

## THE SPECTRAL PROPERTIES OF CERTAIN LINEAR OPERATORS AND THEIR EXTENSIONS

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ABSTRACT. Let  $H$  be a Hilbert space with inner-product  $\langle x, y \rangle$ , and let  $R$  be a bounded positive operator on  $H$  which determines an inner-product,  $\langle x, y \rangle = (Rx, y)$ ,  $x, y \in H$ . Denote by  $H^-$  the completion of  $H$  with respect to the norm  $\|x\| = \langle x, x \rangle^{1/2}$ . In this paper, operators having certain relationships with  $R$  are studied. In particular, if  $T = SR^{1/2}$  where  $S \in B(H)$ , then  $T$  has an extension  $T^- \in B(H^-)$ , and  $T$  and  $T^-$  have essentially the same spectral and Fredholm properties.

### INTRODUCTION

Throughout this paper,  $H$  is a Hilbert space with inner-product  $\langle x, y \rangle$  and norm  $\|x\|_H = \langle x, x \rangle^{1/2}$ . Assume that  $\langle x, y \rangle$  is a bounded inner-product on  $H$ , so there exists  $c > 0$  such that for all  $x, y \in H$ :  $|\langle x, y \rangle| \leq c\|x\|_H\|y\|_H$ . Let  $\|x\| = \langle x, x \rangle^{1/2}$ , and let  $H^-$  be the completion of  $H$  with respect to the norm  $\|x\|$ . Since the inner-product  $\langle x, y \rangle$  is bounded, it is well-known that there exists a positive operator  $R \in B(H)$  such that

$$\langle x, y \rangle = (Rx, y) \quad \text{for all } x, y \in H.$$

For future reference we note that

$$(*) \quad \|R^{1/2}x\|_H = \|x\| \quad \text{for all } x \in H.$$

Early work in this setting centered on operators of the form  $T = SR$  where  $S = S^* \in B(H)$ ; see Chapters 15–17 of [Z]. For such  $T$ ,  $\langle Tx, y \rangle = (RSRx, y) = (Rx, SRy) = \langle x, Ty \rangle$  for all  $x, y \in H$ . An operator  $T$  is *symmetrizable* with respect to an inner-product  $\langle x, y \rangle$ , if  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in H$ . Thus the operator  $T = SR$  above is symmetrizable.

The concept of a symmetrizable operator makes sense whenever there is a bounded inner-product on a Banach space. P. Lax studied symmetrizable operators in this more general setting in [L]. He proved that when  $T$  is symmetrizable, then  $T$  has an extension  $T^- \in B(H^-)$  and  $\sigma(T) \supseteq \sigma(T^-)$ . Istratescu's book [I, Chapter 11] is a good source of information about symmetrizable operators and related ideas.

Here we restrict attention to the case where the underlying space is the Hilbert space  $H$ . Our main results concern operators of the form  $T = SR^{1/2}$  where  $S$  is

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an arbitrary operator in  $B(H)$ . It is shown that  $T = SR^{1/2}$  has an extension to an operator  $T^- \in B(H^-)$ , and that  $T$  and  $T^-$  have essentially the same basic operator properties (for example, they have the same spectrum).

### RESULTS

We use the notation from the Introduction in what follows. In particular,  $R$  is the positive operator determined by the bounded inner-product  $\langle x, y \rangle$ . We use the fact that  $\mathbf{R}(R^{1/2})$  is dense in  $H$  (here, and in what follows,  $\mathbf{R}(S)$  denotes the range of the operator  $S$ ).

**Theorem.** (1)–(4) are equivalent for  $T \in B(H)$ :

- (1)  $RTR^{-1}$  is bounded on  $\mathbf{R}(R)$ ;
  - (2) there exists an operator  $S \in B(H)$  such that  $\langle Tx, y \rangle = \langle x, Sy \rangle$  for all  $x, y \in H$ ;
  - (3) there exists an operator  $S \in B(H)$  such that  $RT = SR$ ;
  - (4)  $T^*(\mathbf{R}(R)) \subseteq \mathbf{R}(R)$ .
- (5)–(8) are equivalent for  $T \in B(H)$ :
- (5)  $R^{1/2}TR^{-1/2}$  is bounded on  $\mathbf{R}(R^{1/2})$ ;
  - (6)  $T$  has an extension to a bounded operator  $T^-$  on  $H^-$ ;
  - (7) there exists an operator  $S \in B(H)$  such that  $R^{1/2}T = SR^{1/2}$ ;
  - (8)  $T^*(\mathbf{R}(R^{1/2})) \subseteq \mathbf{R}(R^{1/2})$ .
- (9)–(12) are equivalent for  $T \in B(H)$ :
- (9)  $TR^{-1/2}$  is bounded on  $\mathbf{R}(R^{1/2})$ ;
  - (10)  $T$  has an extension to a bounded operator linear operator  $T^\sim : H^- \rightarrow H$ ;
  - (11) there exists an operator  $S \in B(H)$  such that  $T = SR^{1/2}$ ;
  - (12)  $T^*(H) \subseteq \mathbf{R}(R^{1/2})$ .

*Proof.* Clearly, (3) $\Rightarrow$ (1). Suppose that (1) holds. Let  $S$  denote the bounded extension of  $RTR^{-1}$  to all of  $H$ . It follows that  $RT = SR$ . Therefore (3) holds.

Assume that (2) holds. Then  $(RTx, y) = (Rx, Sy) = (S^*Rx, y)$  for all  $x, y \in H$ . Therefore,  $RT = S^*R$ , so (3) holds. Conversely, if  $RT = S^*R$ , then reversing the argument above, we have that (2) is true.

That (3) $\Rightarrow$ (4) is clear. Now assume that  $T^*(\mathbf{R}(R)) \subseteq \mathbf{R}(R)$ . Then  $\mathbf{R}(T^*R) \subseteq \mathbf{R}(R)$ , so by the Douglas Range Inclusion Theorem [D], it follows that  $T^*R = RS$  for some operator  $S \in B(H)$ . Taking adjoints, we have  $RT = S^*R$ , and thus (3) holds.

Assume that (5) holds. Then there exists  $M > 0$  such that

$$\|R^{1/2}TR^{-1/2}(R^{1/2}x)\|_H \leq M\|R^{1/2}x\|_H$$

for all  $x \in H$ . Thus by (\*),  $\|Tx\| \leq M\|x\|$  for all  $x \in H$ , and this implies (6). Also, this argument is reversible, so (6) $\Rightarrow$ (5).

Again, assume that (5) holds. Let  $S$  be the bounded extension of  $R^{1/2}TR^{-1/2}$  on  $H$ . It follows immediately that  $SR^{1/2} = R^{1/2}T$ . Thus (7) holds. Clearly (7) $\Rightarrow$ (5).

That (7) $\Rightarrow$ (8) is clear. Now assume that  $T^*(\mathbf{R}(R^{1/2})) \subseteq \mathbf{R}(R^{1/2})$ . Then  $\mathbf{R}(T^*R^{1/2}) \subseteq \mathbf{R}(R^{1/2})$ . Again, applying the Range Inclusion Theorem, we have  $T^*R^{1/2} = R^{1/2}S$  for some operator  $S \in B(H)$ . Taking adjoints in this equality we see that (7) is true.

Assume that (9) holds. Then there exists  $M > 0$  such that  $\|TR^{-1/2}(R^{1/2}x)\|_H \leq M\|R^{1/2}x\|_H$  for all  $x \in H$ . Thus by (\*),  $\|Tx\|_H \leq M\|x\|$  for all  $x \in H$ , and this implies (10). Also, this argument is reversible, so (10) $\Rightarrow$ (9).

Again, assume that (9) holds. Let  $S$  be the bounded extension of  $TR^{-1/2}$  on  $H$ . Then  $T = ST^{1/2}$ . Clearly (11) $\Rightarrow$ (9).

Finally, making use of the Range Inclusion Theorem as before, it is straightforward to check that (11) $\Leftrightarrow$ (12).

**Corollary.** Assume  $T = SR^{1/2}$ , where  $S \in B(H)$ .

- (a) The operator  $T$  has a bounded extension  $T^- \in B(H^-)$  with the property that  $T^-(H^-) \subseteq H$ .
- (b) Let  $E: H \rightarrow H^-$  be the continuous embedding map,  $Ex = x$  for all  $x$ . Let  $T^\sim: H^- \rightarrow H$  be as in part (10) of the Theorem. Then

$$T^\sim \in B(H^-, H); \quad T = T^\sim E; \quad T^- = ET^\sim.$$

*Proof.* Since  $R^{1/2}T = (R^{1/2}S)R^{1/2}$ , the operator  $T$  satisfies (7) in the Theorem. Then by (7) $\Rightarrow$ (6),  $T$  has a bounded extension  $T^-$  on  $H^-$ . Also by hypothesis,  $T$  satisfies (11), so by (10),  $T$  has a bounded extension  $T^\sim \in B(H^-, H)$ . Then clearly  $T^- = ET^\sim$ . It follows that  $T^-(H^-) \subseteq H$ . It is also clear that  $T = T^\sim E$ . This verifies both parts (a) and (b) of the Corollary.

For a bounded linear operator  $S$ , we use the notation:

$$\begin{aligned} \sigma(S) &= \text{the usual operator spectrum of } S; \\ \sigma_F(S) &\equiv \text{the Fredholm spectrum of } S \\ &\equiv \{\lambda \in \mathbf{C}: (\lambda - S) \text{ is not a Fredholm operator}\}; \\ \sigma_W(S) &\equiv \text{the Weyl spectrum of } S \\ &\equiv \{\lambda \in \mathbf{C}: (\lambda - S) \text{ is not a Fredholm operator of index zero}\}. \end{aligned}$$

In what follows,  $R, T, T^-$ , and  $T^\sim$  are as in the Corollary.

**Consequences.** In I–III below, assume  $T = SR^{1/2}$ , where  $S \in B(H)$ .

**I.** By part (a) of the Corollary,  $T^-(H^-) \subseteq H$ . Applying [B1, Theorem 4(2)], we have:

- (i)  $\sigma(T) = \sigma(T^-)$ ;
- (ii)  $\sigma_F(T) = \sigma_F(T^-)$ ;
- (iii)  $\sigma_W(T) = \sigma_W(T^-)$ .

Also: (iv) when  $\lambda \neq 0$ ,  $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T^-)$ ; here  $\mathbf{N}(W)$  denotes the null space of the operator  $W$ .

**II.** By part (b) of the Corollary,  $T = T^\sim E$  and  $T^- = ET^\sim$ . Therefore,  $T$  and  $T^-$  have all the common operator properties described in [B3]. In particular, when  $\lambda \neq 0$ :

- (i)  $\lambda - T$  has a pseudoinverse  $\Leftrightarrow \lambda - T^-$  has a pseudoinverse [B3, Theorem 4];
- (ii)  $\lambda - T$  has closed range  $\Leftrightarrow \lambda - T^-$  has closed range [B3, Theorem 5];
- (iii)  $\lambda$  is a pole of finite rank of the resolvent of  $T \Leftrightarrow \lambda$  is a pole of finite rank of the resolvent of  $T^-$  [B3, Theorem 9].

**III.** Let  $\mathcal{K}$  denote the set of all linear operators  $J \in B(H)$  such that  $J$  is compact, and  $J$  has an extension  $J^-$  on  $H^-$  which is also compact. By I (iii),  $\sigma_W(T) = \sigma_W(T^-)$ . It follows from [B2, Theorem 8] that

- (i)  $\sigma(T + J) = \sigma(T^- + J^-)$  for all  $J \in \mathcal{K}$ .

**IV.** Now assume that  $T = SR$  where  $S = S^* \in B(H)$ . Operators of this form are common in applications. As noted in the Introduction,  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in H$ . Since  $T = (SR^{1/2})R^{1/2}$ , consequences I-III hold for  $T$  and  $T^-$ , and in this case,  $T^-$  is selfadjoint.

**Example 1.** We give an example of a common situation in analysis where the results of this paper apply. Let  $\mu$  be a measure defined on some  $\sigma$ -algebra of subsets of a set  $\Omega$ . Let  $w$  be a weight function,  $w \in L^\infty(\mu)$ , with  $w(x) > 0$   $\mu$ -a.e. Set  $H = L^2(\mu)$ . Consider the bounded inner-product on  $H$  defined by

$$\langle f, g \rangle \equiv \int_{\Omega} fg^- w d\mu \quad (f, g \in H).$$

Let  $R$  be the multiplication operator defined by:  $R(f) = wf$ ,  $f \in H$ . Then  $\langle f, g \rangle = \langle Rf, g \rangle$  for all  $f, g \in H$ .

Now let  $K(x, t)$  be a kernel that determines a bounded integral operator  $S$  on  $L^2(\mu)$ :

$$S(f)(x) = \int_{\Omega} K(x, t)f(t) d\mu(t), \quad f \in L^2(\mu).$$

Then consequences I-III apply to the operator  $T = SR$ ,

$$T(f)(x) = \int_{\Omega} K(x, t)w(t)f(t) d\mu(t), \quad f \in L^2(\mu).$$

When in addition  $S = S^*$ , then IV also applies to the operator  $T$ .

**Example 2.** There exist symmetrizable operators  $T$  for which  $\sigma(T)$  and  $\sigma(T^-)$  can be very different. Now we modify an example due to J. Nieto in [N] to verify this in our particular setting. Let  $H$  be the weighted  $l^2$ -space of sequences  $\{a_k\}_{k \geq 1}$  such that  $\sum_1^\infty 4^k |a_k|^2 < \infty$ . The inner-product on  $H$  is:

$$\langle \{a_k\}, \{b_k\} \rangle = \sum_1^\infty 4^k a_k b_k^-.$$

Consider the inner-product on  $H$  defined by:

$$\langle \{a_k\}, \{b_k\} \rangle = \sum_1^\infty a_k b_k^-.$$

It is easy to check that this inner-product is bounded on  $H$ , and that the positive operator  $R$  such that  $\langle Ra, b \rangle = \langle a, b \rangle$  is the multiplication operator  $R(\{a_k\}) = \{4^{-k}a_k\}$ .

Let  $S$  and  $B$  be the shift and backward shift on  $H$ , so  $S(a_1, a_2, a_3, \dots) = (0, a_1, a_2, \dots)$ ;  $B(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$ . Let  $T = S + B$ , and note that  $T$  is selfadjoint relative to the inner-product  $\langle a, b \rangle$ . Let  $H^-$  be the completion of  $H$  relative to the norm determined by the inner-product  $\langle a, b \rangle$ , so  $H^-$  is the usual sequence space  $l^2$ . Let  $S^-$ ,  $B^-$ , and  $T^-$  denote the extensions of  $S$ ,  $B$ , and  $T$  to  $l^2$ . The extension  $T^-$  has real spectrum (in fact,  $\sigma(T^-) = [-2, 2]$ ). Now we compute the spectrum of  $T$  in  $B(H)$ . Let  $W: H \rightarrow l^2$  be defined by  $W(\{a_k\}) = \{2^k a_k\}$ . Note that  $W$  is a linear isometry that maps  $H$  onto  $l^2$ . A straightforward computation verifies that:

$$WSW^{-1} = 2S^-; \quad WBW^{-1} = \frac{1}{2}B^-; \quad \text{so, } WTW^{-1} = 2S^- + \frac{1}{2}B^-.$$

These operators act on  $l^2$ . The spectrums of these operators have been computed; see [N, Prop. 2]. Using this result, we have that  $\sigma(WTW^{-1}) = \Gamma \equiv \{\text{all the numbers in the complex plane which are inside or on the ellipse } \frac{4}{25}x^2 + \frac{4}{9}y^2 = 1\}$ . Thus,

$$\sigma(T) = \Gamma \supseteq [-2, 2] = \sigma(T^-).$$

We note that in contrast, the weighted shift and weighted backward shift,  $SR^{1/2}$  and  $BR^{1/2}$ , have the same spectral properties on  $H$  and  $H^-$ .

**Example 3.** Let  $H, H^-$ , and  $R$  be as in Example 2. We construct an example of an operator  $W \in B(H)$  such that  $T = WR^{1/2}$  does not have an adjoint in  $B(H)$  with respect to the inner-product  $\langle a, b \rangle$ . Not only does  $T$  have a bounded extension  $T^-$  on  $H^-$  [Corollary], but also  $T$  and  $T^-$  satisfy the consequences I, II, and III.

Let  $e_k$  denote the vector in  $H$  with  $k$ th coordinate 1 and with all other coordinates 0. Note that  $\|e_k\|_H = 2^k$  for all  $k$ , and that the sequence  $\{2^{-k}e_k\}_{k \geq 1}$  is an orthonormal basis for  $H$ . Define  $W$  on this sequence by  $W(e_k) = 0$  if  $k \neq m^2$  for  $m \geq 1$ , and  $W(2^{-m^2}e_{m^2}) = 2^{-m}e_m$ , otherwise. Clearly  $W \in B(H)$ . Now we show that  $RWR^{1/2}R^{-1}$  is not bounded on  $\mathbf{R}(R)$ , so part (1) of the Theorem cannot hold for  $T$ . Since  $R^{-1/2}(e_k) = 2^k e_k$ ,

$$RWR^{-1/2}(2^{-m^2}e_{m^2}) = RW(e_{m^2}) = R(2^{(m^2-m)}e_m) = 4^{-m}2^{(m^2-m)}e_m.$$

Finally,  $\|4^{-m}2^{(m^2-m)}e_m\|_H = 4^{-m}2^{m^2} \rightarrow \infty$  as  $m \rightarrow \infty$ . This proves that  $T = WR^{1/2}$  does not have an adjoint in  $B(H)$  with respect to the inner-product  $\langle a, b \rangle$ .

#### REFERENCES

- [B1] B. Barnes, *The spectral and Fredholm theory of extensions of bounded linear operators*, Proc. Amer. Math. Soc. **105** (1989), 941–949. MR **89i**:47008
- [B2] B. Barnes, *Essential spectra in a Banach algebra applied to linear operators*, Proc. of the Royal Irish Acad. 90A (1990), 73–82. MR **93b**:46087
- [B3] B. Barnes, *Common operator properties of the linear operators  $RS$  and  $SR$* , Proc. Amer. Math. Soc. **126** (1998), 1055–1061. MR **98f**:47003
- [D] R. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415. MR **34**:3315
- [I] V. Istratescu, *Introduction to Linear Operator Theory*, Marcel-Dekker, New York, 1981. MR **83d**:47002
- [L] P. Lax, *Symmetrizable linear transformations*, Comm. Pure and Applied Math. **7** (1954), 633–647. MR **16**:823d
- [N] J. Nieto, *On the essential spectrum of symmetrizable operators*, Math. Annalen **178** (1968), 145–153. MR **38**:1544
- [Z] A. C. Zaanen, *Linear Operators*, North-Holland Pub. Co., Amsterdam, 1953.

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