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GENERALIZED MATLIS DUALITY

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Abstract. Let R be a commutative noetherian ring and let E be the minimal injective cogenerator of the category of R-modules. A module M is said to be reflexive with respect to E if the natural evaluation map from M to $\operatorname{Hom}_R(\operatorname{Hom}_R(M,E),E)$ is an isomorphism. We give a classification of modules which are reflexive with respect to E. A module M is reflexive with respect to E if and only if M has a finitely generated submodule S such that M/S is artinian and $R/\operatorname{ann}(M)$ is a complete semi-local ring.

Matlis and Gabriel in [7] and [5] considered modules over a complete local ring R. They showed that if the dual of an R-module is taken with respect to $E_R(k)$ (the injective envelope of the residue field k of R), then finitely generated and artinian modules are reflexive.

Various authors have considered related questions. For example, dropping the condition that R be complete or weakening local to semilocal ([1], [2], [6], [8], [9]).

In this paper we let R be any commutative noetherian ring and let E be the minimal injective cogenerator of the category of R-modules. We give a classification of modules which are reflexive with respect to E. The result is that a module M is reflexive with respect to E if and only if M has a finitely generated submodule S such that M/S is artinian and if R/I is a complete semilocal ring where $I = \operatorname{ann}(M)$.

We denote by Ω the maximal spectrum of R, and we let $E = \bigoplus_{m\in\Omega} E_R(R/m)$ be the minimal injective cogenerator in the category of R-modules. For an R-module M we let $M^\vee = \operatorname{Hom}_R(M,E)$ and call $M^\vee$ the Matlis dual of M. If the canonical map $M \to M^{\vee\vee}$ is an isomorphism we say that M is (Matlis) reflexive. We note that for any M, the map $M \to M^{\vee\vee}$ is an injection. From this it is easy to conclude that $\operatorname{ann}(M) = \operatorname{ann}(M^\vee)$.

If $S \subset R$ is a multiplicative set and the canonical map $M \to S^{-1}M$ is an isomorphism, we write $M = S^{-1}M$. If $M = S^{-1}M$, then also $M^\vee = S^{-1}(M^\vee)$. When $S = R - P$ is the complement of a prime ideal P of R we use the usual notation M_P.

If R is a local ring we let $\hat{R}$ denote its completion. If M is finitely generated we note that $\hat{R} \otimes_R M \cong \hat{M}$ (the completion of M). We write $\hat{R} \otimes_R M \cong \hat{M} = M$ to mean that $M \to \hat{R} \otimes_R M \cong \hat{M}$ is an isomorphism.

We note that if $m \in \Omega$ and M is a finitely generated R-module M, then $\operatorname{Hom}_R(M, E(R/m)) \neq 0$ if and only if $\operatorname{ann}(M) \subset m$.

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If R is a complete local ring, then all finitely generated and all artinian R-modules are reflexive. If R is a complete semilocal ring, then since R is the product of a finite number of complete local rings, we still have finitely generated and artinian modules over R are reflexive.

Lemma 1. *If $I \subset R$ is an ideal and $IM = 0$ for an R-module M, then M is reflexive as an R-module if and only if M is reflexive as an R/I-module.*

Proof. This follows from the fact that $\mathrm{Hom}_R(R/I, E)$ is a minimal injective cogenerator over R/I and that, since $M \otimes_R R/I \cong M$, we have

$$\mathrm{Hom}_{R/I}(M, \mathrm{Hom}_R(R/I, E)) \cong \mathrm{Hom}_R(M, E).$$

□

Lemma 2. *If $S \subset R$ is multiplicative and $S^{-1}M = M$ for an R-module M, then M is reflexive as an R-module if and only if M is reflexive as an $S^{-1}R$-module.*

Proof. As in Lemma 1, $\mathrm{Hom}_R(S^{-1}R, E)$ is a minimal injective cogenerator over $S^{-1}R$ and

$$\mathrm{Hom}_{S^{-1}R}(M, \mathrm{Hom}_R(S^{-1}R, E)) \cong \mathrm{Hom}_R(M, E).$$

□

The following result strengthens Theorem 2(i) in [1].

Theorem 3. *Let R be a local ring and M a finitely generated R-module. Then M is reflexive if and only if $\hat{R} \otimes_R M = M$.*

Proof. For any such M we have the commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \mathrm{Hom}_R(\mathrm{Hom}_R(M, E_R(k)), E_R(k)) \arrow[d] \\
\hat{M} = \hat{R} \otimes_R M \arrow[r] & \mathrm{Hom}_{\hat{R}}(\mathrm{Hom}_{\hat{R}}(\hat{M}, E_{\hat{R}}(k)), E_{\hat{R}}(k))
\end{tikzcd}$$

where k is the residue field of R and of $\hat{R}$. The bottom horizontal arrow is an isomorphism since $\hat{M}$ is a reflexive $\hat{R}$-module.

But we claim that the right vertical arrow is an isomorphism. For $E_R(k) = E_{\hat{R}}(k)$ and

$$\mathrm{Hom}_{\hat{R}}(\hat{M}, E_R(k)) = \mathrm{Hom}_{\hat{R}}(\hat{R} \otimes_R M, E_R(k))$$

$$= \mathrm{Hom}_R(M, \mathrm{Hom}_{\hat{R}}(\hat{R}, E_R(k))) = \mathrm{Hom}_R(M, E_R(k)).$$

Since $A = \mathrm{Hom}_R(M, E_R(k))$ and $E_R(k)$ are artinian R and $\hat{R}$-modules, we have $\mathrm{Hom}_{\hat{R}}(A, E_R(k)) = \mathrm{Hom}_R(A, E_R(k))$. It follows that $M \to \hat{R} \otimes_R M$ is an isomorphism if and only if M is reflexive. □

Corollary 4. *If I is an ideal in a local ring R, then R/I is reflexive as an R-module (or as an R/I-module) if and only if R/I is a complete local ring.*

Proof. This follows from the fact that $\hat{R} \otimes_R R/I = \widehat{R/I}$. □

Lemma 5. *If M is an R-module and $S \subset M$ is a submodule, then M is reflexive if and only if S and M/S are reflexive.*

Proof. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & (M/S)^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

has exact rows and the vertical maps are all injections. So the result follows. □

Lemma 6. *No infinite direct sum of nonzero R-modules is reflexive.*

Proof. Let $\bigoplus_{i\in I} M_i$ be reflexive where I is infinite and $M_i \neq 0$ for all $i \in I$. We have $(\bigoplus_{i\in I} M_i)^\vee = \prod_{i\in I}(M_i^\vee)$, and the canonical map

$$\bigoplus_{i\in I} M_i \to \left(\prod_{i\in I}(M_i^\vee)\right)^\vee \cong \left(\bigoplus_{i\in I} M_i\right)^{\vee\vee}$$

sends $(x_i)_{i\in I}$ to the map $(\phi_i)_{i\in I} \mapsto \sum_{i\in I} \phi_i(x_i)$. So the image of $(x_i)_{i\in I}$ is 0 on $\bigoplus_{i\in I}(M_i^\vee) \subset \prod_{i\in I}(M_i^\vee)$ if and only if $x_i = 0$ for all i, i.e., if and only if $(x_i)_{i\in I} = 0$. But $\bigoplus_{i\in I}(M_i^\vee) \neq \prod_{i\in I}(M_i^\vee)$ so there is a nonzero linear $f : \prod_{i\in I}(M_i^\vee) \to E$ which is 0 on $\bigoplus_{i\in I}(M_i^\vee)$. By the above no such f can be the image of an $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ and so $\bigoplus_{i\in I} M_i$ is not reflexive. □

Corollary 7. *If M is reflexive there is a finitely generated submodule S such that M/S is artinian.*

Proof. If $M = 0$ this is trivial. If $M \neq 0$ there is a finitely generated $S_1 \subset M$ such that $\mathrm{Soc}(M/S_1) \neq 0$. If $\mathrm{Soc}(M/S_1)$ is essential in M/S_1, it is well known that M/S_1 (and in fact $E(M/S_1)$) is artinian. If it is not essential, let $N/S_1 \cap \mathrm{Soc}(M/S_1) = 0$ with $S_1 \subset N$, $S_1 \neq N$. Then there is a finitely generated S_2 with $S_1 \subset S_2 \subset N$ and $\mathrm{Soc}(N/S_2) \neq 0$. But then $\mathrm{Soc}(M/S_1) \to \mathrm{Soc}(M/S_2)$ is injective but not surjective. We repeat the procedure and see that it must stop, for otherwise if $T = \bigcup S_n$, then $\mathrm{Soc}(M/T)$ is an infinite direct sum. This is impossible by Lemma 6. □

The argument above is taken from Enochs [3], Proposition 1.3, and is included here for completeness.

Proposition 8. *Let M be an R-module and suppose that for some $m \in \Omega$, $\mathrm{Hom}_R(M, E(R/n)) = 0$ for $n \in \Omega$, $n \neq m$. Then if M is reflexive, $M_m = M$ and $\mathrm{Hom}_R(M^\vee, E(R/n)) = 0$ for $n \neq m$.*

Proof. Let M be reflexive and $m \in \Omega$ be such that $\mathrm{Hom}_R(M, E(R/n)) = 0$ for $n \neq m$. If $M \neq 0$, we have a natural nonzero homomorphism $M \to M^{\vee\vee} = \mathrm{Hom}_R(M^\vee, E)$ $(n \in \Omega)$. So if $\mathrm{Hom}_R(M^\vee, E(R/n)) \neq 0$, then the projection $E \to E(R/n)$ induces $\mathrm{Hom}_R(M^\vee, E) \to \mathrm{Hom}_R(M^\vee, E(R/n))$ and so we have a nonzero homomorphism $M \to \mathrm{Hom}_R(M^\vee, E(R/n))$. So then

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(M^\vee, E(R/n))) \neq 0.$$

But

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(M^\vee, E(R/n))) \cong \mathrm{Hom}_R(M^\vee, \mathrm{Hom}_R(M, E(R/n)))$$

so $\mathrm{Hom}_R(M, E(R/n)) \neq 0$. Hence $n = m$.

Since $M^\vee = \mathrm{Hom}_R(M, E(R/m))$, it follows easily using properties of $E(R/m)$ that $(M^\vee)_m = M^\vee$. But then $(M^{\vee\vee})_m = M^{\vee\vee}$ and so $M_m = M$. It now follows by Lemma 2 that M is a reflexive R_m-module as well as a reflexive R-module. □

In the next result we use the fact that if A is an artinian R-module, then there are distinct maximal ideals $n_1, n_2, \ldots, n_t \in \Omega$ and a decomposition

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_t$$

such that $(A_i)_{n_i} = A_i$. The $n_1, \ldots, n_t$ and the decomposition are unique in the obvious sense. If $B \subset A$ is a submodule, then

$$B = (B \cap A_1) \oplus (B \cap A_2) \oplus \cdots \oplus (B \cap A_t)$$

gives the corresponding decomposition of B. We note that $\mathrm{Hom}_R(A_i, E(R/m)) = 0$ for $m \in \Omega$, $m \notin \{n_1, \ldots, n_t\}$.

Theorem 9. *Let M be a finitely generated R-module and let $I = \mathrm{ann}(M)$. Then M is reflexive if and only if R/I is a complete semilocal ring.*

Proof. We have $M^\vee = \mathrm{Hom}_R(M, \bigoplus E(R/m)) \cong \bigoplus \mathrm{Hom}_R(M, E(R/m))$ (the direct sum over all $m \in \Omega$) since M is finitely generated. Since $M^\vee$ is also reflexive, we see by Proposition 8 that $\mathrm{Hom}_R(M, E(R/m)) = 0$ except for a finite number of $m \in \Omega$. Let $n_1, n_2, \ldots, n_t \in \Omega$ be distinct elements of Ω such that $\mathrm{Hom}_R(M, E(R/m)) = 0$ for $m \notin \{n_1, n_2, \ldots, n_t\}$, so $M^\vee = \displaystyle\bigoplus_{i=1}^{t} \mathrm{Hom}_R(M, E(R/n_i))$. We can assume $\mathrm{Hom}_R(M, E(R/n_i)) \neq 0$ for $i = 1, 2, \ldots, t$. If M_i denotes $\mathrm{Hom}_R(M, E(R/n_i))$, then since $\mathrm{Hom}_R(M_i, E(R/m)) = 0$ for $m \neq n_i$, we see $\mathrm{Hom}_R(R/I_i, E(R/m)) = 0$ for $m \neq n_i$ where $I_i = \mathrm{ann}(M_i)$. Hence I_i is contained in only one maximal ideal, namely n_i. Hence R/I_i is a local ring and $(R/I_i)_{n_i} = R/I_i$.

Since R/I_i is also reflexive and finitely generated, we get that R/I_i is a reflexive R_{n_i}-module by Lemma 2. But then by Theorem 3, $\widehat{R_{n_i}} \otimes_{R_{n_i}} R/I_i = R/I_i$. This means that R/I_i is a complete local ring.

But now if $I = \mathrm{ann}(M)$, we have $I = \displaystyle\bigcap_{i=1}^{t} I_i$. The $I_1, \ldots, I_t$ are pairwise comaximal so by the Chinese remainder theorem $R/I \cong R/I_1 \times R/I_2 \times \cdots \times R/I_t$. Hence R/I is a complete semilocal ring.

Now assume that M is a finitely generated R-module and that R/I is a complete semilocal ring where $I = \mathrm{ann}(M)$. Then M is reflexive as an R/I-module and so by Lemma 1 is reflexive as an R-module. □

Corollary 10. *If $I \subset R$ is an ideal, then R/I is reflexive if and only if R/I is a complete semilocal ring.*

Proof. Immediate. □

Corollary 11. *If $I, J \subset R$ are ideals such that R/I and R/J are complete semilocal rings, then R/IJ is a complete semilocal ring.*

Proof. We only need argue that R/IJ is a reflexive R-module. We have the exact sequence

$$0 \to I/IJ \to R/IJ \to R/I \to 0.$$

But R/I is a reflexive R-module, and since I/IJ is a quotient of $(R/J)^n$ for some $n \geq 1$ it is a reflexive R-module. Hence R/IJ is a reflexive R-module. (These claims all use Lemma 5.) $\square$

We now give the complete classification of reflexive modules.

Theorem 12. *An R-module M is reflexive if and only if it has a finitely generated submodule S such that M/S is artinian and if R/I is a complete semilocal ring where $I = \operatorname{ann}(M)$.*

Proof. Assume that M is reflexive. By Corollary 7 we have the finitely generated S with M/S artinian. Since S is reflexive and finitely generated, R/J is a complete semilocal ring where $J = \operatorname{ann}(S)$ by Theorem 9. Since M/S is artinian and reflexive, $(M/S)^\vee$ is noetherian, i.e., finitely generated. Hence, if $K = \operatorname{ann}(M/S)^\vee$, then R/K is complete semilocal. But $\operatorname{ann}(M/S)^\vee = \operatorname{ann}(M/S)$. Since $I = \operatorname{ann}(M) \supset JK$ and since R/JK is complete semilocal by Corollary 11, we see that R/I is complete semilocal.

Conversely assume that M has a finitely generated submodule $S \subset M$ with M/S artinian and that R/I is a complete semilocal ring where $I = \operatorname{ann}(M)$.

Then, since $J = \operatorname{ann}(S) \supset \operatorname{ann}(M) = I$, we see that R/J is complete and semilocal. Hence S is reflexive.

Since M/S is artinian and $K = \operatorname{ann}(M/S) \supset \operatorname{ann}(M) = I$, we see that R/K is complete and semilocal. Then the artinian R/K-module is reflexive as an R/K-module. By Lemma 1, this implies that M/S is a reflexive R-module. But then by an appeal to Lemma 5 it follows that M is a reflexive R-module. $\square$

Examples.

1. If k is an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ is such that $k[x_1, \ldots, x_n]/I$ is semilocal, then in fact $k[x_1, \ldots, x_n]/I$ is artinian. Hence a finitely generated reflexive module M over $k[x_1, \ldots, x_n]$ has finite length. Since the dual $A^\vee$ of an artinian reflexive $k[x_1, \ldots, x_n]$-module A is finitely generated, we see that A also has finite length. So by Theorem 12 the reflexive $k[x_1, \ldots x_n]$-modules are exactly those of finite length.
2. If k is any field and $R = k[[x]][y]$, then $R/(y)$ is a complete local ring and so is a reflexive R-module which is not of finite length.

Remark. We recall that a module M is said to be cotorsion if $\mathrm{xt}^1(F, M) = 0$ for all flat R-modules F. In [4], finitely generated cotorsion modules were characterized over a commutative noetherian ring R of finite Krull dimension. In effect, over such a ring R, a finitely generated module M is cotorsion if and only if it is reflexive. We do not know if this holds if we allow R to have infinite Krull dimension.

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