PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 5, Pages 1523–1525
S 0002-9939(99)05205-3
Article electronically published on August 3, 1999

MISTAKE IN HIRSCH'S PROOF OF THE BROUWER FIXED POINT THEOREM

KAPIL D. JOSHI

(Communicated by Ralph Cohen)

Abstract. A mistake in Hirsch's proof of the Brouwer Fixed Point Theorem, based on the Simplicial Approximation Theorem, is pointed out along with some elaboration of its nature and possible ways to salvage Hirsch's proof.

1. Introduction

In [5], Hirsch claims to give an elementary proof of the Brouwer Fixed Point Theorem. This proof appears in the 'Shorter Notes' section of the Proceedings of the American Mathematical Society, whose declared purpose is to publish very short papers of an 'unusually elegant and polished character'. This proof is described as 'very short, very elegant' in a review by Dugundji [2]. Bing [1] also calls this proof 'beautiful, elementary' in his famous expository article, 'The Elusive Fixed Point Property'.

There is, however, a serious mistake in Hirsch's argument. The purpose of this note is to point it out, to elaborate its nature and to comment on possible ways to circumvent it.

2. The mistake

Instead of proving the Brouwer Fixed Point Theorem directly, Hirsch tries to prove an equivalent statement, viz., that there is no retraction of a closed n-simplex E onto its boundary ∂E. (The equivalence of these two statements is quite well-known; see e.g. Joshi [6] or Spanier [10].) Appealing to the Simplicial Approximation Theorem ([3], p.64), Hirsch claims that if at all there is a retraction of E onto ∂E, then there is a simplicial retraction $f : E \longrightarrow \partial E$. He then considers $f^{-1}(a)$ where a is the barycenter of some $(n-1)$-simplex A contained in ∂E. He shows that $f^{-1}(a)$ is a compact, one-dimensional manifold whose boundary is contained in ∂E and gets a contradiction because the component of $f^{-1}(a)$ containing a has both its end-points going to a under f.

Hirsch's argument breaks down right at the start. Although the Simplicial Approximation Theorem gives a simplicial approximation to every continuous map (between finite polyhedra), nowhere does it say that a simplicial approximation to a retraction is, or can be chosen to be, a simplicial retraction.

Received by the editors November 4, 1997 and, in revised form, March 24, 1998 and June 19, 1998.
1991 *Mathematics Subject Classification.* Primary 55M20, 54H25.

3. Nature of the mistake

The catch is that ∂E plays a double role here, one as the codomain of the map f and the other as a subcomplex of the domain complex E. When the Simplicial Approximation Theorem is applied, the latter gets subdivided but not the former. We could, of course, subdivide the codomain ∂E, too. But then the approximation will no longer be simplicial (although it will continue to be piecewise linear).

It is interesting to note that this difficulty does not arise for approximation by smooth maps. As Hirsch himself remarks, if E is a compact differentiable manifold, then one may use the Differential Approximation Theorem in place of the Simplicial Approximation Theorem. The proof can then be completed by taking a to be a regular value (which exists by Sard's theorem, see e.g. Guillemin and Pollack [4]). What is needed here is a relative version of the differentiable approximation theorem. If $f : X \longrightarrow Y$ is continuous, where X, Y are differentiable manifolds, then there exists a differentiable map $g : X \longrightarrow Y$ which approximates f. But more is true. If Z is a submanifold of X such that f/Z is differentiable, then g may be so chosen that $g/Z = f/Z$. In particular, if f is a continuous retraction, then g can be taken to be a differentiable retraction. It is tempting to think that the Simplicial Approximation Theorem also has a similar relative version. Lefschetz [8], too, remarks that the relative case of the Simplicial Approximation Theorem is an 'immediate generalization' of the absolute case! However, it was only much later that Zeeman [11] proved a relative version of the Simplicial Approximation Theorem and its proof is far from an immediate generalization of the absolute case. Moreover, this version is still not so well-known as the classical absolute version (possibly because, as Zeeman points out, the absolute version is generally sufficient in applications).

It may also be noted that there is an easier way than Hirsch's to show that there cannot exist a simplicial retraction from E onto ∂E. For, if there were, then it would induce a retraction of the *mod* 2 simplicial chain complex of E onto that of ∂E. So any chain in ∂E which is a boundary in E would also be a boundary in ∂E. But this is not the case for the chain consisting of the totality of all $(n-1)$-simplexes of ∂E, a contradiction. Note that this argument needs nothing beyond the definition and functoriality of the *mod* 2 simplicial chain complexes. In particular no homology theory is needed. In fact, the very triviality of this argument suggests that something is wrong in Hirsch's proof, for one cannot expect the Brouwer Fixed Point Theorem to follow so instantaneously from the Simplicial Approximation Theorem.

4. Salvaging Hirsch's proof

If instead of the classical Simplicial Approximation Theorem, one uses Zeeman's [11] relative version, then the existence of a continuous retraction of E onto ∂E would imply the existence of a simplicial retraction from E onto ∂E. One can then apply Hirsch's argument or the alternate argument given in §3 above. Unfortunately, Zeeman's result needs rather intricate constructions and a proof based on it can hardly be called 'elementary' except possibly in a special sense (as, for example, in 'elementary number theory').

An alternate approach is to start with a supposed continuous retraction $f : E \longrightarrow \partial E$ and to modify the proof of the classical Simplicial Approximation Theorem so as to get a piecewise linear retraction $g : E \longrightarrow \partial E$. It can then be shown that by

further subdividing both the domain and the codomain, the map g can be made a simplicial retraction (see Chapter 2 of Rourke and Sanderson [9] for details). Once this is done, the desired contradiction comes instantaneously as shown above in §3. Unfortunately, this approach, too, is still somewhat technical.

A third, and probably the simplest, way to salvage Hirsch's argument is to get a piecewise linear retraction $g : E \longrightarrow \partial E$ exactly as in the second approach above. Thereafter, to get the desired contradiction using Hirsch's argument, one needs a regular value of g. In the smooth case this is done by theorems like Sard's. In the case of a PL map the equivalent work has to be done using results from linear algebra. Considering that another well-known, elementary combinatorial proof of the Brouwer Fixed Point Theorem, using Sperner's lemma (see e.g. Joshi [7] or Spanier [10]), is already available, it is probably not worthwhile to carry this out elaborately. It would, however, be a good exercise for courses in combinatorial topology.

It would, of course, be interesting to know if there is some way to salvage Hirsch's proof without sacrificing its brevity, elegance *and* elementary character.

I am thankful to the referees for pointing out possible ways (especially the second and the third) to salvage Hirsch's proof.

References

[1] R. H. Bing, *The elusive fixed point property* , Amer. Math. Monthly, 76(1969) 119-132. MR **38:**5201

[2] J. Dugundji *in* N. Steenrod, *Reviews of Papers in Algebraic and Differential Topology, Topological Groups and Homological Algebra,* p. 565, American Mathematical Society, Providence, R. I., 1968.

[3] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, N. J., 1952. MR **14:**398b

[4] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, N.J., 1974. MR **50:**1276

[5] M. W. Hirsch, *A proof of the non-retractability of a cell onto its boundary*, Proc. Amer. Math. Soc., 14(1963), 364-365.

[6] K. D. Joshi, *Introduction to General Topology*, New Age International, New Delhi, 1983. MR **84f:**54001

[7] K. D. Joshi, *Applied Discrete Structures*, New Age International, New Delhi, 1997

[8] S. Lefschetz, *Topology*, American Mathematical Society Colloquium Publication, vol. XII, New York, 1930

[9] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise Linear Topology*, Springer-Verlag, New York 1982. MR **83g:**57009

[10] E. H. Spanier, *Algebraic Topology*, McGraw Hill, New York, 1966. MR **35:**1007

[11] E. C. Zeeman, *Relative simplicial approximation*, Proc. Camb. Phil. Soc., 60(1964) 39-43. MR **28:**1626

Department of Mathematics, Indian Institute of Technology, Powai, Mumbai, 400 076 India

E-mail address: kdjoshi@math.iitb.ernet.in