THE SPECTRAL PROPERTIES OF CERTAIN LINEAR OPERATORS AND THEIR EXTENSIONS

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Abstract. Let $H$ be a Hilbert space with inner-product $(x, y)$, and let $R$ be a bounded positive operator on $H$ which determines an inner-product, $(x, y) = (Rx, y)$, $x, y \in H$. Denote by $H^-$ the completion of $H$ with respect to the norm $\|x\| = (x, x)^{1/2}$. In this paper, operators having certain relationships with $R$ are studied. In particular, if $T = SR^{1/2}$ where $S \in B(H)$, then $T$ has an extension $T^- \in B(H^-)$, and $T$ and $T^-$ have essentially the same spectral and Fredholm properties.

Introduction

Throughout this paper, $H$ is a Hilbert space with inner-product $(x, y)$ and norm $\|x\|_H = (x, x)^{1/2}$. Assume that $(x, y)$ is a bounded inner-product on $H$, so there exists $c > 0$ such that for all $x, y \in H$: $|\langle x, y \rangle| \leq c\|x\|_H\|y\|_H$. Let $\|x\| = (x, x)^{1/2}$, and let $H^-$ be the completion of $H$ with respect to the norm $\|x\|$. Since the inner-product $(x, y)$ is bounded, it is well-known that there exists a positive operator $R \in B(H)$ such that

\[ \langle x, y \rangle = (Rx, y) \quad \text{for all } x, y \in H. \]

For future reference we note that

\[ \|R^{1/2}x\|_H = \|x\| \quad \text{for all } x \in H. \]

Early work in this setting centered on operators of the form $T = SR$ where $S = S^* \in B(H)$; see Chapters 15–17 of [Z]. For such $T$, $(Tx, y) = (RSRx, y) = (Rx, SRy) = \langle x, Ty \rangle$ for all $x, y \in H$. An operator $T$ is symmetrizable with respect to an inner-product $(x, y)$, if $(Tx, y) = \langle x, Ty \rangle$ for all $x, y \in H$. Thus the operator $T = SR$ above is symmetrizable.

The concept of a symmetrizable operator makes sense whenever there is a bounded inner-product on a Banach space. P. Lax studied symmetrizable operators in this more general setting in [L]. He proved that when $T$ is symmetrizable, then $T$ has an extension $T^- \in B(H^-)$ and $\sigma(T) \supseteq \sigma(T^-)$. Istratescu’s book [I, Chapter 11] is a good source of information about symmetrizable operators and related ideas.

Here we restrict attention to the case where the underlying space is the Hilbert space $H$. Our main results concern operators of the form $T = SR^{1/2}$ where $S$ is
an arbitrary operator in \( B(H) \). It is shown that \( T = SR^{1/2} \) has an extension to an operator \( T^{-} \in B(H^{-}) \), and that \( T \) and \( T^{-} \) have essentially the same basic operator properties (for example, they have the same spectrum).

**Results**

We use the notation from the Introduction in what follows. In particular, \( R \) is the positive operator determined by the bounded inner-product \( \langle x,y \rangle \). We use the fact that \( R(R^{1/2}) \) is dense in \( H \) (here, and in what follows, \( R(S) \) denotes the range of the operator \( S \)).

**Theorem.** (1)-(4) are equivalent for \( T \in B(H) \):

1. \( RT^{-1}R \) is bounded on \( R(R) \);
2. there exists an operator \( S \in B(H) \) such that \( \langle Tx,y \rangle = \langle x, Sy \rangle \) for all \( x,y \in H \);
3. there exists an operator \( S \in B(H) \) such that \( RT = SR \);
4. \( T(R(R)) \subseteq R(R) \).

(5)-(8) are equivalent for \( T \in B(H) \):

5. \( R^{1/2}T^{-1/2} \) is bounded on \( R(R^{1/2}) \);
6. \( T \) has an extension to a bounded operator \( T^{-} \) on \( H^{-} \);
7. there exists an operator \( S \in B(H) \) such that \( R^{1/2}T = SR^{1/2} \);
8. \( T^{*}(R(R^{1/2})) \subseteq R(R^{1/2}) \).

(9)-(12) are equivalent for \( T \in B(H) \):

9. \( TR^{-1/2} \) is bounded on \( R(R^{1/2}) \);
10. \( T \) has an extension to a bounded operator linear operator \( T^{-} : H^{-} \to H \);
11. there exists an operator \( S \in B(H) \) such that \( T = SR^{1/2} \);
12. \( T^{*}(H) \subseteq R(R^{1/2}) \).

**Proof.** Clearly, (3) \( \Rightarrow \) (1). Suppose that (1) holds. Let \( S \) denote the bounded extension of \( RT^{-1}R \) to all of \( H \). It follows that \( RT = SR \). Therefore (3) holds.

Assume that (2) holds. Then \( (RTx,y) = (Rx, Sy) = (S^{*}Rx, y) \) for all \( x,y \in H \). Therefore, \( RT = S^{*}R \), so (3) holds. Conversely, if \( RT = S^{*}R \), then reversing the argument above, we have that (2) is true.

That (3) \( \Rightarrow \) (4) is clear. Now assume that \( T^{*}(R(R)) \subseteq R(R) \). Then \( R(T^{*}R) \subseteq R(R) \), so by the Douglas Range Inclusion Theorem \([D]\), it follows that \( T^{*}R = RS \) for some operator \( S \in B(H) \). Taking adjoints, we have \( RT = S^{*}R \), and thus (3) holds.

Assume that (5) holds. Then there exists \( M > 0 \) such that

\[
\|R^{1/2}TR^{-1/2}(R^{1/2}x)\|_{H} \leq M\|R^{1/2}x\|_{H}
\]

for all \( x \in H \). Thus by [4], \( \|Tx\| \leq M\|x\| \) for all \( x \in H \), and this implies (6). Also, this argument is reversible, so (6) \( \Rightarrow \) (5).

Again, assume that (5) holds. Let \( S \) be the bounded extension of \( R^{1/2}TR^{-1/2} \) on \( H \). It follows immediately that \( SR^{1/2} = R^{1/2}T \). Thus (7) holds. Clearly (7) \( \Rightarrow \) (5).

Again, apply the range inclusion theorem. It follows immediately that \( T^{*}R^{1/2} = R^{1/2}S \) for some operator \( S \in B(H) \). Taking adjoints in this equality we see that (7) is true.

Assume that (9) holds. Then there exists \( M > 0 \) such that \( \|TR^{-1/2}(R^{1/2}x)\|_{H} \leq M\|R^{1/2}x\|_{H} \) for all \( x \in H \). Thus by [4], \( \|Tx\|_{H} \leq M\|x\| \) for all \( x \in H \), and this implies (10). Also, this argument is reversible, so (10) \( \Rightarrow \) (9).
Again, assume that (9) holds. Let $S$ be the bounded extension of $TR^{-1/2}$ on $H$. Then $T = ST^{1/2}$. Clearly (11)$\Leftrightarrow$(9).

Finally, making use of the Range Inclusion Theorem as before, it is straightforward to check that (11)$\Leftrightarrow$(12).

**Corollary.** Assume $T = SR^{1/2}$, where $S \in B(H)$.

- (a) The operator $T$ has a bounded extension $T^- \in B(H^-)$ with the property that $T^-(H^-) \subseteq H$.
- (b) Let $E: H \to H^-$ be the continuous embedding map. $Ex = x$ for all $x$. Let $T^\sim: H^- \to H$ be as in part (10) of the Theorem. Then

  $$T^\sim \in B(H^-,H); \quad T = T^\sim E; \quad T^- = ET^\sim.$$ 

**Proof.** Since $R^{1/2}T = (R^{1/2}S^{1/2})R^{1/2}$, the operator $T$ satisfies (7) in the Theorem. Then by (7)$\Rightarrow$(6), $T$ has a bounded extension $T^-$ on $H^-$. Also by hypothesis, $T$ satisfies (11), so by (10), $T$ has a bounded extension $T^\sim \in B(H^-,H)$. Then clearly $T^\sim = ET^\sim$. It follows that $T^-(H^-) \subseteq H$. It is also clear that $T = T^\sim E$. This verifies both parts (a) and (b) of the Corollary.

For a bounded linear operator $S$, we use the notation:

- $\sigma(S)$ = the usual operator spectrum of $S$;
- $\sigma_F(S)$ = the Fredholm spectrum of $S$
  
  $$\equiv \{ \lambda \in \mathbb{C}: (\lambda - S) \text{ is not a Fredholm operator} \};$$
- $\sigma_W(S)$ = the Weyl spectrum of $S$
  
  $$\equiv \{ \lambda \in \mathbb{C}: (\lambda - S) \text{ is not a Fredholm operator of index zero} \}.$$

In what follows, $R, T, T^-$, and $T^\sim$ are as in the Corollary.

**Consequences.** In I–III below, assume $T = SR^{1/2}$, where $S \in B(H)$.

**I.** By part (a) of the Corollary, $T^-(H^-) \subseteq H$. Applying [B1] Theorem 4(2)], we have:

- (i) $\sigma(T) = \sigma(T^-)$;
- (ii) $\sigma_F(T) = \sigma_F(T^-)$;
- (iii) $\sigma_W(T) = \sigma_W(T^-)$.

Also: (iv) when $\lambda \neq 0$, $N(\lambda - T) = N(\lambda - T^-)$; here $N(W)$ denotes the null space of the operator $W$.

**II.** By part (b) of the Corollary, $T = T^\sim E$ and $T^- = ET^\sim$. Therefore, $T$ and $T^-$ have all the common operator properties described in [B3]. In particular, when $\lambda \neq 0$:

- (i) $\lambda - T$ has a pseudoinverse $\Leftrightarrow \lambda - T^-$ has a pseudoinverse [B3] Theorem 4];
- (ii) $\lambda - T$ has closed range $\Leftrightarrow \lambda - T^-$ has closed range [B3] Theorem 5];
- (iii) $\lambda$ is a pole of finite rank of the resolvent of $T$ $\Leftrightarrow \lambda$ is a pole of finite rank of the resolvent of $T^-$ [B3] Theorem 9].

**III.** Let $K$ denote the set of all linear operators $J \in B(H)$ such that $J$ is compact, and $J$ has an extension $J^\sim$ on $H^-$ which is also compact. By I (iii), $\sigma_W(T) = \sigma_W(T^-)$. It follows from [B2] Theorem 8] that

- (i) $\sigma(T + J) = \sigma(T^- + J^-)$ for all $J \in K$. 

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IV. Now assume that $T = SR$ where $S = S^* \in B(H)$. Operators of this form are common in applications. As noted in the Introduction, $(Tx, y) = (x, Ty)$ for all $x, y \in H$. Since $T = (SR^{1/2})R^{1/2}$, consequences I–III hold for $T$ and $T^-$, and in this case, $T^-$ is selfadjoint.

**Example 1.** We give an example of a common situation in analysis where the results of this paper apply. Let $\mu$ be a measure defined on some $\sigma$-algebra of subsets of a set $\Omega$. Let $w$ be a weight function, $w \in L^\infty(\mu)$, with $w(x) > 0 \mu$-a.e. Set $H = L^2(\mu)$. Consider the bounded inner-product on $H$ defined by

$$\langle f, g \rangle \equiv \int_\Omega fg^- w \, d\mu \quad (f, g \in H).$$

Let $R$ be the multiplication operator defined by: $R(f) = wf, f \in H$. Then $\langle f, g \rangle = (Rf, g)$ for all $f, g \in H$.

Now let $K(x, t)$ be a kernel that determines a bounded integral operator $S$ on $L^2(\mu)$:

$$S(f)(x) = \int_\Omega K(x, t)f(t) \, d\mu(t), \quad f \in L^2(\mu).$$

Then consequences I–III apply to the operator $T = SR$,

$$T(f)(x) = \int_\Omega K(x, t)w(t)f(t) \, d\mu(t), \quad f \in L^2(\mu).$$

When in addition $S = S^*$, then IV also applies to the operator $T$.

**Example 2.** There exist symmetrizable operators $T$ for which $\sigma(T)$ and $\sigma(T^-)$ can be very different. Now we modify an example due to J. Nieto in [N] to verify this in our particular setting. Let $H$ be the weighted $l^2$-space of sequences $\{a_k\}_{k \geq 1}$ such that $\sum_1^\infty 4^k|a_k|^2 < \infty$. The inner-product on $H$ is:

$$\langle \{a_k\}, \{b_k\} \rangle = \sum_1^\infty 4^k a_k b_k^-.$$

Consider the inner-product on $H$ defined by:

$$\langle \{a_k\}, \{b_k\} \rangle = \sum_1^\infty a_k b_k^-.$$

It is easy to check that this inner-product is bounded on $H$, and that the positive operator $R$ such that $(Ra, b) = \langle a, b \rangle$ is the multiplication operator $R(\{a_k\}) = \{4^{-k}a_k\}$.

Let $S$ and $B$ be the shift and backward shift on $H$, so $S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots); B(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$. Let $T = S + B$, and note that $T$ is selfadjoint relative to the inner-product $\langle a, b \rangle$. Let $H^-$ be the completion of $H$ relative to the norm determined by the inner-product $\langle a, b \rangle$, so $H^-$ is the usual sequence space $l^2$. Let $S^-, B^-$, and $T^-$ denote the extensions of $S, B$, and $T$ to $l^2$. The extension $T^-$ has real spectrum (in fact, $\sigma(T^-) = [-2, 2]$). Now we compute the spectrum of $T$ in $B(H)$. Let $W$: $H \to l^2$ be defined by $W(\{a_k\}) = \{2^k a_k\}$. Note that $W$ is a linear isometry that maps $H$ onto $l^2$. A straightforward computation verifies that:

$$WSW^{-1} = 2S^-; \quad WBW^{-1} = \frac{1}{2}B^-; \quad \text{so,} \quad WTW^{-1} = 2S^- + \frac{1}{2}B^-.$$
These operators act on $l^2$. The spectrums of these operators have been computed; see [N] Prop. 2. Using this result, we have that $\sigma(WTW^{-1}) = \Gamma' \equiv \{\text{all the numbers in the complex plane which are inside or on the ellipse } \frac{x^2}{25} + \frac{y^2}{4} = 1\}$. Thus,

$$\sigma(T) = \Gamma \supseteq [-2, 2] = \sigma(T^-).$$

We note that in contrast, the weighted shift and weighted backward shift, $SR^{1/2}$ and $BR^{1/2}$, have the same spectral properties on $H$ and $H^-$.  

**Example 3.** Let $H, H^-$, and $R$ be as in Example 2. We construct an example of an operator $W \in B(H)$ such that $T = WR^{1/2}$ does not have an adjoint in $B(H)$ with respect to the inner-product $\langle a, b \rangle$. Not only does $T$ have a bounded extension $T^-$ on $H^- [\text{Corollary}]$, but also $T$ and $T^-$ satisfy the consequences I, II, and III.

Let $e_k$ denote the vector in $H$ with $k$th coordinate 1 and with all other coordinates 0. Note that $\|e_k\|_H = 2^k$ for all $k$, and that the sequence $\{2^{-k}e_k\}_{k \geq 1}$ is an orthonormal basis for $H$. Define $W$ on this sequence by $W(e_k) = 0$ if $k \neq m^2$ for $m \geq 1$, and $W(2^{-m^2}e_m) = 2^{-m}e_m$, otherwise. Clearly $W \in B(H)$. Now we show that $WR^{1/2}R^{-1}$ is not bounded on $R(R)$, so part (1) of the Theorem cannot hold for $T$. Since $R^{-1/2}(e_k) = 2^k e_k$,

$$WR^{-1/2}(2^{-m^2}e_m) = RW(2^{-m^2}e_m) = R(2^{m^2}e_m) = 4^{-m^2}e_m.$$  

Finally, $\|4^{-m^2}e_m\|_H = 4^{-m^2} \to \infty$ as $m \to \infty$. This proves that $T = WR^{1/2}$ does not have an adjoint in $B(H)$ with respect to the inner-product $\langle a, b \rangle$.

**References**


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