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BASIC FORMS FOR TRANSVERSELY INTEGRABLE SINGULAR RIEMANNIAN FOLIATIONS

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ABSTRACT. Basic forms for a transversely integrable singular Riemannian foliation with compact leaves are in one-to-one correspondence with "Weyl"–invariant differential forms on a generalized section of the foliation.

In two recent papers, cf. [3, 4], Peter Michor demonstrated that for a proper isometric action of a Lie group G on a smooth Riemannian manifold admitting a section S the restriction of differential forms induces an isomorphism

$$\Omega^p_{hor}(M)^G \to \Omega^p(S)^{W(S)}$$

between the space of horizontal G–invariant differential forms on M and the space of all differential forms on S which are invariant under the action of the generalized Weyl group $W(S)$ of the section S.

The existence of a section assures that on the regular part (the set of points of orbits of the maximal dimension) of the G–manifold the orthogonal distribution is completely integrable, i.e. it defines a foliation. In [7] it has been conjectured that the integrability of the orthogonal distribution on the regular part implies the existence of immersed submanifolds which meet orthogonally all orbits. This conjecture has been proved true by H. Boualem in [2] for singular Riemannian foliations (SRF), thus for a larger class of manifolds.

In this short note we propose to demonstrate the following generalization of the Michor theorem for transversely integrable SRF.

Theorem 1. *Let $\mathcal{F}$ be a transversely integrable SRF with compact leaves on a compact Riemannian manifold M. Let S be a generalized section with the Weyl pseudogroup $\mathcal{W}(S)$. Then the restriction mapping defines an isomorphism $\Omega(M;\mathcal{F}) \to \Omega(S)^{\mathcal{W}(S)}$ where $\Omega(M;\mathcal{F})$ is the algebra of basic forms of the foliated manifold $(M,\mathcal{F})$ and $\Omega(S)^{\mathcal{W}(S)}$ is the algebra of $\mathcal{W}(S)$–invariant forms on the generalized section S.*

The compactness is used in two places: to prove the existence of a generalized section and in the proof of the surjectivity. In our case (compact leaves) the completeness of the Riemannian metric would be sufficient to prove the existence of a generalized section. We also think that using a more delicate topological argument

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the compactness assumption can be replaced by the completeness in the proof of the surjectivity.

1. Preliminaries

First, let us recall some results concerning transversely integrable SRF, cf. [2], which will allow us to define the Weil pseudogroup.

Proposition 1. *Let $\mathcal{F}$ be a transversely integrable SRF with leaves of compact closure. Through any point of M passes an immersed connected submanifold which meets orthogonally all leaves of $\mathcal{F}$.*

Boualem takes a leaf of the horizontal distribution (on the regular part). Owing to a result of Blumenthal–Hebda, cf. [1], this leaf meets any "regular" leaf of $\mathcal{F}$ orthogonally. The idea is to extend this leaf over the singular set of the foliation $\mathcal{F}$. To achieve this Boualem studies a tubular neighbourhood of a singular leaf. It turns out that any connected component of the trace of such a leaf of the horizontal distribution is contained in a fibre of a suitably chosen tubular neighbourhood and moreover, it is a part of the trace of a linear subspace of the fibre. Therefore by taking the whole trace of this linear subspace he was able to extend the leaf. However, it could happen that the extension would intersect itself (it might consist of the trace of several linear subspaces). Therefore the extension of a leaf (of the horizontal distribution) is only an immersed submanifold and it can contain several leaves of the horizontal distribution.

In the regular part any $\mathcal{F}$–leaf curve defines a local isometry between the leaf of $\mathcal{F}^{\perp}$ passing through the starting point of the curve and the leaf passing through the end point of the curve, cf. [1]. Moreover, Boualem proved that any such a local isometry in a suitable small (tubular) neighbourhood of a singular leaf can be extended to a local isometry of generalized sections, cf. Proposition 2.1.1 of [2]. The result is formulated for a very special singular leaf (a point), but obviously it is true for any leaf.

In [6] P. Molino described in detail a new class of SRF called orbit–like foliations. The main characteristic feature of these foliations is the fact that any point x has an adapted neighbourhood in which the foliation is the product of an open neighbourhood of x in the leaf L_x passing through this point and the foliation defined by an action of a group of isometries on a suitable vector space (the fibre N_x of the normal bundle $N(L_x) = TL_x^{\perp}$ at the point x). This foliation is called the infinitesimal transverse model of $\mathcal{F}$ at x. For a suitable small tubular neighbourhood the holonomy transformation defined by a leaf curve beginning and ending in N_x can be extended to a global transformation of N_x. In our case when leaves are compact the traces of leaves of $\mathcal{F}$ on a suitable small ball in N_x are the orbits of a compact group of isometries G_x, whose action generates all holonomy transformations. The restriction of the action of G_x to a section generates its Weyl (pseudo)group.

2. Proof of Theorem 1

A) The restriction to the chosen section is injective.

The foliation $\mathcal{F}$ is regular on an open and dense subset M_0 of M, cf. [5]. The orthogonal distribution $\mathcal{F}^{\perp}$ on M_0 is integrable and its leaves are connected components of the trace of generalized sections on M_0. Any such a leaf intersects all leaves of $\mathcal{F}$ in M_0. Thus any two basic forms whose restrictions to a leaf of $\mathcal{F}^{\perp}$ (in

M_0) are equal are themselves equal on M_0. By continuity they are equal on the whole manifold M.

B) The restriction to the chosen section is surjective.

First let us look at the local problem. Let us choose a point x and let L be the leaf passing through the point x. Let $N(L)$ be a suitably chosen small tubular neighbourhood of L. Any (generalized) section passing through the point x is of the form $exp_x(W)$ where W is a vector subspace of $T_x\mathcal{F}^\perp$ and the trace of the foliation $\mathcal{F}$ on $exp_x(T_x\mathcal{F}^\perp) = N_x$ is given by an action of a Lie group G_x.

Basic forms on $N(L)$ are in one–to–one correspondence with G_x–invariant horizontal forms on N_x. Any section S_x of the foliation $\mathcal{F}$ passing through x in $N(L)$ is also a section of the action of the group G_x. Therefore, according to Michor, basic forms on $N(L)$ are in one-to-one correspondence with $W(S_x)$–invariant forms on S_x.

The main difficulty in passing from the local case to the global one is the fact that a generalized section can pass many times through a given neighbourhood.

Let S be our generalized section. The trace $M_0 \cap S$ of S on the regular part M_0 consists of leaves $\{S_\alpha\}$, $\alpha \in A$, of the foliation $\mathcal{F}^\perp$. The Weyl pseudogroup $\mathcal{W}_\alpha$ of any S_α consists of local isometries and it is the holonomy pseudogroup $\mathcal{H}_\alpha$ of the complete transverse manifold S_α of $(M_0, \mathcal{F})$. Therefore any $\mathcal{W}_\alpha$–invariant form ω can be extended to a basic form ω_0 on M_0. Moreover, the restriction of this form to $\bigcup S_\alpha = M_0 \cap S$ must be $\mathcal{W}(S)$–invariant—directly from the fact that the form ω_0 is basic and that the pseudogroup $\mathcal{W}(S)|M_0$ is generated by holonomy mappings along leaf curves.

It remains to extend the form ω_0 over the singular set Σ. Let L be a leaf in Σ and let us choose a component S_L of the trace of the generalized section S in a sufficiently small tubular neighbourhood $N(L)$ of L. Denote by $\mathcal{W}(L)$ the restriction of the Weyl pseudogroup to S_L. The form $\omega|S_L$ is $\mathcal{W}(L)$–invariant. So it can be extended to a basic form ω_L on $N(L)$. On $N(L) \cap M_0$ the basic forms ω_0 and ω_L coincide, so $\omega_0 \cup \omega_L$ is a smooth basic extension of the form ω. After a finite number of such steps we obtain a global basic form $\tilde{\omega}$ extending the $\mathcal{W}(S)$–invariant form ω. We have just proved the surjectivity of our restriction mapping and thus have completed the proof of the theorem. □

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