IWASAWA INVARIANTS AND CLASS NUMBERS
OF QUADRATIC FIELDS FOR THE PRIME 3

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Abstract. Let \( d \) be a square-free integer with \( d \equiv 1 \pmod{3} \) and \( d > 0 \). Put \( k^+ = \mathbb{Q}(\sqrt{d}) \) and \( k^- = \mathbb{Q}(\sqrt{-3d}) \). For the cyclotomic \( \mathbb{Z}_3 \)-extension \( k^+_\infty \) of \( k^+ \), we denote by \( k^+_n \) the \( n \)-th layer of \( k^+_\infty \) over \( k^+ \). We prove that the 3-Sylow subgroup of the ideal class group of \( k^+_n \) is trivial for all integers \( n \geq 0 \) if and only if the class number of \( k^- \) is not divisible by the prime 3. This enables us to show that there exist infinitely many real quadratic fields in which 3 splits and whose Iwasawa \( \lambda_3 \)-invariant vanishes.

1. Introduction

For a number field \( k \) and a prime number \( p \), we denote by \( \lambda_p(k) \) the Iwasawa \( \lambda \)-invariant associated to the cyclotomic \( \mathbb{Z}_p \)-extension of \( k \), \( \mathbb{Z}_p \) being the ring of \( p \)-adic integers. When R. Greenberg visited Tokyo in May 1994, he asked the question whether there exist infinitely many real quadratic fields with \( p \) split and \( \lambda_p(k) \) vanishing, for a given odd prime number \( p \). This is still an open problem.

Concerning the existence of infinite families of real abelian fields \( k \) with \( \lambda_p(k) = 0 \), some results are well-known. For a number field \( k \), we denote by \( h(k) \) the class number of \( k \). First, genus theory implies that for a given prime number \( p \), there are infinitely many real cyclic fields \( k \) of degree \( p \) such that \( p \) does not divide \( h(k) \) and that \( p \) does not split in \( k \). Hence it follows from a theorem of Iwasawa [Iw1] that there exist infinitely many real cyclic fields \( k \) of degree \( p \) with \( \lambda_p(k) = 0 \). Further, Nakagawa and Horie gave in [NH] a positive lower bound on the density of real quadratic fields \( k \) such that the prime 3 does not divide \( h(k) \) and that 3 does not split in \( k \). Hence, in a similar way, there exist infinitely many real quadratic fields \( k \) with \( \lambda_3(k) = 0 \). Recently, Kraft [Kr] also showed the existence of such an infinite family of real quadratic fields by using a result of Jochnowitz.

Results also exist for the case where the given prime may be split. In [Iw3, page 185], for a given odd prime number \( p \), Iwasawa gave a certain infinite family of real cyclic fields \( k \) of degree \( p \) with \( \lambda_p(k) = 0 \) such that either \( p \) divides \( h(k) \) or \( p \) splits in \( k \). Also, for the prime 2, Ozaki and the author gave in [OT] some

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infinite families of real quadratic fields \( k \) with \( \lambda_2(k) = 0 \) such that either 2 divides \( h(k) \) or 2 splits in \( k \). Recently, Yamamoto \[Ya\] and Fukuda \[Fu\] gave results on the vanishing of \( \lambda_p(k) \) for certain \( (p, p) \)-extensions and certain cyclic extensions of degree \( p \), respectively, for an odd prime number \( p \). These also enable us to give some infinite families of real abelian fields of type \( (p, p) \), and of real cyclic fields of degree \( p \), with \( \lambda_p(k) = 0 \). However, for \( p \) a fixed prime, we do not know whether there exist infinitely many real abelian fields \( k \) with degree prime to \( p \) such that \( p \) splits completely in \( k \) and \( \lambda_p(k) = 0 \).

Let \( \mathbb{Q} \) denote the field of rational numbers. In this paper, we first show the following theorem in Section 2. For the proof, we use a formula (Lemma 2 in Section 2) for the order of the \( p \)-Sylow subgroup of the ambiguous ideal class group in the cyclotomic \( \mathbb{Z}_p \)-extension of a real quadratic field in terms of a special value of a \( p \)-adic \( L \)-function (see also Remark 2 following the statement of Theorem 1).

**Theorem 1.** Let \( d \) be a square-free integer with \( d \equiv 1 \) (mod 3) and \( d > 0 \). Put \( k^+ = \mathbb{Q}(\sqrt{d}) \) and \( k^- = \mathbb{Q}(\sqrt{-3d}) \). For the cyclotomic \( \mathbb{Z}_3 \)-extension \( k^+ \), denote by \( k^+_n \) the \( n \)-th layer in \( k^+ / k^+ \) and by \( A^+_n \) the 3-Sylow subgroup of the ideal class group of \( k^+ \). Then \( A^+_n \) is trivial for all integers \( n \geq 0 \) if and only if the class number \( h(k^-) \) of \( k^- \) is not divisible by 3.

**Remark 1.** Let \( d \) be a square-free integer with \( d \neq 1 \) (mod 3) and \( d > 0 \). Put \( k^+ = \mathbb{Q}(\sqrt{d}) \) and \( k^- = \mathbb{Q}(\sqrt{-3d}) \). In this case, Theorem 1 does not always hold. In fact, since 3 does not split in \( k^+ \), a theorem of Iwasawa \[Iw1\] says that if \( h(k^+) \) is not divisible by 3, then \( A^+_n \) is trivial for all integers \( n \geq 0 \), even in the case where \( h(k^-) \) is a multiple of 3. For example, \( h(\mathbb{Q}(\sqrt{29})) = 1 \) but \( h(\mathbb{Q}(\sqrt{-3 \cdot 29})) = 6 \) for \( d = 29 \equiv 2 \) (mod 3), \( h(\mathbb{Q}(\sqrt{69})) = 1 \) but \( h(\mathbb{Q}(\sqrt{-3 \cdot 69})) = h(\mathbb{Q}(\sqrt{-23})) = 3 \) for \( d = 69 \equiv 6 \) (mod 9), and \( h(\mathbb{Q}(\sqrt{93})) = 1 \) but \( h(\mathbb{Q}(\sqrt{-3 \cdot 93})) = h(\mathbb{Q}(\sqrt{-31})) = 3 \) for \( d = 93 \equiv 3 \) (mod 9).

**Remark 2.** Theorem 1 can be also proved by a purely algebraic argument. This goes roughly as follows. Let \( k^+ \), \( k^- \) and \( k^+ \) be as in Theorem 1 and let \( k^- \) be the cyclotomic \( \mathbb{Z}_3 \)-extension of \( k^- \). Denote by \( M^+ \) the maximal abelian pro-3-extension of \( k^- \) unramified outside of the primes above 3, and by \( L^- \) the maximal unramified abelian pro-3-extension of \( k^- \). Then, by using the Kummer pairing \[Iw2, \S 1\] and \[Iw2, \S 2, \S 3\] to \( k^+ \), we see that \( \text{Gal}(M^+ / k^+) \) is pseudo-isomorphic to \( \text{Gal}(L^- / k^-) \), where \( X \) is \( X \) with a twisted action of \( Z_3[[\text{Gal}(k^+/k^+)]] \) for \( Z_3[[\text{Gal}(k^+/k^+)]] \)-module \( X \) (see \[Iw2, \S 1\] page 278). Since \( 3 \) splits in \( k^+ \), we find that \( M^+ \) is an unramified extension over \( k^+ \), where \( M^+ \) is the maximal abelian pro-3-extension of \( k^+ \) unramified outside of the primes above 3. Hence it follows from a certain relation between \( M^+ \) and \( M^+ \) (see \[Iw2, \S 2, \S 3\]) and Nakayama’s lemma that \( \text{Gal}(L^- / k^-) \) is trivial if and only if \( \text{Gal}(M^+/k^+) \) is trivial, where \( L^+ \) is the maximal unramified abelian pro-3-extension of \( k^+ \). Since \( \text{Gal}(L^- / k^-) \) and \( \text{Gal}(M^+/k^+) \) have no non-trivial finite \( Z_3[[\text{Gal}(k^+/k^+)]] \)- and \( Z_3[[\text{Gal}(k^+/k^+)]] \)-submodules, respectively, we see by the above pseudo-isomorphism that \( \text{Gal}(L^- / k^-) \) is trivial if and only if \( \text{Gal}(L^- / k^-) \) is trivial. Therefore, the assertion of Theorem 1 follows immediately.

Further, we mention here that Theorem 1 is also regarded as a special case of a recent result of Ozaki \[Oz\] Theorem 1]. Our proof given in Section 2 uses different methods.

**Remark 3.** Let \( d \) be a square-free integer with \( d > 0 \). Put \( k^+ = \mathbb{Q}(\sqrt{d}) \) and \( k^- = \mathbb{Q}(\sqrt{-3d}) \). Then, by the pseudo-isomorphism mentioned in Remark 2 we
have \( \lambda_3(k^+) \leq \lambda_3(k^-) \). In particular, if \( \lambda_3(k^-) = 0 \), then \( \lambda_3(k^+) = 0 \). However, in the case where \( d \equiv 6 \pmod{9} \), we always have \( \lambda_3(k^-) \geq 1 \) because the 3-adic \( L \)-function \( L_3(s, \chi) \) has a trivial zero. Here \( \chi \) is the non-trivial Dirichlet character associated to \( k^+ \). In such a case, Kraft [KG] showed that if \( \lambda_3(k^-) = 1 \), then \( h(k^+) \) is not divisible by 3, in particular \( \lambda_3(k^+) = 0 \) (see also [IS] (5.9) in Section 2).

For a finite set \( G \), we denote its cardinality by \#\( G \). Theorem \([\text{I}]\) and results of Nakagawa and Horie in [NH] enable us to give the following lower bound on the density of \( k^+ \) such that \( p \) splits in \( k^+ \) and \( A_n^3 \) is trivial for all integers \( n \geq 0 \). This is proved in Section 3.

**Theorem 2.** Let \( k \) be a real quadratic field with discriminant \( \Delta_k > 0 \). Denote by \( k_n \) the \( n \)-th layer in the cyclotomic \( \mathbb{Z}_3 \)-extension of \( k \) and by \( A_n \) the 3-Sylow subgroup of the ideal class group of \( k_n \). Then

\[
\liminf_{x \to \infty} \frac{\#\{ k \mid \Delta_k < x, \Delta_k \equiv 1 \pmod{3}, A_n = \{1\} \text{ for all } n \geq 0 \}}{\#\{ k \mid \Delta_k < x \}} \geq \frac{3}{16}.
\]

By Theorem \([\text{I}]\) we can give an affirmative answer for \( p = 3 \) to the question mentioned in the beginning of this section. Namely, we obtain the following as a corollary.

**Corollary 1.** There exist infinitely many real quadratic fields \( k \) with the prime 3 splitting in \( k \) and with \( \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0 \), where \( \mu_3(k) \) and \( \nu_3(k) \) denote the Iwasawa \( \mu \)- and \( \nu \)-invariants, respectively, associated to the cyclotomic \( \mathbb{Z}_3 \)-extension of \( k \).

We also give in Section 3 an additional result on density in this direction (see Theorem \([\text{I}]\)).

Finally we mention some topics related to the problem treated here. It is natural to consider the problem whether there exist infinitely many prime numbers \( p \) splitting completely in \( k \) such that \( \lambda_p(k) = 0 \) for a given real abelian field \( k (\neq \mathbb{Q}) \), by fixing the number field instead of fixing the prime number. This problem was also stated by Greenberg when he came to Tokyo. Recently, Ichimura [Ich] studied the problem and showed that for a certain real abelian field \( k \), the problem can be solved if the abc conjecture for \( k \) is valid. He also remarked that if \( k \) is cyclic and its degree is a composite number, or if \( k \) is non-cyclic, then the problem can be solved without assuming the abc conjecture.

There is a stronger conjecture on the Iwasawa invariants of totally real fields, known as Greenberg’s conjecture, which states that both of \( \lambda_p(k) \) and \( \mu_p(k) \) always vanish for any totally real field \( k \) and any prime number \( p \) (cf. [Gr], also [Iw2], page 316]). Several authors gave a number of criteria for the validity of Greenberg’s conjecture (for recent work, see the paper [IS] and its references), but the conjecture is at present unsolved except for the simplest case \( k = \mathbb{Q} \). On the other hand, concerning \( \mu_p \)-invariants, it is well-known that \( \mu_p(k) = 0 \) for any abelian field \( k \) (not necessarily totally real) and any prime number \( p \) by the theorem of Ferrero and Washington [FW].

**2. Proof of Theorem 1**

First, we recall some properties of \( p \)-adic \( L \)-functions. Let \( p \) be an odd prime number and \( \mathbb{C}_p \) the completion of the algebraic closure of the field of \( p \)-adic numbers. We denote by \( \omega \) the Teichmüller character of \( p \) and by \( \chi \) a primitive even Dirichlet
character of conductor $f_\chi$. Then the $p$-adic $L$-function $L_p(s, \chi)$ associated to $\chi$ is the unique $p$-adic meromorphic function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ satisfying

$$L_p(1-n, \chi) = -(1-\chi \omega^{-n}(p)p^{n-1}) \frac{B_{n,\chi\omega^{-n}}}{n}$$

for all integers $n \geq 1$, where $B_{n,\chi\omega^{-n}}$ denotes, as usual, the generalized Bernoulli number (cf. [Wa Theorem 5.11]). When $n = 1$, they can be defined by

$$B_{1,\chi\omega^{-1}} = \frac{1}{f_{\chi\omega^{-1}}} \sum_{a=1}^{f_{\chi\omega^{-1}}} \chi\omega^{-1}(a) a,$$

where $f_{\chi\omega^{-1}}$ is the conductor of $\chi\omega^{-1}$. Assume that $\chi$ is non-trivial and that $f_\chi$ is not divisible by $p^2$. Then we have

$$L_p(m, \chi) \equiv L_p(n, \chi) \pmod{p}$$

for any two integers $m$ and $n$, where both numbers can be shown to be $p$-integral (cf. [Wa Corollary 5.13]). Further, if $f_\chi$ is prime to $p$, then $\chi \omega^{-n}(p) = 0$. Hence we have the following.

**Lemma 1.** Let $p$, $\chi$ and $\omega$ be as above. Assume that $\chi$ is a non-trivial character with conductor prime to $p$. Then

$$L_p(1, \chi) \equiv -B_{1,\chi\omega^{-1}} \pmod{p}.$$

On the other hand, letting $k$ be a real quadratic field, we know that there is a certain relationship between a $p$-adic $L$-function associated to $k$ and the $p$-parts of the ambiguous class groups in the cyclotomic $\mathbb{Z}_p$-extension of $k$. Namely, we showed the following fact in [Ta]:

**Lemma 2** (Proposition 1 in [Ta]). Let $k$ be a real quadratic field with non-trivial Dirichlet character $\chi$ and let $p$ be an odd prime number. For the cyclotomic $\mathbb{Z}_p$-extension $k_\infty$ of $k$ with Galois group $\Gamma = \text{Gal}(k_\infty/k)$, denote by $A_n$ the $p$-Sylow subgroup of the ideal class group of the $n$-th layer in $k_\infty/k$, and by $A_n^\Gamma$ the subgroup of $A_n$ consisting of ideal classes which are invariant under the action of $\Gamma$, namely, the $p$-part of the ambiguous class group of the $n$-th layer in $k_\infty/k$. Assume that $p$ splits in $k$. Then

$$\# A_n^\Gamma = p^{v_p(L_p(1, \chi))}$$

for all sufficiently large integers $n$, where $v_p$ denotes the $p$-adic valuation normalized by $v_p(p) = 1$.

Now we prove Theorem [Ta] We assume that $p = 3$. Let $d$ be a square-free integer with $d \equiv 1 \pmod{3}$ and $d > 0$, and put $k^+ = \mathbb{Q}(\sqrt{d})$ and $k^- = \mathbb{Q}(\sqrt{-3d})$, as in Theorem [Ta] Let $\chi$ be the non-trivial (even) Dirichlet character associated to $k^+$. Then $\chi \omega^{-1}$ is the non-trivial (odd) Dirichlet character associated to $k^-$. Since the conductor $f_{\chi \omega^{-1}}$ has at least two prime factors, we have $k^- \neq \mathbb{Q}(\sqrt{-1})$ and $k^- \neq \mathbb{Q}(\sqrt{-3})$. Hence the analytic class number formula (cf. [Wa Theorem 4.17]) says that $h(k^-) = -B_{1,\chi\omega^{-1}}$. Therefore it follows from Lemma [Ta] that

$$L_3(1, \chi) \equiv h(k^-) \pmod{3}.$$  

(2.1)

Let $\Gamma$ be the Galois group of $k_\infty^+$ over $k^+$, and let $(A_n^+)^\Gamma$ be the subgroup of $A_n^+$ consisting of ideal classes which are invariant under the action of $\Gamma$. Since 3 splits
in \( k^+ \), it follows from Lemma \( 2 \) that
\[
\#(A_n^+) = 0 \pmod{3} \quad \text{for sufficiently large } n \iff L_3(1, \chi) = 0 \pmod{3}.
\]
Therefore, combining (2.1) with (2.2), we see that \( h(A^-_n) \) is not divisible by 3 if and only if \( (A_n^+)^T = \{1\} \) for all sufficiently large integers \( n \). The latter assertion is equivalent to the assertion that \( A_n^+ = \{1\} \) for all integers \( n \geq 0 \), because \( (A_n^+)^T = \{1\} \) if and only if \( A_n^+ = \{1\} \) and because the primes lying above the prime 3 are totally ramified in \( k^-_n/k^+ \). This completes the proof of Theorem 1. \( \square \)

3. Proof of Theorem 2

First, we recall some results of Nakagawa and Horie in [NH]. Let \( m \) and \( N \) be two positive integers satisfying the following condition:

(\*) If an odd prime number \( p \) is a common divisor of \( m \) and \( N \), then \( p^2 \) divides \( N \) but not \( m \). Further if \( N \) is even, then (i) 4 divides \( N \) and \( m \equiv 1 \pmod{4} \), or (ii) 16 divides \( N \) and \( m \equiv 8 \) or 12 \pmod{16}.

For any integer \( x \geq 0 \), we denote by \( K^+(x) \) the set of real quadratic fields with discriminant \( \Delta_k < x \) and by \( K^-(x) \) the set of imaginary quadratic fields with the absolute value of discriminant \( |\Delta_k| < x \), and put
\[
K^+(x, m, N) = \{ k \in K^+(x) \mid \Delta_k \equiv m \pmod{N} \},
K^-(x, m, N) = \{ k \in K^-(x) \mid \Delta_k \equiv m \pmod{N} \}.
\]

Moreover, for a quadratic field \( k \), we denote by \( h^*_3(k) \) the number of ideal classes of \( k \) whose cubes are principal. Then Nakagawa and Horie showed the following formulas for \( h^*_3(k) \) in [NH] Theorem 1:
\[
(3.1) \quad \sum_{k \in K^+(x, m, N)} h^*_3(k) \sim \frac{4}{3} \#K^+(x, m, N) \quad (x \to \infty),
(3.2) \quad \sum_{k \in K^-(x, m, N)} h^*_3(k) \sim 2 \#K^-(x, m, N) \quad (x \to \infty),
\]
and the following estimate on \( K^+(x, m, N) \) in [NH] Proposition 2:
\[
(3.3) \quad \#K^+(x, m, N) \sim \#K^-(x, m, N) \sim \frac{3x}{\pi^2 \varphi(N)} \prod_{p|N} \frac{q}{p + 1} \quad (x \to \infty),
\]
where \( X \sim Y \ (x \to \infty) \) means \( \lim_{x \to \infty} \frac{X}{Y} = 1 \), and \( \varphi \) is the Euler function, \( \pi \) is the circular constant, and \( p \) runs over all prime factors of \( N \), and further, \( q = 4 \) or \( p \) according as \( p = 2 \) or not. In particular, since \( K^+(x) = K^+(x, 1, 1) \), we have
\[
(3.4) \quad \#K^+(x) \sim \#K^-(x) \sim \frac{3x}{\pi^2} \quad (x \to \infty)
\]
by (3.3).

Let us put
\[
K^+_n(x, m, N) = \{ k \in K^+(x, m, N) \mid h^*_3(k) = 1 \},
K^-_n(x, m, N) = \{ k \in K^-(x, m, N) \mid h^*_3(k) = 1 \}.
\]
Namely,

\[ K^+(x, m, N) = \{ k \in K^+(x, m, N) \mid h(k) \not\equiv 0 \pmod{3} \}, \]

\[ K^-(x, m, N) = \{ k \in K^-(x, m, N) \mid h(k) \not\equiv 0 \pmod{3} \}. \]

Since

\[ \#K^+_n(x, m, N) + 3(\#K^+_n(x, m, N) - \#K^+_n(x, m, N)) \leq \sum_{k \in K^+(x, m, N)} h^*_3(k), \]

it follows that

\[ \#K^*_n(x, m, N) \geq \frac{3}{2} \#K^+(x, m, N) - \frac{1}{2} \sum_{k \in K^+(x, m, N)} h^*_3(k). \]

Furthermore, concerning the right-hand side of the inequality above, we have

\[
\begin{align*}
\frac{3}{2} \#K^+(x, m, N) - \frac{1}{2} \sum_{k \in K^+(x, m, N)} h^*_3(k) & \sim \frac{5}{6} \#K^+(x, m, N) \quad (x \to \infty), \\
\frac{3}{2} \#K^-(x, m, N) - \frac{1}{2} \sum_{k \in K^-(x, m, N)} h^*_3(k) & \sim \frac{1}{2} \#K^-(x, m, N) \quad (x \to \infty),
\end{align*}
\]

by \(3.1\) and \(3.2\).

Now, we prove Theorem 2. For a real quadratic field \(k\), we denote by \(k_n\) the \(n\)-th layer in the cyclotomic \(\mathbb{Z}_3\)-extension of \(k\), and let \(A_n\) be the 3-Sylow subgroup of the ideal class group of \(k_n\) as in Theorem 2. First, Theorem 1 tells us that we have a one-to-one correspondence between certain real and imaginary quadratic fields:

\[
\{ k \in K^+(x, 1, 3) \mid \#A_n = 1 \text{ for all } n \geq 0 \} \longleftrightarrow K^*_n(3x, 6, 9) = \{ \mathbb{Q}(\sqrt{-3}) \}
\]

for any integer \(x \geq 0\). Since \((m, N) = (6, 9)\) satisfies condition \((\ast)\), it follows from \(3.3\), \(3.4\) and \(3.5\) that

\[
\liminf_{x \to \infty} \frac{\#K^*_n(3x, 6, 9)}{\#K^-(x)} \geq \liminf_{x \to \infty} \frac{\#K^-(3x, 6, 9)}{2 \#K^-(x)} = \frac{3}{16}.
\]

Therefore, by \(3.4\) and \(3.7\), this implies that

\[
\liminf_{x \to \infty} \frac{\#\{ k \in K^+(x, 1, 3) \mid \#A_n = 1 \text{ for all } n \geq 0 \}}{\#K^+(x)} \geq \frac{3}{16}.
\]

This completes the proof of Theorem 2.

Finally, we give the following theorem, which is a slight improvement of a result of Nakagawa and Horie [NH, Theorem 3].

**Theorem 3.** Let \(k\) be a real quadratic field with discriminant \(\Delta_k > 0\). Then

\[
\liminf_{x \to \infty} \frac{\#\{ k \mid \Delta_k < x, \lambda_3(k) = 0 \}}{\#\{ k \mid \Delta_k < x \}} \geq \frac{17}{24}.
\]
Proof. A theorem of Iwasawa [Iw1] says that if $h(k)$ is not divisible by a prime $p$ not split in $k$, then the $p$-Sylow subgroup of the ideal class group in the cyclotomic $\mathbb{Z}_p$-extension of $k$ is trivial, in particular $\lambda_p(k) = 0$. Using this, Nakagawa and Horie [NH, Theorem 3] showed the following:

$$\liminf_{x \to \infty} \frac{\# \left\{ k \mid \Delta_k < x, \Delta_k \not\equiv 1 \pmod{3}, \lambda_3(k) = 0 \right\}}{\# \left\{ k \mid \Delta_k < x \right\}} \geq \frac{25}{48}.$$  

In fact, by (3.3), (3.4) and (3.5), we have

$$\liminf_{x \to \infty} \frac{\# \left\{ k \mid \Delta_k < x, h(k) \equiv 2 \pmod{3}, h(k) \not\equiv 0 \pmod{3} \right\}}{\# \left\{ k \mid \Delta_k < x \right\}} \geq \frac{5}{16},$$

$$\liminf_{x \to \infty} \frac{\# \left\{ k \mid \Delta_k < x, h(k) \equiv 3 \pmod{9}, h(k) \not\equiv 0 \pmod{3} \right\}}{\# \left\{ k \mid \Delta_k < x \right\}} \geq \frac{5}{48},$$

$$\liminf_{x \to \infty} \frac{\# \left\{ k \mid \Delta_k < x, h(k) \equiv 6 \pmod{9}, h(k) \not\equiv 0 \pmod{3} \right\}}{\# \left\{ k \mid \Delta_k < x \right\}} \geq \frac{5}{48}.$$  

Therefore the desired result follows immediately from this and Theorem 2. This completes the proof of Theorem 3.

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