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PARALLELIZABILITY OF COMPLEX PROJECTIVE STIEFEL MANIFOLDS

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ABSTRACT. The question of parallelizability of the complex projective Stiefel manifolds is settled.

1. INTRODUCTION

The motivation for this note derives from our work [2], where a description of the cohomology of the complex projective Stiefel manifolds $PW_{n,k}$ is given, and from our interest in the question of parallelizability of certain families of manifolds [3].

The manifold $PW_{n,k}$ is the quotient space of the free circle action on the complex Stiefel manifold $W_{n,k}$ of orthonormal k-frames in complex n-space given by $z(v_1, \dots, v_k) = (zv_1, \dots, zv_k)$. Our result is as follows.

Theorem. *If $k < n-1$, then $PW_{n,k}$ is not stably parallelizable; $PW_{n,n-1}$ is parallelizable, except $PW_{2,1} = S^2$; and $PW_{n,n}$ is the projective unitary group, and so is parallelizable.*

2. PROOF

Consider the standard inclusion of $U(k) \times U(n-k)$ in $U(n)$ and regard $S^1 \times U(n-k)$ as the subgroup

$$\left\{ \begin{pmatrix} zI & 0 \\ 0 & A \end{pmatrix} \Big| z \in S^1 \text{ and } A \in U(n-k) \right\}$$

of $U(k) \times U(n-k)$. Then the map $U(n) \longrightarrow W_{n,k}$ that takes the first k columns induces a diffeomorphism

$$PW_{n,k} \cong U(n)/S^1 \times U(n-k)$$

and we obtain a principal bundle

$$PU(k) \longrightarrow PW_{n,k} \xrightarrow{q} Gr_k(\mathbb{C}^n) \tag{2.1}$$

where $Gr_k(\mathbb{C}^n)$ is the Grassmann manifold of complex k–planes in $\mathbb{C}^n$ and q sends a given point of $PW_{n,k}$ to the k–plane generated by any k–frame representing the given point. Let ξ and $\xi^\perp$ be the canonical k–plane and $(n-k)$–plane bundles on

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$Gr_k(\mathbb{C}^n)$, so that $\xi \oplus \xi^\perp \cong n$, the trivial complex bundle of dimension n. There is an isomorphism of complex bundles

$$TGr_k(\mathbb{C}^n) \cong \operatorname{Hom}_{\mathbb{C}}(\xi, \xi^\perp), \tag{2.2}$$

where T denotes the tangent bundle. This is easy to prove as in [5, p. 169]. If P is a k–plane in $\mathbb{C}^n$ define a map

$$\operatorname{Hom}_{\mathbb{C}}(P, P^\perp) \longrightarrow Gr_k(\mathbb{C}^n)$$

by $\varphi \mapsto$ graph of φ. The differential at the origin is an isomorphism

$$\operatorname{Hom}_{\mathbb{C}}(P, P^\perp) = T_0 \operatorname{Hom}_{\mathbb{C}}(P, P^\perp) \xrightarrow{\cong} T_P Gr_k(\mathbb{C}^n)$$

and this determines a bundle isomorphism (2.2). Since (2.1) is a principal bundle, the bundle of tangents along the fibre is trivial, and we have an isomorphism of real vector bundles

$$TPW_{n,k} \cong q^* TGr_k(\mathbb{C}^n) \oplus (k^2 - 1). \tag{2.3}$$

Here $k^2 - 1$ denotes a real trivial vector bundle of dimension $k^2 - 1$.

Let L be the complex line bundle on $PW_{n,k}$ associated with the bundle $\pi : W_{n,k} \to PW_{n,k}$; let L_i be the complex line bundle whose fibre over a point of $PW_{n,k}$ is the subspace generated by the ith vector on any frame representing the point.

The total space of L is the quotient $W_{n,k} \times_{S^1} \mathbb{C}$ of $W_{n,k} \times \mathbb{C}$ by the equivalence relation that identifies $(v_1, \ldots, v_k, w)$ with $(zv_1, \ldots, zv_k, z^{-1}w)$. The map

$$\varphi_i : W_{n,k} \times_{S^1} \mathbb{C} \longrightarrow PW_{n,k} \times \mathbb{C}^n$$

given by $\varphi_i(v_1, \ldots, v_k, w)$ ((line generated by v_i), wv_i) is a vector bundle monomorphism with $\operatorname{im} \varphi_i = L_i$. Thus $L_i \cong L$ for all $1 \leq i \leq k$, and so

$$q^* \xi \cong kL, \tag{2.4}$$

the Whitney sum of k copies of L, since it is clear that $q^*\xi = L_1 \oplus \cdots \oplus L_k$. Then, if $\xi^* = \operatorname{Hom}_{\mathbb{C}}(\xi, \mathbb{C})$,

$$q^* TGr_k(\mathbb{C}^n) \cong q^* \operatorname{Hom}_{\mathbb{C}}(\xi, \xi^\perp) \cong q^*(\xi^* \otimes_{\mathbb{C}} \xi^\perp) \cong kL^* \otimes_{\mathbb{C}} q^* \xi^\perp.$$

Since $\xi \oplus \xi^\perp \cong n$ we have from (2.4)

$$kL \oplus q^* \xi^\perp \cong n \tag{2.5}$$

and so, in $KU(PW_{n,k})$,

$$\begin{aligned} q^* TGr_k(\mathbb{C}^n) &= kL^*(n - kL) \\ &= nkL^* - k^2. \end{aligned} \tag{2.6}$$

Thus, we see from (2.3) that $TPW_{n,k}$ is stably equivalent to $nkr(L^*)$, where $r(L^*)$ denotes the "realification" of L^*. Then the Pontrjagin class

$$p(PW_{n,k}) = p(nkr(L^*)) = (1 + x_0^2)^{nk} \tag{2.7}$$

where $x_0 = -c_1(L^*)$.

Assume $k < n - 1$. Then $W_{n,k}$ is 4–connected, so the Gysin sequence of the sphere bundle $\pi : W_{n,k} \longrightarrow PW_{n,k}$ shows that $H^4(PW_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$ is generated by x_0^2. From (2.7),

$$p_1(PW_{n,k}) = nkx_0^2 \neq 0,$$

so $PW_{n,k}$ is not stably parallelizable.

Consider the case $k = n-1$. Then (2.5) becomes

$$(n-1)L \oplus q^*\xi^{\perp} \cong n$$

and this implies

$$nL^* \cong (n-1) \oplus E \tag{2.8}$$

where $E = L^* \otimes_{\mathbb{C}} q^*\xi^{\perp}$. Applying the second exterior power operator λ^2 to (2.8) we obtain

$$\begin{aligned} \tbinom{n}{2}(L^*)^2 \cong \lambda^2(nL^*) &\cong \lambda^2((n-1) \oplus E) \\ &\cong \tbinom{n-1}{2} \oplus (n-1)E. \end{aligned} \tag{2.9}$$

Let $u = L^* - 1 \in KU(PW_{n,n-1})$. Then a straightforward calculation using (2.8) and (2.9) shows that

$$\tbinom{n}{2}u^2 = 0.$$

Let

$$c : KO(PW_{n,n-1}) \longrightarrow KU(PW_{n,n-1})$$

be the "complexification" homomorphism; it satisfies $cr(F) = F + F^*$ for any complex vector bundle F.

Then

$$\begin{aligned} cr(u) &= L^* + L - 2 \\ &= 1 + u + \frac{1}{1+u} - 2 \\ &= \frac{u^2}{1+u}, \end{aligned}$$

so that $c(\binom{n}{2}r(u)) = 0$. Since $\ker c$ is well known to consist of elements of order 2 we obtain

$$n(n-1)r(u) = 0$$

in $KO(PW_{n,n-1})$. By (2.3) and (2.6) we now have

$$TPW_{n,n-1} - \dim PW_{n,n-1} = n(n-1)r(u) = 0,$$

so $PW_{n,n-1}$ is stably parallelizable.

To show $PW_{n,n-1}$ is parallelizable for $n > 2$ we appeal to a theorem proved in [6] and [4] that states that if an $(m-1)$–dimensional manifold is stably parallelizable, then either it is parallelizable or it admits exactly the same number of linearly independent tangent vector fields as the $(m-1)$–sphere. This immediately implies the parallelizability of $PW_{3,2}$, since it has dimension seven and S^7 is parallelizable. By [1] the $(m-1)$–sphere admits exactly $\rho(m) - 1$ linearly independent tangent vector fields, where ρ is the numerical function given by $\rho(m) = 8a + 2^b$ if $m = 2^{4a+b} \cdot$ (odd integer) with $0 \le b \le 3$. The parallelizability of $PW_{n,n-1}$ for $n > 3$ is a consequence of (2.3) and the following lemma.

Lemma 2.10. *If $n > 3$, then $\rho(n^2 - 1) < (n-1)^2$.*

Proof. Note first that if n is even, then $\rho(n^2 - 1) = 1$, so the lemma holds in this case. If $n = 2m + 1$, then $m > 1$ and we must prove that

$$\rho(4m(m+1)) < 4m^2.$$

Let ν be the numerical function defined by $t = 2^{\nu(t)}\cdot$(odd integer). It is easy to verify that $\rho(t) \leq 2\nu(t) + 2$ and $\nu(t) \leq t - 1$ for all $t > 0$. Then

$$\begin{aligned}\rho(4m(m+1)) &\leq 2\nu(4m(m+1)) + 2\\ &= 2(2 + \nu(m) + \nu(m+1)) + 2\\ &\leq 4m + 4\\ &< 4m^2,\end{aligned}$$

where the last inequality holds because $m > 1$. □

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