

A GENERALIZATION OF KELLEY'S THEOREM FOR C -SPACES

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ABSTRACT. We prove that if an open map $f : X \rightarrow Y$ of compacta X and Y has perfect fibers and Y is a C -space, then there exists a 0-dimensional compact subset of X intersecting each fiber of f . This is a stronger version of a well-known theorem of Kelley. Applications of this result and related topics are discussed.

1. INTRODUCTION

All spaces are assumed to be separable metrizable. By a perfect set we mean a set with no isolated point.

Definition 1.1. A space X is said to be a C -space if for every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of families of disjoint open sets such that $\bigcup \mathcal{V}_i$ covers X and \mathcal{V}_i refines \mathcal{U}_i for every i .

C -spaces have been of considerable interest in infinite dimensional topology. On the one hand they pick up some nice properties of finite dimensional spaces. On the other hand the class of C -spaces is significantly larger than the class of finite dimensional spaces; it includes countable dimensional spaces and even infinite dimensional spaces which are not countable dimensional. If a space is not a C -space, then it behaves in many aspects as a strongly infinite dimensional space. For example if a compactum X is not a C -space, then it contains a compactum which is hereditarily not a C -space (see [2] and [3]).

C -spaces are weakly infinite dimensional and it is still unknown if the class of C -spaces coincides with the class of weakly infinite dimensional spaces. For more information on C -spaces see [1].

The main result of this paper is:

Theorem 1.2. *Let X and Y be compacta and let a map $f : X \rightarrow Y$ be open and onto with perfect fibers and such that Y is a C -space. Then there exists a 0-dimensional compact subset Z of X such that $f(Z) = Y$. Moreover, almost all sets of $\mathcal{M} = \{M \in 2^X : f(M) = Y\}$ are 0-dimensional (where almost all=all but a set of first category).*

Theorem 1.2 generalizes a well-known theorem of Kelley [7] which assumes that f is open monotone and onto with non-trivial fibers and Y is finite dimensional.

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Theorem 1.2 applies to prove:

Theorem 1.3. *Let X and Y be compacta and let a map $f : X \rightarrow Y$ be open and onto with perfect fibers and such that Y is a C -space. Then there exists a map $g : X \rightarrow I = [0, 1]$ such that for each $y \in Y$, $g(f^{-1}(y)) = I$.*

Theorem 1.3 was proved by Bula [8] for a finite dimensional Y . Actually, we will show that if an open map with perfect fibers satisfies the conclusions of Theorem 1.2, then it also satisfies the conclusions of Theorem 1.3. Note that Dranishnikov [9] constructed an open map $f : X \rightarrow Y$ of compacta with fibers homeomorphic to a Cantor set and such that there are no disjoint closed sets F and H in X with $f(F) = f(G) = Y$. For a map satisfying the conclusions of Theorem 1.3 such sets always exist: $F = g^{-1}(0)$ and $G = g^{-1}(1)$. Thus with no restriction on Y , Theorem 1.3 and hence Theorem 1.2 do not hold. Apparently the space Y in Dranishnikov's example is not a C -space.

Another example of a map not satisfying the conclusions of Theorem 1.2 was given by Kelley [7]. He showed that for an open monotone map $f : X \rightarrow Y$ of a hereditarily indecomposable continuum X of $\dim \geq 2$ such that the fibers of f are non-trivial and sufficiently small, there is no closed 0-dimensional subset of X intersecting each fiber. By Theorem 1.2, Y must be not a C -space.

The approach for proving Theorem 1.2 can be combined with the approaches and results of [6] and [4] to obtain the following theorem which we state without a proof.

Theorem 1.4. *Let X be a 2-dimensional continuum. Then the hyperspace $\mathcal{C}(X)$ (=the space of subcontinua of X) is not a C -space. Moreover, there exists a 1-dimensional subcontinuum Y of X such that $\mathcal{C}(Y)$ is not a C -space.*

The question whether or not $\mathcal{C}(X)$ is infinite dimensional for a 2-dimensional continuum X was a long standing open problem which was answered in the affirmative in [6]. Theorem 1.4 improves this result.

2. PROOFS

Let us say that a sequence $l = (l_1, l_2, \dots)$ of integers dominates a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of families of subsets of a space X if each element of \mathcal{V}_i intersects at most l_j elements of \mathcal{V}_j for every $j \leq i$. We also say that l dominates X if for every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of families of open sets dominated by l and such that $\bigcup \mathcal{V}_i$ covers X and \mathcal{V}_i refines \mathcal{U}_i .

Proposition 2.1. *There exists a sequence $l = (l_1, l_2, \dots)$ which dominates every C -compactum.*

Proof of Proposition 2.1. Let $r(m)$ be an integer such that each m -dimensional compactum admits an arbitrarily small cover such that every element of the cover intersects at most $r(m)$ elements of the cover. $r(m)$ exists, since each m -dimensional compactum can be embedded in Euclidean space of dimension $2m + 1$.

Let X be a C -compactum and let \mathcal{U}_i be a sequence of open covers of X . Then there exists a finite sequence $\mathcal{V}_1, \dots, \mathcal{V}_n$ of finite families of closed disjoint sets of X such that $\mathcal{V} = \bigcup \mathcal{V}_i$ covers X and \mathcal{V}_i refines \mathcal{U}_i .

We are going to define a sequence $l = (l_1, l_2, \dots)$ and to replace each $V \in \mathcal{V}_i$ by a finite family of closed sets \mathcal{F}_V such that $\bigcup \{F : F \in \mathcal{F}_V\} = V$ and for $\mathcal{V}'_i = \bigcup \{\mathcal{F}_V : V \in \mathcal{V}_i\}$ the sequence $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_n$ is dominated by l .

We construct \mathcal{V}'_i and l_i by induction. Set $\mathcal{V}'_1 = \mathcal{V}_1$ and $l_1 = 1$, and assume that $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_i$ are constructed to be dominated by (l_1, l_2, \dots, l_i) .

For each $W \in \mathcal{W} = \mathcal{V}'_1 \cup \mathcal{V}'_2 \cup \dots \cup \mathcal{V}'_i \cup \mathcal{V}_{i+1}$ take an open set W' such that $W \subset W'$, for every $1 \leq j \leq i$ $\text{order}\{W' : W \in \mathcal{V}'_j\} = \text{order}\mathcal{V}'_j \leq l_j$ and $\text{order}\{W' : W \in \mathcal{W}\} = \text{order}\mathcal{W} \leq m = l_1 + l_2 + \dots + l_i + 1$.

Let $f_W : X \rightarrow I = [0, 1]$ be such that $W = f_W^{-1}(1)$ and $f_W^{-1}(0) = X \setminus W'$. Then for the product map $f = \prod\{f_W : W \in \mathcal{W}\} : X \rightarrow I^{|\mathcal{W}|}$, $f(\mathcal{W}) = \{f(W) : W \in \mathcal{W}\}$ is of $\text{order} \leq m$, $\text{order}f(\mathcal{V}'_j) \leq l_j$, $j \leq i$, each point of $M = f(X)$ has at most m non-zero coordinates and hence $\dim M \leq m$. Take a small finite closed cover \mathcal{M} of M such that each element of \mathcal{M} intersects at most l_j elements of $f(\mathcal{V}'_j)$, $j \leq i$, and at most $r(m)$ elements of \mathcal{M} .

For each $V \in \mathcal{V}_{i+1}$ define $\mathcal{F}_V = \{V \cap f^{-1}(F) : F \in \mathcal{M}, F \cap f(V) \neq \emptyset\}$ and set $l_{i+1} = r(m)$. Then $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_{i+1}$ is dominated by $(l_1, l_2, \dots, l_{i+1})$ and the proposition follows. □

Proposition 2.2. *Let $f : X \rightarrow Y$ be an open map of compacta with perfect fibers. Then for every $\epsilon > 0$ there exists a sequence $\epsilon > \epsilon_1 > \epsilon_2 > \dots$ such that for any point $x \in X$ and any finite family of sets A_1, A_2, \dots, A_n with $\text{diam} A_i \leq \epsilon_i$ there is a point $x' \in f^{-1}f(x)$ such that $d(x, x') < \epsilon$ and $d(x', A_1 \cup A_2 \cup \dots \cup A_n) > \epsilon_n$.*

Proof of Proposition 2.2. We will construct ϵ_i by induction. Assume that we have determined $\epsilon_1, \epsilon_2, \dots, \epsilon_i$ such that for any point $x \in X$ and any family of sets A_1, A_2, \dots, A_i with $\text{diam} A_j \leq \epsilon_j$ there is a point $x' \in f^{-1}f(x)$ such that $d(x, x') < (1 - 1/2^i)\epsilon$ and $d(x', A_1 \cup A_2 \cup \dots \cup A_i) > \epsilon_i$. The following construction for ϵ_{i+1} also applies to ϵ_1 .

Set $\epsilon_0 = \epsilon$. Since f is open with perfect fibers and X is compact, one can take ϵ_{i+1} so small that $0 < \epsilon_{i+1} < (1/2^{i+1})\epsilon$ and for each $x' \in X$ there exists $x'' \in f^{-1}f(x')$ such that $3\epsilon_{i+1} < d(x'', x') < (1/2)\epsilon_i$.

Fix a point $x \in X$ and take a family of sets $A_1, A_2, \dots, A_i, A_{i+1}$ with $\text{diam} A_j \leq \epsilon_j$. Then by the induction assumption there is a point $x' \in f^{-1}f(x)$ such that $d(x, x') < (1 - 1/2^i)\epsilon$ and $d(x', A_1 \cup A_2 \cup \dots \cup A_i) > \epsilon_i$. For $i = 0$ assume that $x' = x$. Now ϵ_{i+1} is determined such that there is a point $x'' \in f^{-1}f(x)$ with $3\epsilon_{i+1} < d(x'', x') < (1/2)\epsilon_i$. Hence $d(x'', A_1 \cup A_2 \cup \dots \cup A_i) > (1/2)\epsilon_i > 3\epsilon_{i+1}$ and either $d(x', A_{i+1}) > \epsilon_{i+1}$ or $d(x'', A_{i+1}) > \epsilon_{i+1}$. $d(x, x'') < (1 - 1/2^{i+1})\epsilon$ and we are done. □

Proof of Theorem 1.2. Following [7] we will show that for $\epsilon > 0$ the family $\mathcal{M}_\epsilon = \{M \in 2^X : \text{the components of } M \text{ are of } \text{diam} < \epsilon \text{ and } f(M) = Y\}$ is dense in $\mathcal{M} = \{M \in 2^X : f(M) = Y\}$. Since \mathcal{M}_ϵ is open in \mathcal{M} , the density of \mathcal{M}_ϵ proves the theorem.

By Proposition 2.1 Y is dominated by some $l = (l_1, l_2, \dots)$. Fix $M \in \mathcal{M}$. Let $\epsilon > 0$ and let a sequence $\epsilon_1, \epsilon_2, \dots$ satisfy the conclusion of Proposition 2.2. Denote $\alpha_i = \frac{1}{2}\epsilon_{l_1+l_2+\dots+l_{i+1}}$. Take covers \mathcal{U}_i of Y such that for every $U \in \mathcal{U}_i$ and every $x \in f^{-1}(U)$ we have that $U \subset f(V(\alpha_i, x))$, where $V(\alpha, x)$ = the α -neighborhood of x .

Let $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n$ be finite families of closed sets such that \mathcal{V}_i refines \mathcal{U}_i , $\mathcal{W} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_n$ covers Y and each element of \mathcal{V}_i meets at most l_j elements of \mathcal{V}_j for $1 \leq j \leq i$. Arrange the elements of $\mathcal{W} = \{W_1, W_2, \dots\}$ such that the elements of \mathcal{V}_{i_1} appear before the elements of \mathcal{V}_{i_2} for $i_1 < i_2$. Let us assign for every $W \in \mathcal{W}$ the index $i(W)$ for which $W \in \mathcal{V}_{i(W)}$.

For each W_j we will choose by induction a closed set G_j such that $f(G_j) = W_j$, $\text{diam}G_j \leq 2\alpha_{i(W_j)}$, $G_j \subset V(2\epsilon, M)$ and $G_{j_1} \cap G_{j_2} = \emptyset$ for $j_1 \neq j_2$.

Assume that we have constructed G_1, G_2, \dots, G_j . The following construction for G_{j+1} is also applicable for choosing G_1 .

Take $x \in M \cap f^{-1}(W_{j+1})$. Let $i = i(W_{j+1})$. W_{j+1} intersects at most $m = l_1 + l_2 + \dots + l_i$ preceding elements W_{j_1}, \dots, W_{j_m} of \mathcal{W} . Then $W_{j_1}, \dots, W_{j_m} \in \mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_i$ and among the sets W_{j_1}, \dots, W_{j_m} there are at most l_k elements of \mathcal{V}_k for each $1 \leq k \leq i$. Hence the family G_{j_1}, \dots, G_{j_m} consists of at most l_1 sets of $\text{diam} \leq 2\alpha_1 \leq \epsilon_{l_1}$, at most l_2 sets of $\text{diam} \leq 2\alpha_2 \leq \epsilon_{l_1+l_2}, \dots$ and at most l_i sets of $\text{diam} \leq \epsilon_{l_1+l_2+\dots+l_i}$.

By Proposition 2.2 there is a point $x' \in V(\epsilon, x) \cap f^{-1}f(x)$ such that $\text{cl}V(\alpha_i, x')$ does not intersect $G_{j_1} \cup G_{j_2} \cup \dots \cup G_{j_m}$. $f(V(\alpha_i, x'))$ contains W_{j+1} and hence $G_{j+1} = \text{cl}V(\alpha_i, x') \cap f^{-1}(W_{j+1})$ will satisfy the required properties.

Denote $M_\epsilon = \bigcup G_j$. Then $M_\epsilon \in \mathcal{M}_\epsilon$ and $M_\epsilon \subset V(2\epsilon, M)$. By adding a finite number of points to M_ϵ we may assume that $d_H(M_\epsilon, M) < 2\epsilon$, where d_H is the Hausdorff metric in 2^X . Thus \mathcal{M}_ϵ is dense in \mathcal{M} and the theorem follows. \square

Proof of Theorem 1.3. Let $\epsilon > 0$ and let \mathcal{V} be an open finite cover of X with $\text{mesh} < \epsilon$. Let V_ϵ be the open ϵ -neighborhood of $V \in \mathcal{V}$. Denote

$$M_V = (X \setminus f^{-1}(f(V_\epsilon))) \cup \text{cl}V_\epsilon$$

and define

$$\sigma = \inf\{d(x, y) : x \in X \setminus f^{-1}(f(V_\epsilon)), y \in \text{cl}f^{-1}(f(V))\}.$$

$f(M_V) = Y$, $\sigma > 0$ and by Theorem 1.2 there is a 0-dimensional closed set $Z_V \subset X$ such that $d_H(Z_V, M_V) < \min\{\sigma, \epsilon\}$ and $f(Z_V) = Y$. Then for every $x \in V$ there exists $x' \in f^{-1}(f(x)) \cap Z_V$ such that $d(x, x') < 3\epsilon$. Thus $Z_\epsilon = \bigcup\{Z_V : V \in \mathcal{V}\}$ is closed and 0-dimensional, and for every $x \in X$ there exists $x' \in f^{-1}(f(x)) \cap Z_\epsilon$ such that $d(x, x') < 3\epsilon$.

Denote $Z = \bigcup Z_{1/n}$. Z is 0-dimensional and F_σ . Hence there exists a map $h : X \rightarrow I = [0, 1]$ such that $h|_Z$ is 1-to-1. Since for each $y \in Y$, $Z \cap f^{-1}(y)$ is dense in $f^{-1}(y)$ we have that $C_y = h(f^{-1}(y))$ is a perfect subset of I .

Define a decomposition q_y of $f^{-1}(y)$ as follows. Two points $x_1, x_2 \in f^{-1}(y)$ are in the same element of the decomposition if either $h(x_1) = h(x_2)$ or $h(x_1)$ and $h(x_2)$ are adjacent end points of C_y , that is, for $a = \min\{h(x_1), h(x_2)\}$ and $b = \max\{h(x_1), h(x_2)\}$, $(a, b) \cap C_y = \emptyset$.

The decompositions $q_y, y \in Y$, form the corresponding decomposition of X which we denote by q . q is an upper semi-continuous decomposition and we will regard q as the quotient map $q : X \rightarrow Q$ to the quotient space Q . Clearly for each $y \in Y$, $q(f^{-1}(y))$ is an arc.

A map p of a compactum P is said to be a Bing map if every continuum contained in a fiber of p is hereditarily indecomposable. It is proved in [5] that every compactum admits a Bing map to the unit interval I (moreover, almost all maps in the function space $C(P, I)$ are Bing maps).

Let $p : Q \rightarrow I$ be a Bing map. Since for $y \in Y$, $q(f^{-1}(y))$ is an arc, $p(q(f^{-1}(y)))$ is not a singleton. Thus $p(q(f^{-1}(y))) = [a, b]$ with $a < b$. Let $L_y : [a, b] \rightarrow I$ be the linear transformation sending a and b to 0 and 1 respectively. Define $g : X \rightarrow I$ by $g(x) = L_y(p(q(x)))$ for $x \in f^{-1}(y)$. Then g has the required property and the theorem is proved. \square

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