

A WILD MINIMAL PLANE IN \mathbb{R}^3

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ABSTRACT. The main object of this article is to construct a complete minimal immersed plane in \mathbb{R}^3 whose closure has nonempty interior but it is not dense in the whole space. Furthermore, its Gaussian curvature is bounded.

1. INTRODUCTION

In [Rsb], Rosenberg constructed a dense minimal surface in \mathbb{R}^3 by reflection of a fundamental domain. For this he used a result of Jenkins-Serrin [J-S]. Inspired by Rosenberg's example, Jorge [Jor] asked whether a complete minimal surface in \mathbb{R}^3 with bounded Gaussian curvature is either proper or dense. The answer is negative. The purpose of this note is to construct a complete minimal immersed surface in \mathbb{R}^3 which is conformal to a plane and whose closure has nonempty interior. Moreover it is not dense in the whole space and the Gaussian curvature is bounded. For this purpose we will need the following result from [And]. Here, we identify \mathbb{R}^3 with $\mathbb{C} \times \mathbb{R}$.

Theorem 1. *Let h be a harmonic function in an open set $\Omega \subset \mathbb{C}$. Suppose that L and H are holomorphic functions in Ω such that*

$$(1.1) \quad L'H' = \left(\frac{\partial h}{\partial z} \right)^2 \quad \text{and} \quad |L'(z)| + |H'(z)| > 0,$$

for $z \in \Omega$. Then the mapping

$$\chi : \Omega \rightarrow \mathbb{C} \times \mathbb{R}, \quad \chi(z) = \left(L(z) - \overline{H(z)}, h(z) \right)$$

is a conformal minimal immersion.

We call a map χ satisfying equation (1) an *Enneper immersion*. In [And], we prove this result and show that any immersed minimal surface in $\mathbb{C} \times \mathbb{R}$ can be parametrized by an Enneper immersion.

Theorem 2. *There exists a complete immersed minimal plane in $\mathbb{C} \times \mathbb{R}$ whose closure contains the xy -plane in its interior. Furthermore, it is not dense in the whole space and its Gaussian curvature is bounded.*

To show the existence of that plane we consider a dense curve in a cylindrical shell of revolution and solve the Plateau problem for it. The author wishes to thank his colleague at UFC, L. Jorge, for clarifying conversations.

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The paper is organized into three sections. Section 1 is this introduction. For sake of completeness, in Section 2 we present the proof of Theorem 1. In Section 3 we construct the example.

2. PROOF OF THEOREM 1

We will use the standard inner product; given $u = (a_1 + ib_1, c_1)$ and $v = (a_2 + ib_2, c_2) \in \mathbb{C} \times \mathbb{R}$ we define

$$(2.1) \quad \langle u, v \rangle = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Lemma 1. *If $\chi : \Omega \rightarrow \mathbb{C} \times \mathbb{R}$, $\chi(z) = (a(z) + ib(z), c(z))$, is a C^1 map, then*

$$(2.2) \quad (E - G) - 2iF = 4 \left[\left(\frac{\partial a}{\partial z} \right)^2 + \left(\frac{\partial b}{\partial z} \right)^2 + \left(\frac{\partial c}{\partial z} \right)^2 \right],$$

where $E = \langle \chi_u, \chi_u \rangle$, $G = \langle \chi_v, \chi_v \rangle$ and $F = \langle \chi_u, \chi_v \rangle$.

Proof. A straightforward computation shows that

$$(2.3) \quad (E - G) - 2iF = (a_u - ia_v)^2 + (b_u - ib_v)^2 + (c_u - ic_v)^2.$$

Since

$$(2.4) \quad a_u - ia_v = 2 \frac{\partial a}{\partial z}, \quad b_u - ib_v = 2 \frac{\partial b}{\partial z} \quad \text{and} \quad c_u - ic_v = 2 \frac{\partial c}{\partial z},$$

we easily complete the proof of the lemma. \square

Proof (Theorem 1). Assume that $\chi(z) = (L(z) - \overline{H(z)}, h(z))$ is a map such that the functions $L(z) = p(z) + iq(z)$ and $H(z) = f(z) + ig(z)$ are holomorphic and h is harmonic. The Cauchy-Riemann equations allow us to write the derivatives of L and H as

$$(2.5) \quad \frac{1}{2}L' = \frac{\partial p}{\partial z} = i \frac{\partial q}{\partial z} \quad \text{and} \quad \frac{1}{2}H' = \frac{\partial f}{\partial z} = i \frac{\partial g}{\partial z}.$$

Since $\chi = ((p - f) + i(q + g), h)$, from Lemma 1, we may write the identities

$$\begin{aligned} (E - G) - 2iF &= 4 \left[\left(\frac{\partial(p-f)}{\partial z} \right)^2 + \left(\frac{\partial(q+g)}{\partial z} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 \right] \\ &= 4 \left[\frac{1}{4} (L' - H')^2 + \frac{1}{4} (-iL' - iH')^2 + \left(\frac{\partial h}{\partial z} \right)^2 \right] \\ &= 4 \left[-L'H' + \left(\frac{\partial h}{\partial z} \right)^2 \right]. \end{aligned}$$

From now on, let us suppose that the functions L , H and h satisfy the differential equation (1.1). This implies that $E = G$ and $F = 0$. Since $\|\chi_u\| = \|\chi_v\|$, to prove that χ is a conformal immersion it is sufficient to show that the partial derivative χ_u does not vanish. Indeed, assume that $\chi_u(z_0) = (0, 0, 0) = \chi_v(z_0)$ for some point $z_0 \in \Omega$; in other words, assume that

$$(2.6) \quad p_u(z_0) - f_u(z_0) = 0, \quad q_u(z_0) + g_u(z_0) = 0 \quad \text{and} \quad h_u(z_0) = 0 = h_v(z_0).$$

It is immediate to verify that $H'(z_0) = \overline{L'(z_0)}$ and $h_u(z_0) = 0$. From (1.1), it follows that $4|L'(z_0)|^2 = 0$. Thus $|L'(z_0)| = 0 = |H'(z_0)|$, which is a contradiction because $|L'(z)| + |H'(z)| \neq 0$ for any $z \in \Omega$. So, χ is a conformal immersion. It still

remains to be shown that χ is minimal. Let N be the unit vector field normal to the immersion. Since the mean curvature \mathcal{H} is given by

$$(2.7) \quad \mathcal{H} = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

where $l = \langle N, \chi_{uu} \rangle$, $n = \langle N, \chi_{vv} \rangle$ and $m = \langle N, \chi_{uv} \rangle$ and since each coordinate of χ is a harmonic function we have $\chi_{uu} = -\chi_{vv}$, which implies that $l = -n$. With those conditions, it is easy to verify that the mean curvature is zero. This completes the proof of Theorem 1. \square

The Weierstrass data of χ can be easily obtained from the usual formulas. Following the notations of [B-C], we have

$$(S, g, w) = \left(\Omega, -\frac{L'}{\frac{\partial h}{\partial z}}, 2\frac{\partial h}{\partial z} dz \right).$$

The first fundamental form and the Gauss curvature induced on Ω are, respectively,

$$(2.8) \quad ds = (|L'| + |H'|) |dz| \quad \text{and} \quad K = -4 \left[\frac{\left| \frac{\partial h}{\partial z} \right| \left| \frac{dg}{g} \right|}{(|L'| + |H'|)^2} \right]^2.$$

3. THE EXAMPLE

Let d, r_1, r_2 be fixed real numbers with $r_2 \neq 0$. We will call *trochoid* a parametric curve in the complex plane given by

$$(3.1) \quad x(v) = (r_1 - r_2) \cos(v) + d \cos \left(\left(\frac{r_1}{r_2} - 1 \right) v \right),$$

$$(3.2) \quad y(v) = (r_1 - r_2) \sin(v) - d \sin \left(\left(\frac{r_1}{r_2} - 1 \right) v \right),$$

$v \in \mathbb{R}$. To avoid remarks about degenerate cases we will assume that

$$(3.3) \quad r_1 \neq r_2 \quad \text{and} \quad r_1 r_2 d \neq 0.$$

Recall that, for $r_1 > r_2 > 0$ and $d > 0$, the mechanical plane curve generated by a point P attached to a circle C_2 with radius r_2 rolling about the inside of a fixed circle C_1 with radius r_1 is described by the above parametrization where d denotes the distance from the point P to the center of the circle $C(r_2)$ (see [Law]). Here, the notation $C(r)$ for $r > 0$ means the circle $C(r) = \{z \in \mathbb{C}, |z| = r\}$ endowed with the usual orientation. Observe that we do not make any restriction on the sign of the parameters (r_1, r_2, d) of a trochoid. To construct the example we need to study that family of curves. Denote by $A(s_1, s_2) \subset \mathbb{C}$ the compact annulus with center at the origin and radii $s_1 > s_2$, where

$$(3.4) \quad s_1 = |r_1 - r_2| + |d|,$$

$$(3.5) \quad s_2 = \left| |r_1 - r_2| - |d| \right|.$$

The set $\Gamma = A(s_1, s_2) \times [-l, l] \subset \mathbb{C} \times \mathbb{R}$ will be called a cylindrical shell of revolution. We are going to construct a dense curve in Γ .

Lemma 2. *Suppose that ϕ is an irrational number, $l > 0$ and $\mu \in S^1$. The sequence*

$$(a_k, b_k) = \left(d\mu e^{-2k\phi\pi i}, l e^{k\frac{\phi}{2}\pi i} \right), \quad k \in \mathbb{Z},$$

is dense in an embedded circle of type $(p, q) = (-4, 1)$ on the torus $C(|d|) \times C(l)$.

Proof. Consider the function

$$(3.6) \quad f : [0, 4] \rightarrow C(|d|) \times C(|l|), \quad f(t) = (d\mu e^{-2\pi ti}, l e^{\pi \frac{1}{2}i})$$

and the sequence $\{c_k\}$ in the domain of f defined by

$$(3.7) \quad c_k \equiv \phi k, \quad \text{mod } 4, \quad k \in \mathbb{Z}.$$

Of course, this sequence is dense in the interval because ϕ is irrational. Since $f(c_k) = (a_k, b_k)$ and f is a C^∞ parametrization of an embedded circle of type $(-4, 1)$, we conclude that the sequence image is dense in this embedded circle. \square

Lemma 3. *Suppose that $l > 0$. The parametric curve given by*

$$\begin{aligned} x(v) &= (r_1 - r_2) \cos(v) + d \cos\left(\left(\frac{r_1}{r_2} - 1\right)v\right), \\ y(v) &= (r_1 - r_2) \sin(v) - d \sin\left(\left(\frac{r_1}{r_2} - 1\right)v\right), \\ z(v) &= l \sin\left(\frac{r_1}{2r_2}v\right), \end{aligned}$$

$v \in \mathbb{R}$, is in the cylindrical shell of revolution $\Gamma = A(s_1, s_2) \times [-l, l] \subset \mathbb{C} \times \mathbb{R}$. If $\frac{r_1}{r_2} \in \mathbb{Q}$, then the curve is periodic; otherwise, it is dense in that cylindrical shell of revolution.

Proof. The xy -projection of the curve is a trochoid in \mathbb{C} and it can be written as

$$x(v) + iy(v) = (r_1 - r_2)e^{iv} + d e^{-i\left(\frac{r_1}{r_2} - 1\right)v}.$$

So, from the triangular inequalities we obtain that

$$(3.8) \quad \left| |r_1 - r_2| - |d| \right| \leq |x(v) + iy(v)| \leq |r_1 - r_2| + |d|.$$

This shows that the trochoid is in the annulus $A(s_1, s_2)$ and it is immediate to conclude that the curve $(x(v) + iy(v), z(v))$ is in the cylindrical shell of revolution Γ . Now, if $\frac{r_1}{r_2} = \frac{p}{q} \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, one can see that the curve is periodic because

$$(x(v + 4q\pi) + iy(v + 4q\pi), z(v + 4q\pi)) = (x(v) + iy(v), z(v)).$$

Let us assume that $\frac{r_1}{r_2}$ is not rational.

First case: Suppose that $|r_1 - r_2| \geq |d|$.

Given a point $(w_0, b) \in A(s_1, s_2) \times [-l, l]$, choose a point z_1 in the core circle $C(|r_1 - r_2|) \subset A(s_1, s_2)$ whose distance from w_0 is $|d|$. If $0 \leq \theta = \arg(z_1) < 2\pi$, we have that $z_1 = |r_1 - r_2| e^{i v_k}$ where $v_k = \theta + 2k\pi$. So,

$$\begin{aligned} x(v_k) + iy(v_k) &= |r_1 - r_2| e^{i v_k} + d e^{-\left(\frac{r_1}{r_2} - 1\right)i v_k} \\ &= z_1 + d e^{-\left(\frac{r_1}{r_2} - 1\right)(\theta + 2k\pi)i} \\ &= z_1 + d \mu e^{-2\frac{r_1}{r_2} k \pi i}, \end{aligned}$$

for some unit complex number μ . Therefore the point

$$\begin{aligned} a_k &= x(v_k) + iy(v_k) - z_1 \\ &= d \mu e^{-2\frac{r_1}{r_2} k \pi i} \end{aligned}$$

and $w_0 - z_1$ belong to the same circle $C(|d|)$, for each k . Now, take the sequence

$$b_k = l e^{\left(\frac{r_1}{2r_2} k \pi i\right)}$$

in the circle $C(l)$. Since $\frac{r_1}{r_2} \notin \mathbb{Q}$, from Lemma 2 we conclude that the sequence $\{(a_k, b_k)\}$ is dense in a circle of type $(-4, 1)$ of the torus $C(|d|) \times C(l)$. Let $(w_0 - z_1, \beta_0)$ be a point in this embedded circle such that the y -projection $\Pi : C(l) \rightarrow [-l, l]$ of β_0 is the given point $b \in [-l, l]$. It is possible to choose β_0 because the embedded circle is transverse to the fibers $C(|d|) \times \{\beta\}$. If (a_{k_i}, b_{k_i}) is a subsequence which converges to $(w_0 - z_1, \beta_0)$, then

$$(x(v_{k_i}) + iy(v_{k_i}), z(v_{k_i})) = (z_1 + a_{k_i}, \Pi(b_{k_i}))$$

converges to (w_0, b) . This shows that the curve is dense in the cylindrical shell of revolution $\Gamma = A(s_1, s_2) \times [-l, l]$.

Second case: Suppose that $|r_1 - r_2| < |d|$.

In this case, the core circle is $C(|d|) \subset A(s_1, s_2)$. Given a point $(w_0, b) \in A(s_1, s_2) \times [-l, l]$, choose a point z_1 in the core circle whose distance from w_0 is $|r_1 - r_2|$. If $0 \leq \theta = \arg(z_1) < 2\pi$, we have that $z_1 = d e^{i v_k}$ where $v_k = \theta + 2k\pi$. So, we proceed in the same way as before to find a sequence in the trochoid which converges to (w_0, b) . \square

Proof (Theorem 2). Consider the map

$$(3.9) \quad \chi : \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{R}, \quad \chi(z) = \left(L(z) - \overline{H(z)}, h(z) \right)$$

where

$$\begin{aligned} L(z) &= (r_1 - r_2) e^z, \\ H(z) &= -d e^{\left(\frac{r_1}{r_2} - 1\right)z}, \\ h(z) &= 4 \left(\frac{d}{r_2}\right)^{\frac{1}{2}} \left| \frac{r_2}{r_1} \right| |r_1 - r_2| \operatorname{Re} \left(i e^{\frac{r_1}{2r_2} z} \right). \end{aligned}$$

As before, the parameters satisfy the conditions $r_1 \neq r_2$ and $r_1 r_2 d \neq 0$; however, we now impose the extra restrictions

$$(3.10) \quad r_2 > 0, \quad \frac{r_1}{r_2} \notin \mathbb{Q}, \quad \frac{r_1}{r_2} < 1 \quad \text{and} \quad d = r_2 - r_1.$$

By using the identity

$$\frac{\partial}{\partial z} \operatorname{Re}(P(z)) = \frac{1}{2} P'(z),$$

for any analytic function $P(z)$, one can easily show that the map χ satisfies the hypothesis of Theorem 1. So, it is a conformal minimal immersion of the plane. Fixed $u \in \mathbb{R}$, the xy -projection of the curve $v \mapsto \chi(u + iv)$ is a (nondegenerate) trochoid, namely,

$$\begin{aligned} x(v) &= (R_1(u) - R_2(u)) \cos(v) + D(u) \cos \left(\left(\frac{R_1(u)}{R_2(u)} - 1 \right) v \right), \\ y(v) &= (R_1(u) - R_2(u)) \sin(v) - D(u) \sin \left(\left(\frac{R_1(u)}{R_2(u)} - 1 \right) v \right), \end{aligned}$$

whose data is

$$(3.11) \quad R_1(u) = r_1 e^u, \quad R_2(u) = r_2 e^u \quad \text{and} \quad D(u) = -d e^{(\frac{r_1}{r_2}-1)u}.$$

The main property is that the trochoid obtained by projection satisfies the same irrationality condition; namely, for each u we have

$$\frac{R_1(u)}{R_2(u)} = \frac{r_1}{r_2} \notin \mathbb{Q}.$$

From Lemma 3, we conclude that the path $v \mapsto \chi(u + iv)$ is dense in the vertical cylindrical shell of revolution $\Gamma_u = A(s_1(u), s_2(u)) \times [-l(u), l(u)]$, whose dimensions are (see (3.4), (3.5) and Lemma 3)

$$\begin{aligned} s_1(u) &= |r_1 - r_2| \left(e^u + e^{(\frac{r_1}{r_2}-1)u} \right), \\ s_2(u) &= |r_1 - r_2| |e^u - e^{(\frac{r_1}{r_2}-1)u}|, \\ l(u) &= 4 \left(\frac{d}{r_2} \right)^{\frac{1}{2}} \left| \frac{r_2}{r_1} \right| |r_1 - r_2| e^{\frac{r_1}{2r_2}u}. \end{aligned}$$

It is clear that the image of the map χ is dense in the region

$$\Gamma = \bigcup_{u \in \mathbb{R}} \Gamma_u.$$

The xy -plane is contained in the interior of Γ , since the height of the vertical cylindrical shell of revolution Γ_u is $2l(u) > 0$, the smaller radius satisfies $s_2(0) = 0$ and $s_2(u) \rightarrow +\infty$ when $u \rightarrow +\infty$. The region Γ is not dense in $\mathbb{C} \times \mathbb{R}$ because there exist infinitely many cylinders which do not intersect Γ , for example, the cylinder

$$R = A(0, s_1(1)) \times [49l(1), +\infty).$$

The restrictions $\frac{r_1}{r_2} < 1$, $d = r_2 - r_1$ and the inequality $e^u + e^{(\frac{r_1}{r_2}-1)u} \geq 2$ allow us to write that

$$\begin{aligned} ds &= \left(|r_1 - r_2| e^u + d e^{(\frac{r_1}{r_2}-1)u} \right) |dz| \\ &\geq 2|r_1 - r_2| |dz|. \end{aligned}$$

It is sufficient to show that the immersed plane is complete. To study the Gaussian curvature K , we observe that

(3.12)

$$L'(z) = k_1 e^z, \quad H'(z) = k_2 e^{(\frac{r_1}{r_2}-1)z}, \quad \frac{g'(z)}{g(z)} = k_3, \quad \text{and} \quad \frac{\partial h}{\partial z} = k_4 e^{\frac{r_1}{2r_2}z}$$

where k_i , $i = 1, 2, 3, 4$, are nonzero constants. From (2.8), we get

$$K(u + iv) = -4 \left[\frac{|k_3 k_4|}{\left(|k_1| e^{(1-\frac{r_1}{4r_2})u} + |k_2| e^{(\frac{3r_1}{4r_2}-1)u} \right)^2} \right]^2.$$

Since the exponents of $e^{(1-\frac{r_1}{4r_2})u}$ and $e^{(\frac{3r_1}{4r_2}-1)u}$ have opposite signs, we may easily conclude that K is bounded. This completes the proof of Theorem 2. \square

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