma above holds for the finite semigroup \( S \), it suffices to check whether or not the canonical map \( X \rightarrow X \otimes S T(S^3) \) is injective for all right \( S \)-sets \( X \) with generators of its number equal to or less than \( k \). Note that there exist only finitely many right \( S \)-sets with generators of its number equal to or less than \( k \), up to \( S \)-isomorphism. Hence the main theorem follows.

As far as we know, the following problems remain open.

**Problem.** Is it decidable whether or not a finite semigroup has any one of the following properties?

1. being an amalgamation base for semigroups,
2. being an amalgamation base for finite semigroups.

**Acknowledgments**

I thank John Meakin for his generous hospitality and stimulating conversation without which this work would not have been realized. I thank Tom Hall for reading the manuscript and correcting it. Finally, I thank the referee for good advice.
References


Department of Mathematics, Shimane University, Matsue, Shimane, 690-8503 Japan

E-mail address: ksho@math.shimane-u.ac.jp