

## A REMARK ON ANALYTIC CONTINUATION

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ABSTRACT. A simple example is given to show that the space of germs obtained by analytic continuation of a given germ need not be a covering space in the topological sense.

There is a significant difference between the definition of a covering surface used by complex analysts and that used by topologists. In complex analysis one constructs the space  $G$  of all germs that can be obtained by analytic continuation of a given analytic function  $f$ , and then  $G$  is a covering space of the natural domain of the extended function in the sense that every germ  $g$  in  $G$  has a neighbourhood on which the natural map (taking a germ at  $z_0$  to the point  $z_0$ ) is a homeomorphism. In topology, however, a covering surface  $\tilde{X}$  of a surface  $X$  is given by a covering map  $p: \tilde{X} \rightarrow X$  with the property that each  $x$  in  $X$  has a neighbourhood  $N$  such that the restriction of  $p$  to each component  $N_j$  of  $p^{-1}(N)$  is a homeomorphism of  $N_j$  onto  $N$ . The topological notion of a covering surface is equivalent to what complex analysts often call a regular covering surface (see [1], p.29), although for topologists, a regular covering space has an entirely different meaning (see [2], p. 163). The difference between the two definitions of a covering surface can be seen by comparing the proofs of the two versions of the Monodromy Theorem in the standard references [1] and [2] (and on p.149 and Lemma 3.3, p.152, in [2]).

One way to show that the topologist's version of a covering space is not always applicable to analytic continuation is to construct a function  $f$  whose analytic continuation has infinitely many branches, say  $f_n$ , at some point  $z_0$ , and which is such that the radius of convergence  $r_n$  of the Taylor series for the branch  $f_n$  at  $z_0$  tends to zero as  $n \rightarrow \infty$  (at least on a subsequence). I am not aware of such an example in the literature, and here we give an explicit and very simple example.

Let  $\mathbb{D}$  be the open unit disc in  $\mathbb{C}$ , and let  $a_1, a_2, \dots$  be distinct points in  $\mathbb{D}$  such that  $\sum(1 - |a_n|)$  converges, and such that the  $a_n$  accumulate at every point of the unit circle  $\partial\mathbb{D}$ . With these assumptions, the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \left( \frac{z - a_n}{1 - \bar{a}_n z} \right)$$

is an analytic map of  $\mathbb{D}$  into itself with zeros at and only at the  $a_n$  (each zero being a simple zero), and with natural boundary  $\partial\mathbb{D}$ . Further, as  $B$  is a contraction of

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the hyperbolic metric in  $\mathbb{D}$  (the Schwarz-Pick Lemma), we see that

$$(1) \quad 0 < |B'(a_n)| \leq \frac{1 - |B(a_n)|^2}{1 - |a_n|^2} = \frac{1}{1 - |a_n|^2} \leq \frac{1}{1 - |a_n|}.$$

For each  $n$ ,  $a_n$  is a simple zero of  $B$  so we can define a branch  $f_n$  of  $B^{-1}$  at 0 that satisfies  $f_n(0) = a_n$ , and it is clear that the  $f_n$  are analytic continuations of each other. Now let  $r_n$  and  $\Delta_n$  be the radius of convergence, and the disc of convergence, respectively, of the Taylor series for  $f_n$  about 0, so that  $\Delta_n$  is given by  $|z| < r_n$ . Then  $f_n$  maps  $\Delta_n$  into  $\mathbb{D}$  (as otherwise, we could analytically continue  $B$  beyond  $\mathbb{D}$ ), and hence  $f_n$  is a homeomorphism of  $\Delta_n$  onto  $f_n(\Delta_n)$  with inverse  $B$ . In particular, for each  $n$ , if  $m \neq n$ , then

$$(2) \quad a_m \notin f_n(\Delta_n).$$

Next, as  $f_n$  is univalent on  $\Delta_n$ , Koebe's 1/4-Theorem is applicable; thus  $f_n(\Delta_n)$  contains the open disc with centre  $a_n$  and radius  $r_n|f'_n(0)|/4$ . From this, together with (1), (2) and the fact that  $|f'_n(0)B'(a_n)| = 1$ , we have for  $m \neq n$ ,

$$(3) \quad 4|a_n - a_m| \geq r_n|f'_n(0)| \geq r_n(1 - |a_n|).$$

It is clear that we can construct the  $a_n$  satisfying the earlier assumptions, and also such that (for example) for all  $k$ ,  $|a_{2k} - a_{2k+1}| \leq (1 - |a_{2k}|)^2$ , and this with  $n = 2k$  and  $m = 2k + 1$  in (3) shows that  $r_{2n} \rightarrow 0$ . This concludes the example.

#### REFERENCES

- [1] Ahlfors, L.V. and Sario, L., *Riemann surfaces*, Princeton, 1960. MR **22**:5729
- [2] Massey, W.S., *Algebraic Topology: An Introduction*, GTM 56, Springer-Verlag, 1967. MR **35**:2271

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