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THE NUMBER OF KNOT GROUP REPRESENTATIONS IS NOT A VASSILIEV INVARIANT

MICHAEL EISERMANN

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ABSTRACT. For a finite group G and a knot K in the 3-sphere, let $F_G(K)$ be the number of representations of the knot group into G. In answer to a question of D. Altschuler we show that F_G is either constant or not of finite type. Moreover, F_G is constant if and only if G is nilpotent.

We prove the following, more general boundedness theorem: If a knot invariant F is bounded by some function of the braid index, the genus, or the unknotting number, then F is either constant or not of finite type.

INTRODUCTION

For a knot K in the 3-sphere $\mathbf{S}^3$, we denote by $\pi(K)$ the fundamental group of the knot complement $\mathbf{S}^3 \setminus K$. Since $\pi(K)$ itself is very difficult to deal with, we may look at simpler invariants, for example, the set $\mathrm{Hom}(\pi(K), G)$ of representations in some finite group G or the numerical invariant $F_G(K) = |\mathrm{Hom}(\pi(K), G)|$.

In recent years, invariants of finite type, also called Vassiliev invariants, have attracted much attention (cf. [2]). D. Altschuler [1] has shown that F_G is not of finite type for certain groups G. He raised the question whether, for an arbitrary finite group G, the invariant F_G is either constant or not of finite type. We answer this by proving the following theorems:

Theorem 1. *For any finite group G, the knot invariant F_G is either constant or not of finite type.*

Theorem 2. *The invariant F_G is constant if and only if the group G is nilpotent.*

For example, every group G of prime power order is nilpotent and, consequently, F_G has constant value $|G|$. On the other hand, if G contains a non-abelian simple group or a dihedral group of order $2p$, with p being odd, then the invariant F_G is not of finite type. In particular, the number of p-colorings defined by R.H. Fox is not an invariant of finite type, because p-colorings correspond to dihedral representations.

We prove Theorems 1 and 2 in Section 2. Section 1 is devoted to a more general boundedness result. Let $\mathcal{K}$ be the set of isotopy classes of knots in $\mathbf{S}^3$.

Theorem 3. *Let ν be either the braid index, the genus, or the unknotting number. If a knot invariant $F : \mathcal{K} \to \mathbb{C}$ satisfies $|F(K)| \leq \phi(\nu(K))$ for all knots K and some function $\phi : \mathbb{N} \to \mathbb{N}$, then F is either constant or not of finite type.*

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For example, braid index, genus and unknotting number are themselves not of finite type. Nor is the signature, because it is bounded by twice the unknotting number. The theorem also holds for the bridge number, because every invariant bounded by some function of the bridge number is also bounded by some function of the braid index.

The situation is completely different for the crossing number: D. Bar-Natan proved in [3] that every knot invariant of type m is bounded by some polynomial of degree m in the crossing number.

Theorem 3 does not fully generalize to links. As an example, consider the linking number lk: this is a Vassiliev invariant of type 1 satisfying the inequality $|lk| \leq u$, where u is the unknotting number. If we restrict ourselves to the braid index or the genus, however, the theorem does extend to links (cf. Corollary 10).

1. Boundedness arguments

In this section we prove Theorem 3, using twist sequences as introduced by J. Dean [6] and R. Trapp [12]. There are two types of twist sequences, according to the orientation of the strands involved: A *vertical twist sequence* is a family of knots K_z, indexed by $z \in \mathbb{Z}$, that looks locally like Figure 1 and is identical outside that region. A *horizontal twist sequence* is depicted in Figure 2.

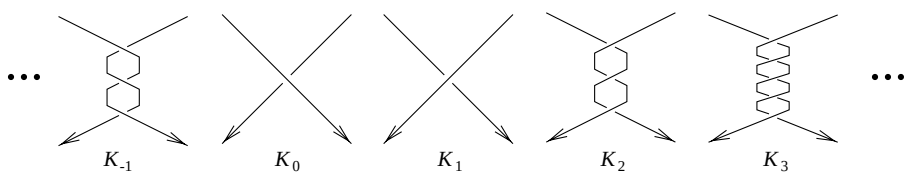

Figure 1. Local picture of a vertical twist sequence

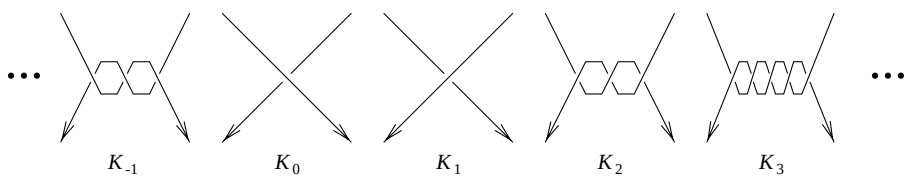

Figure 2. Local picture of a horizontal twist sequence

The only property of Vassiliev invariants needed in this article is the following:

Lemma 4 ([6, 12])**.** *Let $\{ K_z \mid z \in \mathbb{Z} \}$ be a twist sequence. If $F : \mathcal{K} \to \mathbb{C}$ is a Vassiliev invariant of type $\leq m$, then $F(K_z)$ is a polynomial in z of degree $\leq m$.* □

Corollary 5. *If a Vassiliev invariant F is bounded on every vertical (resp. horizontal) twist sequence, then F is constant.*

Proof. Given a knot K, we represent it as a diagram. Around a crossing p we construct a vertical (resp. horizontal) twist sequence $\{ K_z \mid z \in \mathbb{Z} \}$. Since the map $z \mapsto F(K_z)$ is a polynomial and bounded, it must be constant. In particular we have $F(K_0) = F(K_1)$, which means that we can switch the crossing p without changing the value of F. Since we may always switch crossings to connect K to the unknot, this proves that F is constant. □

Proof of Theorem 3. Let ν be the braid index (or the genus, or the unknotting number, respectively). We assume that $F : \mathcal{K} \to \mathbb{C}$ is a Vassiliev invariant which satisfies the inequality $|F(K)| \leq \phi(\nu(K))$ for all knots K.

We will prove in the three lemmas following below, that ν is bounded on any vertical (resp. horizontal) twist sequence. This implies that F is bounded on any vertical (resp. horizontal) twist sequence. By Corollary 5, F is constant. □

Lemma 6. *The braid index is bounded on any vertical twist sequence.*

Proof. Given a vertical twist sequence K_z, we can represent it as a sequence of diagrams as in Figure 1. By a slight generalization of Alexander's Theorem (cf. [4, 10]), we can put these diagrams in braid form without moving the part where the twisting takes place. In this way we find a braid β on n strands such that each knot K_z is represented by the braid $\beta\sigma_i^{2z}$. (Here σ_i is the standard generator of the braid group B_n intertwining strands i and $i+1$ by a half twist.) This yields an upper bound for the braid index, $s(K_z) \leq n$ for all z. □

Alternative proof. For a vertical twist sequence of diagrams as in Figure 1, the number of Seifert circles is constant. By a theorem of S. Yamada [13], this number is an upper bound for the braid index. □

Lemma 7. *The genus is bounded on any horizontal twist sequence.*

Proof. Given a horizontal twist sequence K_z, we can represent it as a sequence of diagrams as in Figure 2. To these we apply Seifert's algorithm (cf. [5]) to construct a Seifert surface from each of the diagrams. All these surfaces have the same Euler characteristic and hence the same genus g_0. This implies an upper bound $g(K_z) \leq g_0$ for all z. □

Lemma 8. *The unknotting number is bounded on any horizontal twist sequence of knots.*

Proof. We will show that any horizontal twist sequence K_z can be uniformly unknotted in the following way: We start with a diagram D_0 for the knot K_0 such that the sequence K_z arises by horizontally twisting around the crossing p as in Figure 2. Travelling along the diagram D_0, starting and ending at the upper strand of the crossing p, we call a crossing *ascending* if the first visit is on the lower strand and the second visit on the upper one. We denote by A the set of ascending crossings of D_0, and by D_z^A the diagram D_z with all crossings of A switched.

We claim that for each z, the diagram D_z^A represents the trivial knot. Thus we conclude $u(K_z) \leq |A|$ for all z.

A proof that the diagrams D_z^A are trivial can be supplied as follows: Any knot diagram D can be parametrized by an immersion $f : \mathbf{S}^1 \to D \subset \mathbb{R}^2$. A *height function* corresponding to the parametrized diagram (D, f) is a continuous map $h : \mathbf{S}^1 \to \mathbb{R}$ such that, at each crossing, the overcrossing strand has a greater height than the undercrossing strand. This is the same as saying that the map $(f, h) : \mathbf{S}^1 \to \mathbb{R}^2 \times \mathbb{R}$ is a parametrized knot which projects to the diagram D.

For any diagram D_z^A as above, one can construct a height function having only one maximum, which means that the resulting knot is trivial. Such a height function is given in Figure 3 for the case $z \geq 1$. For some points on the diagram, their height is indicated. Between these points, let the height function be strictly decreasing while travelling from P to Q, and strictly increasing from Q to P. By definition of D_z^A, this is indeed a height function. □

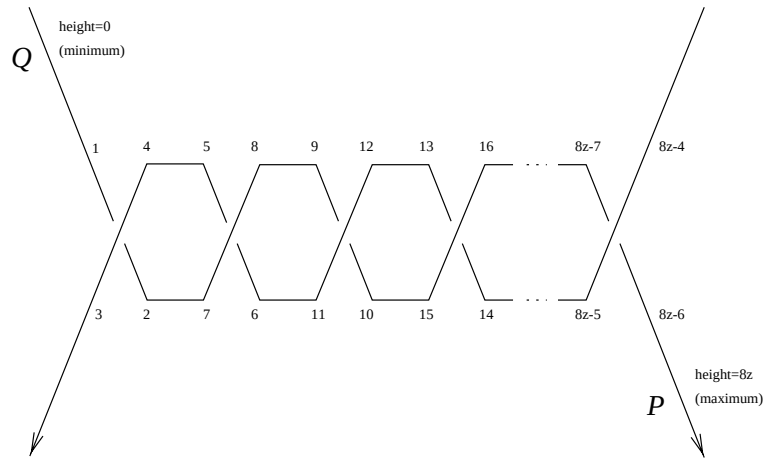

FIGURE 3. A height function for the diagram D_z^A

Remark 9. The choice of twist sequence in the preceding lemmas is not arbitrary: The braid index is bounded on any vertical twist sequence, but in the horizontal case we only have $s(K_z) \leq s_0 + |z|$. Twist knots (i.e. twisted Whitehead doubles of the unknot) show that the linear bound cannot be improved.

The genus and the unknotting number, however, are bounded on any horizontal twist sequence, but in the vertical case we only have $g(K_z) \leq g_0 + |z|$ and $u(K_z) \leq u_0 + |z|$. Linear growth occurs, for example, for the $(2, n)$-torus knots.

For the boundedness of the unknotting number it is essential that we are dealing with a *knot*. Lemma 8 is false for links: Whenever two different components are twisted, their linking number satisfies $lk(L_z) = lk(L_0) + z$. The inequality $u \geq |lk|$ implies that the unknotting number is unbounded.

It is, however, very easy to prove a restricted version of Theorem 3 for links. Let $\mathcal{L}_\mu$ be the set of isotopy classes of links with μ components. Clearly, Lemmas 6 and 7 also hold for the braid index and the genus of links. Thus we have:

Corollary 10. *Let $\nu : \mathcal{L}_\mu \to \mathbb{N}$ be either the braid index or the genus. If a link invariant $F : \mathcal{L}_\mu \to \mathbb{C}$ satisfies $|F(L)| \leq \phi(\nu(L))$ for all links L and some function $\phi : \mathbb{N} \to \mathbb{N}$, then F is either constant or not of finite type.* □

2. Application to knot group representations

In this section we apply the boundedness result of Theorem 3 to the number of knot group representations. Theorem 1 is an immediate consequence whereas the proof of Theorem 2 requires a little bit of group theory.

Proof of Theorem 1. We want to show that, for any finite group G, the knot invariant $F_G(K) = |\operatorname{Hom}(\pi(K), G)|$ is either constant or not of finite type.

Let s be the braid index of the knot K. The Wirtinger presentation, obtained from a closed s-braid representing K, shows that the knot group $\pi(K)$ can be presented with s generators. (For an alternative argument using braid technique see [4], Theorem 2.2.) This implies the inequality $F_G(K) \leq |G|^{s(K)}$. By Theorem 3 we conclude that F_G is either constant or not of finite type. □

Remark 11. The same argument works if only a subset of representations or representations up to some equivalence are counted — in all these cases the boundedness theorem still applies.

2.1. **Homomorphic images of knot groups.** We will characterize the groups G for which F_G is constant. This leads to the question which groups appear as homomorphic images of knot groups. This question was raised by L. P. Neuwirth [11] and first answered by F. Gonzalez-Acuña [7]:

Theorem 12 ([7, 9])**.** *A finite group G is a homomorphic image of some knot group if and only if it is generated by the conjugates of some element $x \in G$.* □

We denote by x^G the orbit of x under conjugation of the group G. The condition of the theorem can then be abbreviated as $G = \langle x^G \rangle$. The necessity of this condition follows from the Wirtinger presentation, because $\pi(K)$ is generated by conjugates of a meridian. To prove sufficiency, D. Johnson [9] has found an elegant way to construct a knot together with an epimorphism $\pi(K) \twoheadrightarrow G$, sending a meridian to the element $x \in G$. Theorem 12 has the following corollary:

Corollary 13. *The invariant F_G is not of finite type if and only if G contains a non-abelian subgroup $H \leq G$ such that $H = \langle x^H \rangle$ for some element $x \in H$.*

Equivalently, the invariant F_G is constant if and only if all subgroups $H \leq G$ satisfying $H = \langle x^H \rangle$ are abelian and hence cyclic.

Proof. For any knot K there are exactly $|G|$ representations which factor through the abelianization $\pi(K)_{ab} \cong \mathbb{Z}$. This implies $F_G(K) \geq F_G(\bigcirc) = |G|$.

If there exist a knot K with $F_G(K) > |G|$, then there must be a homomorphism $\varphi : \pi(K) \to G$ which does not factor through the abelianization. The image $H = \mathrm{Im}(\varphi)$ is necessarily non-abelian and satisfies $H = \langle x^H \rangle$, where x is the image of a meridian.

Conversely, suppose that G contains a non-abelian subgroup $H = \langle x^H \rangle$. By Theorem 12 there exist a knot K and a homomorphism $\pi(K) \twoheadrightarrow H \hookrightarrow G$. This means $F_G(K) > F_G(\bigcirc)$ and the knot invariant F_G cannot be of finite type. □

2.2. **Nilpotent groups.** We rephrase the preceding criterion in more group-theoretic terms. For subgroups $S, H \leq G$, we define $\langle S^H \rangle$ to be the subgroup generated by the elements s^h, where $s \in S$ and $h \in H$. For a cyclic subgroup $S = \langle x \rangle$ we have $\langle S^H \rangle = \langle x^H \rangle$.

Lemma 14. *For a finite group G and a subgroup $S \leq G$ the following two conditions are equivalent:*

1. *The only subgroup $H \leq G$ satisfying $H = \langle S^H \rangle$ is the group S itself.*
2. *The group S is subnormal in G, or more explicitly, there exists a subnormal sequence $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_{n-1} \trianglerighteq G_n = S$.*

Proof. (1⇒2) We can construct a subnormal sequence starting with $G_0 = G$ by inductively setting $G_{k+1} := \langle S^{G_k} \rangle$. Since G is finite, this sequence must stabilize, which means $G_{n+1} = G_n$ for some n. Hence, $G_n = \langle S^{G_n} \rangle$ and by hypothesis (1) we can conclude that $G_n = S$.

(2⇒1) Suppose we have a subnormal sequence as stated in (2) and a subgroup $H = \langle S^H \rangle$. Since $H \leq G_0$, we see that $H = \langle S^H \rangle \leq \langle S^{G_0} \rangle \leq G_1$. The last inclusion holds because G_1 contains S and is normal in G_0. Now we can reiterate this argument: Since $H \leq G_1$, we obtain $H = \langle S^H \rangle \leq \langle S^{G_1} \rangle \leq G_2$. Continuing like this, we arrive at $H \leq G_n = S$, which proves $H = S$. □

Lemma 15. *For a finite group G, the following conditions are equivalent:*

(1) *The group G is nilpotent.*
(2) *Every subgroup is subnormal in G.*
(3) *Every cyclic subgroup is subnormal in G. (Baer condition)*
(4) *For any pair of elements $x, y \in G$, the iterated commutators defined by $x_0 = x$ and $x_{k+1} = [x_k, y]$ vanish for sufficiently large k. (Engel condition)*

Proof. The equivalence (1⇔2) is one of the characterizations of nilpotent groups and may be found, for example, in [8, Chapter III, §2,§7]. The step (2⇒3) is trivial. To prove (3⇒4), let $G = G_0 \trianglerighteq \cdots \trianglerighteq G_n = \langle y \rangle$ be a subnormal sequence. We have $x_0 \in G_0$ and by induction $x_{k+1} = x_k y x_k^{-1} y^{-1} \in \langle y^{G_k} \rangle \leq G_{k+1}$. Thus we arrive at $x_n \in G_n = \langle y \rangle$ and hence $x_{n+1} = [x_n, y] = 1$. The implication (4⇒1) was first observed by M. Zorn [14], and a proof may be found in [8, Chapter III, §6]. □

Proof of Theorem 2. By Corollary 13, the knot invariant F_G is constant if and only if every subgroup $H = \langle x^H \rangle$ is abelian, which means $H = \langle x \rangle$. By Lemma 14, this happens if and only if every cyclic subgroup is subnormal in G. By Lemma 15, this is equivalent to G being nilpotent. □

2.3. **Application to link group representations.** If we consider links instead of knots, then a simpler version of Theorem 2 holds. Let $F_G : \mathcal{L}_\mu \to \mathbb{N}$ be the link invariant defined by $F_G(L) = |\operatorname{Hom}(\pi(L), G)|$.

Theorem 16. *Suppose $\mu \geq 2$. The link invariant $F_G : \mathcal{L}_\mu \to \mathbb{N}$ is constant if G is abelian and not of finite type if G is non-abelian.*

Proof. Let s be the braid index of the link L. The Wirtinger presentation shows that the link group $\pi(L)$ can be presented with s generators. This implies the inequality $F_G(L) \leq |G|^{s(L)}$, exactly as in the case of knots. By Corollary 10 we conclude that F_G is either constant or not of finite type.

For an abelian group G, every representation factors through the abelianization $\pi(L)_{ab} \cong \mathbb{Z}^\mu$, which means that $F_G \equiv |G|^\mu$ is constant.

If G is non-abelian, however, then F_G is not constant. We explain this in the case of two-component links. For the trivial link $\bigcirc^2$, the link group is *free* on two generators, whereas the group of the Hopf-link H is *free abelian* on two generators. This means $F_G(H) < F_G(\bigcirc^2)$ for every non-abelian group G. □

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Mathematisches Institut der Universität Bonn, Beringstr. 1, 53115 Bonn, Germany
E-mail address: `eiserm@math.uni-bonn.de`