

PARALLELIZABILITY OF COMPLEX PROJECTIVE STIEFEL MANIFOLDS

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(Communicated by Ralph Cohen)

ABSTRACT. The question of parallelizability of the complex projective Stiefel manifolds is settled.

1. INTRODUCTION

The motivation for this note derives from our work [2], where a description of the cohomology of the complex projective Stiefel manifolds $PW_{n,k}$ is given, and from our interest in the question of parallelizability of certain families of manifolds [3].

The manifold $PW_{n,k}$ is the quotient space of the free circle action on the complex Stiefel manifold $W_{n,k}$ of orthonormal k -frames in complex n -space given by $z(v_1, \dots, v_k) = (zv_1, \dots, zv_k)$. Our result is as follows.

Theorem. *If $k < n - 1$, then $PW_{n,k}$ is not stably parallelizable; $PW_{n,n-1}$ is parallelizable, except $PW_{2,1} = S^2$; and $PW_{n,n}$ is the projective unitary group, and so is parallelizable.*

2. PROOF

Consider the standard inclusion of $U(k) \times U(n - k)$ in $U(n)$ and regard $S^1 \times U(n - k)$ as the subgroup

$$\left\{ \begin{pmatrix} zI & 0 \\ 0 & A \end{pmatrix} \mid z \in S^1 \text{ and } A \in U(n - k) \right\}$$

of $U(k) \times U(n - k)$. Then the map $U(n) \rightarrow W_{n,k}$ that takes the first k columns induces a diffeomorphism

$$PW_{n,k} \cong U(n)/S^1 \times U(n - k)$$

and we obtain a principal bundle

$$(2.1) \quad PU(k) \longrightarrow PW_{n,k} \xrightarrow{q} Gr_k(\mathbb{C}^n)$$

where $Gr_k(\mathbb{C}^n)$ is the Grassmann manifold of complex k -planes in \mathbb{C}^n and q sends a given point of $PW_{n,k}$ to the k -plane generated by any k -frame representing the given point. Let ξ and ξ^\perp be the canonical k -plane and $(n - k)$ -plane bundles on

Received by the editors June 19, 1998.

1991 *Mathematics Subject Classification.* Primary 57R25.

The second author was partially supported by an NSF grant.

$Gr_k(\mathbb{C}^n)$, so that $\xi \oplus \xi^\perp \cong n$, the trivial complex bundle of dimension n . There is an isomorphism of complex bundles

$$(2.2) \quad TGr_k(\mathbb{C}^n) \cong \text{Hom}_{\mathbb{C}}(\xi, \xi^\perp),$$

where T denotes the tangent bundle. This is easy to prove as in [5, p. 169]. If P is a k -plane in \mathbb{C}^n define a map

$$\text{Hom}_{\mathbb{C}}(P, P^\perp) \longrightarrow Gr_k(\mathbb{C}^n)$$

by $\varphi \mapsto$ graph of φ . The differential at the origin is an isomorphism

$$\text{Hom}_{\mathbb{C}}(P, P^\perp) = T_0 \text{Hom}_{\mathbb{C}}(P, P^\perp) \xrightarrow{\cong} T_P Gr_k(\mathbb{C}^n)$$

and this determines a bundle isomorphism (2.2). Since (2.1) is a principal bundle, the bundle of tangents along the fibre is trivial, and we have an isomorphism of real vector bundles

$$(2.3) \quad TPW_{n,k} \cong q^*TGr_k(\mathbb{C}^n) \oplus (k^2 - 1).$$

Here $k^2 - 1$ denotes a real trivial vector bundle of dimension $k^2 - 1$.

Let L be the complex line bundle on $PW_{n,k}$ associated with the bundle $\pi : W_{n,k} \rightarrow PW_{n,k}$; let L_i be the complex line bundle whose fibre over a point of $PW_{n,k}$ is the subspace generated by the i th vector on any frame representing the point.

The total space of L is the quotient $W_{n,k} \times_{S^1} \mathbb{C}$ of $W_{n,k} \times \mathbb{C}$ by the equivalence relation that identifies (v_1, \dots, v_k, w) with $(zv_1, \dots, zv_k, z^{-1}w)$. The map

$$\varphi_i : W_{n,k} \times_{S^1} \mathbb{C} \longrightarrow PW_{n,k} \times \mathbb{C}^n$$

given by $\varphi_i(v_1, \dots, v_k, w)$ ((line generated by v_i), wv_i) is a vector bundle monomorphism with $\text{im } \varphi_i = L_i$. Thus $L_i \cong L$ for all $1 \leq i \leq k$, and so

$$(2.4) \quad q^*\xi \cong kL,$$

the Whitney sum of k copies of L , since it is clear that $q^*\xi = L_1 \oplus \dots \oplus L_k$. Then, if $\xi^* = \text{Hom}_{\mathbb{C}}(\xi, \mathbb{C})$,

$$q^*TGr_k(\mathbb{C}^n) \cong q^*\text{Hom}_{\mathbb{C}}(\xi, \xi^\perp) \cong q^*(\xi^* \otimes_{\mathbb{C}} \xi^\perp) \cong kL^* \otimes_{\mathbb{C}} q^*\xi^\perp.$$

Since $\xi \oplus \xi^\perp \cong n$ we have from (2.4)

$$(2.5) \quad kL \oplus q^*\xi^\perp \cong n$$

and so, in $KU(PW_{n,k})$,

$$(2.6) \quad \begin{aligned} q^*TGr_k(\mathbb{C}^n) &= kL^*(n - kL) \\ &= nkL^* - k^2. \end{aligned}$$

Thus, we see from (2.3) that $TPW_{n,k}$ is stably equivalent to $nkr(L^*)$, where $r(L^*)$ denotes the ‘‘realification’’ of L^* . Then the Pontrjagin class

$$(2.7) \quad p(PW_{n,k}) = p(nkr(L^*)) = (1 + x_0^2)^{nk}$$

where $x_0 = -c_1(L^*)$.

Assume $k < n - 1$. Then $W_{n,k}$ is 4-connected, so the Gysin sequence of the sphere bundle $\pi : W_{n,k} \rightarrow PW_{n,k}$ shows that $H^4(PW_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$ is generated by x_0^2 . From (2.7),

$$p_1(PW_{n,k}) = nkx_0^2 \neq 0,$$

so $PW_{n,k}$ is not stably parallelizable.

Consider the case $k = n - 1$. Then (2.5) becomes

$$(n - 1)L \oplus q^*\xi^\perp \cong n$$

and this implies

$$(2.8) \quad nL^* \cong (n - 1) \oplus E$$

where $E = L^* \otimes_{\mathbb{C}} q^*\xi^\perp$. Applying the second exterior power operator λ^2 to (2.8) we obtain

$$(2.9) \quad \begin{aligned} \binom{n}{2}(L^*)^2 &\cong \lambda^2(nL^*) \cong \lambda^2((n - 1) \oplus E) \\ &\cong \binom{n-1}{2} \oplus (n - 1)E. \end{aligned}$$

Let $u = L^* - 1 \in KU(PW_{n,n-1})$. Then a straightforward calculation using (2.8) and (2.9) shows that

$$\binom{n}{2}u^2 = 0.$$

Let

$$c : KO(PW_{n,n-1}) \longrightarrow KU(PW_{n,n-1})$$

be the “complexification” homomorphism; it satisfies $cr(F) = F + F^*$ for any complex vector bundle F .

Then

$$\begin{aligned} cr(u) &= L^* + L - 2 \\ &= 1 + u + \frac{1}{1 + u} - 2 \\ &= \frac{u^2}{1 + u}, \end{aligned}$$

so that $c(\binom{n}{2}r(u)) = 0$. Since $\ker c$ is well known to consist of elements of order 2 we obtain

$$n(n - 1)r(u) = 0$$

in $KO(PW_{n,n-1})$. By (2.3) and (2.6) we now have

$$TPW_{n,n-1} - \dim PW_{n,n-1} = n(n - 1)r(u) = 0,$$

so $PW_{n,n-1}$ is stably parallelizable.

To show $PW_{n,n-1}$ is parallelizable for $n > 2$ we appeal to a theorem proved in [6] and [4] that states that if an $(m - 1)$ -dimensional manifold is stably parallelizable, then either it is parallelizable or it admits exactly the same number of linearly independent tangent vector fields as the $(m - 1)$ -sphere. This immediately implies the parallelizability of $PW_{3,2}$, since it has dimension seven and S^7 is parallelizable. By [1] the $(m - 1)$ -sphere admits exactly $\rho(m) - 1$ linearly independent tangent vector fields, where ρ is the numerical function given by $\rho(m) = 8a + 2^b$ if $m = 2^{4a+b}$ (odd integer) with $0 \leq b \leq 3$. The parallelizability of $PW_{n,n-1}$ for $n > 3$ is a consequence of (2.3) and the following lemma.

Lemma 2.10. *If $n > 3$, then $\rho(n^2 - 1) < (n - 1)^2$.*

Proof. Note first that if n is even, then $\rho(n^2 - 1) = 1$, so the lemma holds in this case. If $n = 2m + 1$, then $m > 1$ and we must prove that

$$\rho(4m(m + 1)) < 4m^2.$$

Let ν be the numerical function defined by $t = 2^{\nu(t)}$ (odd integer). It is easy to verify that $\rho(t) \leq 2\nu(t) + 2$ and $\nu(t) \leq t - 1$ for all $t > 0$. Then

$$\begin{aligned} \rho(4m(m+1)) &\leq 2\nu(4m(m+1)) + 2 \\ &= 2(2 + \nu(m) + \nu(m+1)) + 2 \\ &\leq 4m + 4 \\ &< 4m^2, \end{aligned}$$

where the last inequality holds because $m > 1$. □

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