

## A RELATION BETWEEN CERTAIN INTERPOLATED CUNTZ ALGEBRAS AND INTERPOLATED FREE GROUP FACTORS

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ABSTRACT. We investigate von Neumann algebras generated by the real parts of generators of Toeplitz extensions of interpolated Cuntz algebras  $\mathcal{O}_\beta$  on sub-Fock spaces. We show that some of them are isomorphic to interpolated free group factors  $L(F_r)$ . For example, in case of the golden number  $\beta = \frac{1+\sqrt{5}}{2}$  the corresponding number  $r$  is  $\frac{3}{2}$ .

### 1. INTRODUCTION

In this paper we shall investigate von Neumann algebras  $\mathcal{F}_\beta$  generated by the real parts of generators of Toeplitz extensions of interpolated Cuntz algebras  $\mathcal{O}_\beta$  on sub-Fock spaces.

K. Matsumoto introduced a class of  $C^*$ -algebras  $\mathcal{O}_\Lambda$  associated with subshift  $\Lambda$  in [11]. His class of  $C^*$ -algebras is a generalization of that of Cuntz-Krieger algebras which are associated with topological Markov shifts [3]. He showed that many of them are purely infinite and simple. He also calculated K-theory for these  $C^*$ -algebras  $\mathcal{O}_\Lambda$  in [12].

In [10] Y. Katayama, K. Matsumoto and the second-named author defined interpolated Cuntz algebras  $\mathcal{O}_\beta$  as the  $C^*$ -algebras  $\mathcal{O}_{\Lambda_\beta}$  associated with  $\beta$ -shift arising from  $\beta$ -expansion of real-numbers. Evans-Kishimoto [8] showed that the trace scaling automorphisms of a stable AF-algebra  $A$  are outer conjugate if the scaling factors are the same. The crossed product  $A \rtimes \mathbb{Z}$  is stably isomorphic to the  $C^*$ -algebra  $\mathcal{O}_\beta$ .

We shall consider von Neumann algebras  $\mathcal{F}_\beta$  generated by the real parts of generators of Toeplitz extensions of  $\mathcal{O}_\beta$  on sub-Fock spaces. Suppose that  $\beta$  is a positive integer  $n$ . Then  $\mathcal{O}_\beta$  is identified with the Cuntz algebra  $\mathcal{O}_n$  and the von Neumann algebra  $\mathcal{F}_n$  generated by the real parts of the creation operators on a full Fock space is known to be isomorphic to the free group factor  $L(\mathbb{F}_n)$  of  $n$  generators by D. Voiculescu [16].

In this article we shall show that some of these von Neumann algebras  $\mathcal{F}_\beta$  are isomorphic to interpolated free group factors  $L(\mathbb{F}_r)$  introduced by K. Dykema [4] and F. Rădulescu [14]. For example, in case of the golden number  $\beta = \frac{1+\sqrt{5}}{2}$  the corresponding number  $r$  is  $\frac{3}{2}$ .

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The  $q$ -deformation of the Cuntz algebras generated by  $q$ -canonical commutation relations are studied by Jørgensen-Schmitt-Werner [9] and Dykema-Nica [7], and von Neumann algebras generated by the real parts of creation operators in that setting are investigated by Bożejko-Kümmerer-Speicher [1]. Their objects are interesting but different from ours.

2.  $C^*$ -ALGEBRAS  $\mathcal{O}_\beta$  ASSOCIATED WITH  $\beta$ -SHIFT

We recall the construction of  $C^*$ -algebras associated with subshifts. Fix a finite set  $\Sigma = \{1, 2, \dots, n\}$ , and let  $\Sigma^{\mathbb{Z}}$  and  $\Sigma^{\mathbb{N}}$  be the infinite product spaces endowed with the product topology. We denote by  $\sigma$  the shift on  $\Sigma^{\mathbb{Z}}$  and  $\Sigma^{\mathbb{N}}$  defined by  $(\sigma(x))_i = x_{i+1}$ . For a shift invariant closed subset  $\Lambda$  of  $\Sigma^{\mathbb{Z}}$ , the topological dynamical system  $(\Lambda, \sigma|_\Lambda)$  is called a subshift. We denote  $\sigma|_\Lambda$  by  $\sigma$  for simplicity. A finite sequence  $\mu = (\mu_1, \dots, \mu_k)$  of elements  $\mu_j \in \Sigma$  is called a block or a word. We denote by  $|\mu|$  the length  $k$  of  $\mu$ . For  $k \in \mathbb{N}$ , let

$$\Lambda_\beta^k = \{\mu | \mu \text{ is a block with length } k \text{ appearing in some } x \in \Lambda\}$$

and  $\Lambda^* = \bigcup_{k=0}^\infty \Lambda^k$ , where  $\Lambda^0$  denotes the set of the empty word.

Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of a  $n$ -dimensional Hilbert space  $H = \mathbb{C}^n$ . Let  $F^0$  be the one dimensional space  $\mathbb{C}\Omega$  spanned by a vector  $\Omega$ , called a vacuum vector, and  $F^k$  be the Hilbert space spanned by the vectors  $e_\mu = e_{\mu_1} \otimes \dots \otimes e_{\mu_k}$  for  $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$ . Consider a subspace  $F_\Lambda = \bigoplus_{k=0}^\infty F^k$  of the full Fock space  $F(H)$ .

The creation operator  $T_\nu$  by  $e_\nu$  on  $F_\Lambda$  for  $\nu \in \Lambda^*$  is defined by

$$T_\nu \Omega = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu & (\text{if } \nu\mu \in \Lambda^*), \\ 0 & \text{otherwise.} \end{cases}$$

Then  $T_\nu$  is a partial isometry. We put  $T_\phi = 1$  for the empty word  $\phi$ . Let  $\mathbb{P}_\Omega$  be the rank-one projection onto the vacuum vector  $\Omega$ . Then  $\sum_{i=1}^n T_i T_i^* + \mathbb{P}_\Omega = 1$ . The unital  $C^*$ -algebra  $\mathcal{T}_\Lambda$  generated by  $\{T_i | i = 1, \dots, n\}$  is called the Toeplitz algebra and contains the algebra  $\mathcal{K}(F_\Lambda)$  of compact operators on  $F_\Lambda$ . The  $C^*$ -algebra  $\mathcal{O}_\Lambda$  associated with  $\Lambda$  is defined as the quotient  $C^*$ -algebra  $\mathcal{T}_\Lambda / \mathcal{K}(F_\Lambda)$  of  $\mathcal{T}_\Lambda$  by  $\mathcal{K}(F_\Lambda)$ . We denote by  $S_\mu$  the quotient image of  $T_\mu$  for  $\mu \in \Lambda^*$ .

If  $\Lambda$  is a Markov shift determined by a 0-1 matrix  $A$ , then  $\mathcal{O}_\Lambda$  is nothing but the Cuntz-Krieger algebra  $\mathcal{O}_A$ . In particular, if  $\Lambda$  is a full  $n$ -shift,  $\mathcal{O}_\Lambda$  is isomorphic to the Cuntz algebra  $\mathcal{O}_n$ .

Fix an arbitrary real number  $\beta > 1$  unless we specify. We recall the interpolated Cuntz algebra  $\mathcal{O}_\beta$ . Take a natural number  $n$  with  $n < \beta \leq n + 1$ . Put  $\Sigma = \{0, 1, \dots, n\}$ . For a positive number  $t$ , we denote by  $[t]$  the integer part of  $t$ . Let  $f_\beta : [0, 1] \rightarrow [0, 1]$  be a map defined by

$$f_\beta(x) = \beta x - [\beta x], \quad x \in [0, 1].$$

The  $\beta$ -expansion of  $x \in [0, 1]$  is a sequence  $\{d_i(x, \beta)\}_{i \in \mathbb{N}}$  of integers of  $\Sigma$  determined by

$$d_i(x, \beta) = [\beta f_\beta^{i-1}(x)], \quad i \in \mathbb{N}.$$

The numbers  $d_i(x, \beta)$  will be denoted by  $d_i(x)$  for simplicity. By this sequence, we can write  $x$  as

$$x = \sum_{i=1}^{\infty} \frac{d_i(x)}{\beta^i}.$$

We endow the infinite product  $\Sigma^{\mathbb{N}}$  with the product topology and the lexicographical order. Put

$$\zeta_{\beta} = \sup_{x \in [0,1)} (d_i(x))_i \in \Sigma^{\mathbb{N}}.$$

We define the  $\sigma$ -invariant compact subset  $\Lambda_{\beta}$  of  $\Sigma^{\mathbb{N}}$  by

$$\Lambda_{\beta} = \{\omega \in \Sigma^{\mathbb{N}} \mid \sigma^i(\omega) \leq \zeta_{\beta}, i = 0, 1, 2, \dots\}.$$

The subshift  $(\Lambda_{\beta}, \sigma)$  is called the  $\beta$ -shift (cf. [15] and [13]). We denote by  $\mathcal{O}_{\beta}$  the  $C^*$ -algebra  $\mathcal{O}_{\Lambda_{\beta}}$  associated with  $\beta$ -shift  $\Lambda_{\beta}$ .

**Example 1.** If  $\beta = n \in \mathbb{N}$ , then  $\zeta_{\beta} = (n - 1, n - 1, \dots)$  and  $\Lambda_n = \Sigma^{\mathbb{N}}$  is the  $n$ -full shift. Hence  $\mathcal{O}_{\beta}$  is isomorphic to  $\mathcal{O}_n$ .

**Example 2.** If  $\beta = \frac{1+\sqrt{5}}{2}$  (the golden number), then  $\zeta_{\beta} = (1, 0, 1, 0, 1, 0, \dots)$  and  $n = 1$ . We see

$$\Lambda_{\frac{1+\sqrt{5}}{2}} = \{(x_i) \in \prod_{i=1}^{\infty} \{0, 1\} \mid \text{“11” does not appear in } (x_i)\}.$$

Therefore  $\Lambda_{\beta}$  is a topological Markov shift  $\Lambda_A$  determined by the matrix  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ . Hence the  $C^*$ -algebra  $\mathcal{O}_{\beta}$  is isomorphic to the Cuntz-Krieger algebra  $\mathcal{O}_A$ .

### 3. FREE PRODUCTS OF EXTENSIONS OF CUNTZ-KRIEGER ALGEBRAS

Let  $\mathcal{T}_n$  be the Toeplitz algebra for a  $n$ -dimensional Hilbert space  $H_n = \mathbb{C}^n$ , that is, the  $C^*$ -algebra generated by creation operators  $\{T_{\xi} \mid \xi \in H_n\}$  on the full Fock space  $F(H_n)$ . The vacuum state  $\omega_n$  is defined as the vector state of the vacuum vector  $\Omega_n \in F(H_n)$ . In [16] D. Voiculescu pointed out the following isomorphism:

$$(\mathcal{T}_n, \omega_n) * (\mathcal{T}_m, \omega_m) \cong (\mathcal{T}_{n+m}, \omega_{n+m}).$$

Motivated by the fact we shall prepare a similar formula for Toeplitz extensions of Cuntz-Krieger algebras.

Let  $A = [A(i, j)]$  (resp.  $B = [B(i, j)]$ ) be a  $n \times n$  (resp.  $m \times m$ ) matrix with 0-1 entries over an index set  $\Sigma_A$  (resp.  $\Sigma_B$ ). We shall assume that no row and no column of  $A$  and  $B$  is identically zero. We define a new matrix  $A * B$  over the index set  $\Sigma_{A*B} = \Sigma_A \cup \Sigma_B$  (the disjoint union) by

$$A * B = \begin{bmatrix} A & J_1 \\ J_2 & B \end{bmatrix},$$

where  $J_1, J_2$  are the matrices whose entries are all 1.

Before going to the statement we briefly recall the Toeplitz extension (acting on a sub-Fock space) of  $\mathcal{O}_A$  (or  $\mathcal{O}_B, \mathcal{O}_{A*B}$ ). The sub-Fock space  $F_A$  and the Toeplitz algebra (or Toeplitz extension of  $\mathcal{O}_A$  on the sub-Fock space)  $\mathcal{T}_A$  associated with  $A$

are defined as those associated with the topological Markov shift  $\Lambda_A$  determined by the matrix  $A$  (see §2). Notice here that in this case the creation operator  $T_i$  ( $= T_{e_i}$ ) can be explicitly described by using the matrix  $A$  as follows:

$$(1) \quad T_i \Omega_A = e_i, \quad T_i e_\mu = A(i, \mu_1) e_i \otimes e_\mu.$$

The vacuum state  $\omega_A$  is defined as the vector state of the vacuum vector  $\Omega_A$ . Finally, we note that the vacuum state is not faithful but the G.N.S. representation is faithful and unitary equivalent to the identity representation on the sub-Fock space as in the Cuntz algebra case.

**Proposition 1.** *We have*

$$(\mathcal{T}_A, \omega_A) * (\mathcal{T}_B, \omega_B) \cong (\mathcal{T}_{A*B}, \omega_{A*B}).$$

*Proof.* By the definition of  $C := A * B$  we can construct a unitary operator:

$$U : (F_A, \Omega_A) * (F_B, \Omega_B) \rightarrow (F_C, \Omega_C)$$

in such a way that  $U\Omega = \Omega_C$  ( $\Omega$  is denoted by the reference vector of the free product Hilbert space) and

$$U(e_{\alpha_1} \otimes e_{\beta_1} \otimes \cdots) = e_{(\alpha_1, \beta_1, \dots)}, \quad U(e_{\beta_1} \otimes e_{\alpha_1} \otimes \cdots) = e_{(\beta_1, \alpha_1, \dots)}$$

for  $\alpha_1, \dots \in \Lambda_A^*$ ,  $\beta_1, \dots \in \Lambda_B^*$ . The identity representations of  $\mathcal{T}_A$ ,  $\mathcal{T}_B$  and  $\mathcal{T}_C$  on the sub-Fock spaces  $F_A$ ,  $F_B$  and  $F_C$  are denoted by  $\pi_A$ ,  $\pi_B$  and  $\pi_C$  respectively. Denote, for  $X = A$  or  $B$ ,

$$H(X) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \left( \bigoplus_{\substack{X_1 \neq \dots \neq X_n, X_1 \neq X \\ X_j \in \{A, B\}}} F_{X_1}^\circ \otimes \dots \otimes F_{X_n}^\circ \right)$$

( $F_X^\circ := F_X \ominus \mathbb{C}\Omega_X$ ), and define a unitary operator  $V_X : F_X \otimes H(X) \rightarrow F_A * F_B$  as usual (see [16]). Let  $\lambda_A, \lambda_B$  be the free product representations of  $\mathcal{T}_A, \mathcal{T}_B$  on the free product Hilbert space  $F_A * F_B$  defined by

$$\lambda_A(T) = V_A(\pi_A(T) \otimes 1_{H(A)})V_A^*, \quad \lambda_B(S) = V_B(\pi_B(S) \otimes 1_{H(B)})V_B^*$$

for  $T \in \mathcal{T}_A, S \in \mathcal{T}_B$ . Then by direct computations we obtain

$$U\lambda_A(T_i)U^* = \pi_C(T_i), \quad U\lambda_B(T_j)U^* = \pi_C(T_j)$$

for  $i \in \Sigma_A, j \in \Sigma_B$ . For example, let  $\mu = (\mu_1, \dots, \mu_r) \in (\Sigma_A \cup \Sigma_B)^r$ . If  $\mu_1, \dots, \mu_{k-1} \in \Sigma_A$  and  $\mu_k \in \Sigma_B$  ( $2 \leq k \leq r$ ), then  $e_{\mu_1} \otimes \dots \otimes e_{\mu_{k-1}} \in F_A(H_A)^\circ, e_{\mu_k} \otimes \dots \otimes e_{\mu_r} \in H(A)$  and for  $i \in \Sigma_A$

$$\begin{aligned} & U\lambda_A(T_i)U^*(e_{\mu_1} \otimes \dots \otimes e_{\mu_r}) \\ &= U\lambda_A(T_i)((e_{\mu_1} \otimes \dots \otimes e_{\mu_{k-1}}) \otimes (e_{\mu_k} \otimes \dots \otimes e_{\mu_r})) \\ &= U((A(i, \mu_1)e_i \otimes e_{\mu_1} \otimes \dots \otimes e_{\mu_{k-1}}) \otimes (e_{\mu_k} \otimes \dots \otimes e_{\mu_r})) \\ &\quad (\text{since } (\pi_A(T_i)(e_{\mu_1} \otimes \dots \otimes e_{\mu_{k-1}})) \text{ is orthogonal to } \Omega_A \text{ and by (1)}) \\ &= A(i, \mu_1)e_i \otimes e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \\ &= C(i, \mu_1)e_i \otimes e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \quad (\text{since } i, \mu_1 \in \Sigma_A) \\ &= \pi_C(T_i)(e_{\mu_1} \otimes \dots \otimes e_{\mu_r}). \end{aligned}$$

If  $\mu_1, \dots, \mu_r \in \Sigma_B$ , then  $C(i, \mu_1) = 1$  and

$$\begin{aligned} & U\lambda_A(T_i)U^*(e_{\mu_1} \otimes \dots \otimes e_{\mu_r}) \\ &= U(e_i \otimes e_{\mu_1} \otimes \dots \otimes e_{\mu_r}) \quad (\text{since } \pi_A(T_i)\Omega_A \text{ is orthogonal to } \Omega_A) \\ &= C(i, \mu_1)e_i \otimes e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \\ &= \pi_C(T_i)(e_{\mu_1} \otimes \dots \otimes e_{\mu_r}). \end{aligned}$$

The other cases can be similarly checked. Therefore, we obtain the desired isomorphism via  $\text{Ad}U$ . □

**Corollary 2.** *Let  $A_1, \dots, A_k$  be a finite sequence of matrices with 0-1 entries. Then we have*

$$(\mathcal{T}_{A_1}, \omega_{A_1}) * \dots * (\mathcal{T}_{A_k}, \omega_{A_k}) \cong (\mathcal{T}_C, \omega_C).$$

Here  $C$  (denoted by  $A_1 * \dots * A_k$ ) has block diagonals  $A_1, \dots, A_k$  and all the other entries of off block diagonals are 1.

#### 4. VON NEUMANN ALGEBRAS ASSOCIATED WITH CERTAIN $\beta$ -SHIFTS

In this section, we shall investigate von Neumann algebras  $\mathcal{F}_\beta$  generated by the real parts of creation operators associated with Toeplitz extensions of  $\mathcal{O}_\beta$  acting on sub-Fock spaces.

**Theorem 3.** *Let  $\beta > 1$  be a non-integral positive real number and  $n$  be the integer part of  $\beta$ . Suppose that  $\beta$  satisfies an algebraic equation:*

$$\beta^{k+1} = n\beta^k + n\beta^{k-1} + \dots + n\beta + n.$$

Then the von Neumann algebra  $\mathcal{F}_\beta$  generated by the real parts  $\text{Re}T_i$  of creation operators  $T_i$  ( $i = 0, \dots, n$ ) on a sub-Fock space  $F_{\Lambda_\beta}$  associated with  $\beta$ -shift  $(\Lambda_\beta, \sigma)$  is isomorphic to some interpolated free group factor  $L(\mathbb{F}_r)$ . The number  $r$  can be determined as

$$(2) \quad r = n + \sum_{\substack{1 \leq i, j \leq k+1 \\ i \neq j}} \frac{(\sin(\frac{i\pi}{k+2}))^2}{\sum_{m=1}^{k+1} (\sin(\frac{im\pi}{k+2}))^2} \frac{(\sin(\frac{j\pi}{k+2}))^2}{\sum_{m=1}^{k+1} (\sin(\frac{jm\pi}{k+2}))^2}.$$

*Proof.* Since

$$1 = \frac{n}{\beta} + \frac{n}{\beta^2} + \dots + \frac{n}{\beta^{k+1}},$$

the  $\beta$ -expansion of 1 is given by

$$d_n(1) = (\underbrace{n, n, \dots, n}_{k+1 \text{ times}}, 0, 0, \dots).$$

Thus the maximal element  $\zeta_\beta = (\xi_m)_{m=1}^\infty \in \Lambda_\beta$  is given by

$$\xi_m = \begin{cases} n & \text{if } m = 1, 2, \dots, k \pmod{k+1}, \\ n-1 & \text{if } m = 0 \pmod{k+1}, \end{cases}$$

that is,

$$\zeta_\beta = (\underbrace{n, n, \dots, n}_{k \text{ times}}, n-1, \underbrace{n, n, \dots, n}_{k \text{ times}}, n-1, n, \dots).$$



Therefore, by a result of K. Dykema (see [5], [6]) the von Neumann algebra  $\mathcal{F}_\beta$  is isomorphic to some interpolated free group factor  $L(\mathbb{F}_r)$ , and the equation (2) comes from the formula  $r = \text{fdim}(N_0, \omega_0) + \text{fdim}(N_1, \omega_1) + \cdots + \text{fdim}(N_n, \omega_n)$ .  $\square$

Here we give the above  $r$  explicitly in the case that  $k$  is small.

**The case  $k = 1$ :** The above  $N_n$  and the distribution data coming from the vacuum state  $\omega_n$  are

$$(\mathbb{C} \oplus \mathbb{C}), \quad \left(\frac{1}{2}, \frac{1}{2}\right).$$

Thus the free dimension of the pair is  $1 - \|(\frac{1}{2}, \frac{1}{2})\|_2^2 = \frac{1}{2}$ . (See Definition 3.1 of [6].) Therefore, we have  $\mathcal{F}_\beta \cong L(\mathbb{F}_{n+\frac{1}{2}})$ . In case of the golden number, i.e., the case  $n = 1$ , the corresponding  $r$  is  $\frac{3}{2}$ , namely we obtained

$$\mathcal{F}_{\frac{1+\sqrt{5}}{2}} \cong L(\mathbb{F}_{\frac{3}{2}}).$$

**The case  $k = 2$ :** The above  $N_n$  and the distribution data are

$$(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}), \quad \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

Thus  $\mathcal{F}_\beta$  is isomorphic to  $L(\mathbb{F}_{n+\frac{5}{8}})$ .

**The case  $k = 3$ :** The above  $N_n$  and the distribution data are

$$(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}), \quad \left(\frac{5-\sqrt{5}}{20}, \frac{5+\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+\sqrt{5}}{20}\right).$$

Thus  $\mathcal{F}_\beta$  is isomorphic to  $L(\mathbb{F}_{n+\frac{7}{10}})$ .

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