THE SURJECTIVITY OF THE CANONICAL HOMOMORPHISM FROM SINGULAR HOMOLOGY TO ČECH HOMOLOGY

KATSUYA EDA AND KAZUHIRO KAWAMURA

(Communicated by Ralph Cohen)

ABSTRACT. Let X be a locally n-connected compact metric space. Then, the canonical homomorphism from the singular homology group $H_{n+1}(X)$ to the Čech homology group $\check{H}_{n+1}(X)$ is surjective. Consequently, if a compact metric space X is locally connected, then the canonical homomorphism from $H_1(X)$ to $\check{H}_1(X)$ is surjective.

1. INTRODUCTION AND SUMMARY

There exists a natural homomorphism from the singular homology group to the Čech homology group for any space [8, 7]. It is known that it is not an isomorphism in general, but is an isomorphism when the space is locally contractible. In the present paper, we study this homomorphism when the space is locally connected up to certain dimension. The following is our main result.

Theorem 1.1. *Let X be a locally n-connected (LC^n for short) compact metric space. Then, the canonical homomorphism from the singular homology group $H_{n+1}(X)$ to the Čech homology group $\check{H}_{n+1}(X)$ is surjective.*

Corollary 1.2. *If a compact metric space X is locally connected, then the canonical homomorphism $\varphi_* : H_1(X) \to \check{H}_1(X)$ is surjective.*

We recall that the canonical homomorphism is an isomorphism when X is semi-$n+1$-lc^s by [8, Theorem 1] and, in particular, when X is LC^{n+1}. The above theorem holds for the homotopy groups and the Čech homotopy groups if the space X is the limit of an inverse sequence of polyhedra with weak fibrations as bonding maps [9, p. 178]. On the other hand, an example of Mardešić [8, p. 162] shows that this theorem and corollary do not hold for non-metrizable compact Hausdorff spaces.

In the following we review the definition of the above canonical homomorphism and also its dual for cohomology, which will be used for an application in Section 3. For our purpose, it is convenient to use the Vietoris homology groups, which are naturally isomorphic to the Čech homology groups [1]. A natural homomorphism from the singular homology group to the Vietoris homology group is defined

fairly directly, and that simplicity is helpful to keep the geometric idea transparent. The canonical homomorphism of Theorem 1.1 is the composition of the above homomorphism with the natural isomorphism between the Vietoris and the Čech homology groups, so, in the sequel, we investigate the canonical homomorphism from the singular homology group to the Vietoris homology group. First we recall the definition of the Vietoris homology groups and next the homomorphism above.

For an open cover $\mathcal{O}$ of a space X, a subset S of X is said to be $\mathcal{O}$-small, if there exists an element $O \in \mathcal{O}$ such that $S \subset O$. Let $X_{\mathcal{O}}$ be the simplicial complex whose n-simplex is an $(n+1)$-tuple $(x_0, \cdots, x_n)$ such that $\{x_0, \cdots, x_n\}$ is $\mathcal{O}$-small. The set of all n-simplexes of $X_{\mathcal{O}}$ is denoted by $X_{\mathcal{O}}^n$.

The n-dimensional chain group $C_n(X_{\mathcal{O}})$ is the free abelian group generated by $X_{\mathcal{O}}^n$. The boundary operator ∂ is defined by

$$\partial((x_0, \cdots, x_{n+1})) = \sum_{i=0}^{n+1} (-1)^i (x_0, \cdots, \widehat{x_i}, \cdots, x_{n+1}),$$

where $(x_0, \cdots, \widehat{x_i}, \cdots, x_{n+1})$ is the n-tuple obtained by deleting x_i. Then, $H_n(X_{\mathcal{O}})$ is defined as the homology group of this chain complex. When an open cover $\mathcal{P}$ is a refinement of another open cover $\mathcal{O}$, we write $\mathcal{O} \leq \mathcal{P}$. Then, $C_n(X_{\mathcal{P}})$ is a subgroup of $C_n(X_{\mathcal{O}})$ and the inclusion induces a homomorphism $h_{\mathcal{OP}*} : H_n(X_{\mathcal{P}}) \to H_n(X_{\mathcal{O}})$. Now, the n-dimensional Vietoris homology group is the inverse limit $\varprojlim(H_n(X_{\mathcal{O}}), h_{\mathcal{OP}*} : \mathcal{O} \leq \mathcal{P})$, and, as was mentioned earlier, the group is naturally isomorphic to the Čech homology group $\check{H}_n(X)$.

We use standard notations for the singular homology groups. The set of all continuous maps from a space X to Y is denoted by $C(X, Y)$. The standard n-simplex is denoted by Δ_n. The vertices of Δ_n are denoted by $\mathbf{e}_0, \cdots, \mathbf{e}_n$. The simplicial map $\varepsilon_i : \Delta_{n-1} \to \Delta_n$ is defined by $\varepsilon_i(\mathbf{e}_j) = \mathbf{e}_j$ for $j < i$ and by $\varepsilon_i(\mathbf{e}_j) = \mathbf{e}_{j+1}$ for $j \geq i$. The singular chain group, the free abelian group generated by all singular n-simplexes, is denoted by $S_n(X)$. The boundary operator $\partial_{n+1}(= \partial) : S_{n+1}(X) \to S_n(X)$ is defined by $\partial(u) = \sum_{i=0}^{n+1} u \cdot \varepsilon_i$ for $u \in C(\Delta_{n+1}, X)$. Since this operation ∂ is defined as well in the case that u is defined only on $\partial \Delta_{n+1}$, by abuse of notation, we define $\partial(u)$ for $u \in C(\partial \Delta_{n+1}, X)$ by the same formula as above.

Let $\mathcal{O}$ be an open cover of X. A well-known fact [10, p. 178] shows that there exists a chain homotopy equivalence $t_{\mathcal{O}} : \{S_n(X)\} \to \{S_n^{\mathcal{O}}(X)\}$, where $S_n^{\mathcal{O}}(X)$ is the subcomplex of $S_n(X)$ generated by all singular n-simplexes $u \in C(\Delta_n, X)$ such that $\operatorname{Im} u = u(\Delta_n)$ are $\mathcal{O}$-small. Also $t_{\mathcal{O}}$ is the chain homotopy inverse of the inclusion $S_n^{\mathcal{O}}(X) \hookrightarrow S_n(X)$. Hence any singular n-chain is homologous to a chain of the form $\sum_{i=0}^m \lambda_i u_i$ with $\lambda_i = \pm 1$ and $u_k \in C(\Delta_n, X)$ such that each $\operatorname{Im}(u_i)$ is $\mathcal{O}$-small. Define $\varphi_{\mathcal{O}} : S_n^{\mathcal{O}}(X) \to C_n(X_{\mathcal{O}})$ by: $\varphi_{\mathcal{O}}(u) = (u(\mathbf{e}_0), \cdots, u(\mathbf{e}_n))$ for each singular simplex u of $S_n^{\mathcal{O}}(X)$. Then, $\varphi_{\mathcal{O}}$ induces a homomorphism from $H_n(X)$ to $H_n(X_{\mathcal{O}})$, which is denoted by $\varphi_{\mathcal{O}*}$. Since the equality $\varphi_{\mathcal{O}*} = h_{\mathcal{OP}*} \cdot \varphi_{\mathcal{P}*}$ holds for each pair of open covers $\mathcal{O}$ and $\mathcal{P}$ with $\mathcal{O} \leq \mathcal{P}$, we get the canonical homomorphism $\varphi_* : H_n(X) \to \varprojlim(H_n(X_{\mathcal{O}}), h_{\mathcal{OP}*} : \mathcal{O} \leq \mathcal{P}) \simeq \check{H}_n(X)$. For simplicity, we use the following notation: $[z]$ denotes the homology class which contains a cycle z, when the homology group under consideration is clear in the context. When we need to consider a single cycle z in several different homology classes, we shall note which homology group is in question.

For use in Section 3, we recall a definition of the Alexander cohomology group. First let $C^n(X_{\mathcal{O}}) = \operatorname{Hom}(C_n(X_{\mathcal{O}}), \mathbb{Z})$, and let the coboundary map $\delta_n : C^n(X_{\mathcal{O}}) \to$

$C^{n+1}(X_{\mathcal{O}})$ be defined by $\delta_n(h)(u) = h(\partial_n(u))$ for $u \in C_{n+1}(X_{\mathcal{O}})$ and for $h \in C^n(X_{\mathcal{O}})$. The cohomology group of this cochain complex is denoted by $H^n(X_{\mathcal{O}})$. The homomorphism $g_{\mathcal{OP}*} : H^n(X_{\mathcal{O}}) \to H^n(X_{\mathcal{P}})$ is induced by the dual homomorphism $g_{\mathcal{OP}} : C^n(X_{\mathcal{O}}) \to C^n(X_{\mathcal{P}})$ of the inclusion $C_n(X_{\mathcal{P}}) \hookrightarrow C_n(X_{\mathcal{O}})$. The n-dimensional Alexander cohomology group is the direct limit $\varinjlim(H^n(X_{\mathcal{O}}), g_{\mathcal{OP}*} : \mathcal{O} \leq \mathcal{P})$, and, as in the Vietoris homology groups, the group is naturally isomorphic to the Čech cohomology group $\check{H}^n(X)$. Since the singular cohomology group $H^n(X)$ is well-known, we do not repeat the definition here. The homomorphism $\varphi_{\mathcal{O}} : S_n^{\mathcal{O}}(X) \to C_n(X_{\mathcal{O}})$ induces a homomorphism $S_n(X) \to C_n(X_{\mathcal{O}})$ via $t_{\mathcal{O}}$ and consequently we get the dual homomorphism $\varphi^{\mathcal{O}} : \mathrm{Hom}(C_n(X_{\mathcal{O}}), \mathbb{Z}) \to \mathrm{Hom}(S_n(X), \mathbb{Z})$ which commutes the coboundary maps. Thus, $\varphi^{\mathcal{O}}$ induces a homomorphism $\varphi^{\mathcal{O}*} : H^n(X_{\mathcal{O}}) \to H^n(X)$. Since $h_{\mathcal{OP}}^* : H^n(X_{\mathcal{O}}) \to H^n(X_{\mathcal{P}})$ commutes with $\varphi^{\mathcal{O}}$ and $\varphi^{\mathcal{P}}$, we get the canonical homomorphism $\varphi^* : \check{H}^n(X) \to H^n(X)$.

2. Proof of Theorem 1.1

Throughout the present paper, all open covers of spaces are assumed to be finite. Since we restrict our attention only to compact Hausdorff spaces, this assumption loses no generality. The first lemma seems to be known, but we were not able to find this result in the literature so we present a proof for the sake of completeness. (We refer the reader to [8] for the homology version of this lemma.) For an open cover $\mathcal{P}$ of a space X and a subset P of X, $St_{\mathcal{P}}(P)$ denotes the set $\bigcup\{U \in \mathcal{P} : U \cap P \neq \emptyset\}$. For a map $f : X_{\mathcal{P}}^i \to C(\Delta_i, X)$ and each simplex $(x_0, \cdots, x_i)$, we simply write $f(x_0, \cdots, x_i)$ instead of $f((x_0, \cdots, x_i))$.

Lemma 2.1. *Let X be an LC^n compact metric space and $\mathcal{O}$ an open cover of X. Then, there exist a refinement $\mathcal{P}$ of $\mathcal{O}$ and a map $\psi_i : X_{\mathcal{P}}^i \to C(\Delta_i, X)$ for each $0 \leq i \leq n$ with the following properties:*

(1) *For each $P \in \mathcal{P}$, $St_{\mathcal{P}}(P)$ is $\mathcal{O}$-small.*
(2) *$\psi_0(x)(\mathbf{e}_0) = x$ for each 0-simplex $x \in X$.*
(3) *For each $s \in X_{\mathcal{P}}^i$, $\mathrm{Im}\,\psi_i(s)$ is $\mathcal{O}$-small.*
(4) *$\varphi_{\mathcal{O}} \cdot \psi_i$ is the identity on $X_{\mathcal{P}}^i$.*
(5) *The equality $\psi_{i+1}(x_0, \cdots, x_{i+1}) \cdot \varepsilon_j = \psi_i(x_0, \cdots, \widehat{x_j}, \cdots, x_{i+1})$ holds for each $(i+1)$-simplex $(x_0, \cdots, x_{i+1})$ and for each $0 \leq j \leq i+1$.*

Proof. As was mentioned at the beginning of this section, we assume that all covers are finite. We define a sequence of covers $\mathcal{O}_i$ by induction. Let $\mathcal{O}_0 = \mathcal{O}$. For a given cover $\mathcal{O}_i$, take a refinement $\mathcal{O}_{i+1}$ of $\mathcal{O}_i$ satisfying the following condition: for any $U \in \mathcal{O}_{i+1}$, there exists a $V \in \mathcal{O}_i$ such that any map $f \in C(\partial\Delta_{n+1-i}, St_{\mathcal{O}_{i+1}}(U))$ extends to a map $\overline{f} \in C(\Delta_{n+1-i}, V)$. The cover $\mathcal{P}$ is defined by $\mathcal{P} = \mathcal{O}_{n+1}$.

Next, we construct ψ_i also by induction. The map ψ_0 is defined by condition (2). Suppose that $\psi_i : X_{\mathcal{O}_i}^i \to C(\Delta_i, X)$ is defined so as to satisfy conditions (2)–(5), and, in addition, for any $(x_0, \cdots, x_i) \in X_{\mathcal{P}}^i$, $\mathrm{Im}(\psi_i(x_0, \cdots, x_i))$ is $\mathcal{O}_{n+1-i}$-small.

Take any $(i+1)$-simplex $(x_0, \cdots, x_{i+1}) \in X_{\mathcal{P}}^{i+1}$. Since

$$\begin{aligned} \psi_i(x_0, \cdots, \widehat{x_j}, \cdots, x_{i+1}) \cdot \varepsilon_{k-1} &= \psi_{i-1}(x_0, \cdots, \widehat{x_j}, \cdots, \widehat{x_k}, \cdots, x_{i+1}) \\ &= \psi_i(x_0, \cdots, \widehat{x_k}, \cdots, x_{i+1}) \cdot \varepsilon_j \end{aligned}$$

for $0 \leq j < k \leq i+1$, the equations $f \cdot \varepsilon_j = \psi_i(x_0, \cdots, \widehat{x_j}, \cdots, x_{i+1})$ $(j \leq i+1)$ define a map $f \in C(\partial\Delta_{i+1}, X)$. By the induction hypothesis, there are $V_j \in \mathcal{O}_{n+1-i}$ $(0 \leq j \leq l+1)$ such that $\mathrm{Im}(\psi_i(x_0, \cdots, \widehat{x_j}, \cdots, x_{i+1})) \subset V_j$, which imply $\mathrm{Im}(f) \subset$

$St_{\mathcal{O}_{n+1-i}}(V_0)$. Then, there exist a $V \in \mathcal{O}_{n-i}$ and an extension $\psi_{i+1}(x_0, \cdots, x_{i+1}) \in C(\Delta_{i+1}, V)$ of f. Now, ψ_{i+1} has the required properties. $\quad\square$

Notice that any refinement of $\mathcal{P}$ can be used to define ψ_i of the above lemma as well. Since $C_n(X_\mathcal{O})$ is free, the map ψ_i extends to a homomorphism from $C_n(X_\mathcal{O})$ to $S_i(X)$, which is also denoted by ψ_i. We remember that ψ_i maps $X_\mathcal{P}^i$ into $C(\Delta_i, X)$, which is important in the final step of the proof of Theorem 1.1. For the next lemma, we introduce an auxiliary notion. A singular n-cycle $z \in Z_n(X)$ is called a *standard* n-cycle, if there exists a continuous map $u : \partial\Delta_{n+1} \to X$ such that $\partial u = z$.

Lemma 2.2. *Let X be an LC^n compact metric space and $\mathcal{O}$ an open cover of X. Take a refinement $\mathcal{P}$ of $\mathcal{O}$ satisfying the properties of Lemma 2.1. Suppose that a singular cycle $z \in Z_{n+1}(X) \cap S_{n+1}^\mathcal{P}(X)$ satisfies the following conditions:*

(1) $z = \sum_{j=0}^{m_0} \lambda_j \psi_{n+1}(x_{j0}, \cdots, x_{j\,n+1})$ *where* $\lambda_j = \pm 1$;
(2) $\varphi_\mathcal{O}(z) = \sum_{j=0}^{m_0} \lambda_j(x_{j0}, \cdots, x_{j\,n+1}) \in B_{n+1}(X_\mathcal{P})$.

Then, there exist $\mu_k = \pm 1$ and standard cycles $u_k \in S_{n+1}^\mathcal{O}(X)$ $(0 \leq k \leq m_0)$ such that $z = \sum_{k=0}^m \mu_k u_k$ in $S_{n+1}(X)$.

Proof. By condition (2), there exist $(y_{k0}, \cdots, y_{k\,n+2}) \in X_\mathcal{P}^{n+2}$ and $\mu_k = \pm 1$ $(0 \leq k \leq m)$ such that $\varphi_\mathcal{O}(z) = \sum_{k=0}^{m_1} \mu_k \partial(y_{k0}, \cdots, y_{k\,n+2})$. Substituting the equality of (2) and the definition of $\partial(y_{k0}, \cdots, y_{k\,n+2})$ for both sides of this equation, we have

$$\sum_{j=0}^{m_0} \lambda_j(x_{j0}, \cdots, x_{j\,n+1}) = \sum_{k=0}^{m_1} \mu_k \sum_{i=0}^{n+2} (-1)^i(y_{k0}, \cdots, \widehat{y_{ki}}, \cdots, y_{k\,n+2}).$$

Applying ψ_{n+1}, we have

$$z = \sum_{k=0}^{m_1} \mu_k \sum_{i=0}^{n+2} (-1)^i \psi_{n+1}(y_{k0}, \cdots, \widehat{y_{ki}}, \cdots, y_{k\,n+2}).$$

By property (5) of Lemma 2.1 for ψ_{n+1} and ψ_n, we define $v_k \in C(\partial\Delta_{n+2}, X)$ by:

$$v_k \cdot \varepsilon_i = \psi_{n+1}(y_{k0}, \cdots, \widehat{y_{ki}}, \cdots, y_{k\,n+2}) \text{ for each } 0 \leq i \leq n+2.$$

Now, let $u_k = \partial v_k$ for k. Then $\text{Im}(v_k)$ is $\mathcal{O}$-small by property (1) of $\mathcal{P}$ in Lemma 2.1 and hence these u_k's are the desired standard cycles. $\quad\square$

The next lemma is essentially due to Mardešić [8] where the result is proved under a weaker assumption lc_s^n rather than LC^n.

Lemma 2.3. *Let X be an LC^n compact metric space and $\mathcal{O}$ an open cover of X. Then, $\varphi_{\mathcal{O}*}(H_{n+1}(X))$ is equal to the image of the canonical projection $h_\mathcal{O} : \check{H}_{n+1}(X) = \varprojlim\{H_{n+1}(X_\mathcal{O}), h_{\mathcal{O}\mathcal{P}*} : \mathcal{O} \leq \mathcal{P}\} \to H_{n+1}(X_\mathcal{O})$.*

Proof. Take a refinement $\mathcal{P}$ of $\mathcal{O}$ and ψ_i $(0 \leq i \leq n)$ which satisfies the properties of Lemma 2.1. Let $c = \sum_{j=0}^m \lambda_j(x_{j0}, \cdots, x_{j\,n+1})$ be a cycle in $C_n(X_\mathcal{P})$, where

$\lambda_k = \pm 1$. Then,

$$\partial(\sum_{j=0}^{m}\lambda_j\psi_{n+1}(x_{j0},\cdots,x_{j\,n+1})) \;\;=\;\; \sum_{j=0}^{m}\lambda_j\sum_{k=0}^{n+1}(-1)^k\psi_{n+1}(x_{j0},\cdots,x_{j\,n+1})\cdot\varepsilon_k$$

$$=\;\; \sum_{j=0}^{m}\lambda_j\sum_{k=0}^{n+1}(-1)^k\psi_n(x_{j0},\cdots,\widehat{x_{jk}},\cdots,x_{j\,n+1})$$

$$=\;\; 0.$$

The last equality follows from the equation

$$\sum_{j=0}^{m}\lambda_j\sum_{k=0}^{n+1}(-1)^k(x_{j0},\cdots,\widehat{x_{jk}},\cdots,x_{j\,n+1}) = \partial c = 0.$$

Now, we have that $\sum_{j=0}^{m}\lambda_j\psi_{n+1}(x_{j0},\cdots,x_{j\,n+1})$ is a cycle. Also notice that $c = \varphi_{\mathcal{O}*}(\sum_{j=0}^{m}\lambda_j\psi_{n+1}(x_{j0},\cdots,x_{j\,n+1}))$ by Lemma 2.1. These two imply that $\mathrm{Im}(h_{\mathcal{O}*})$ is contained in $\varphi_{\mathcal{O}*}(H_{n+1}(X))$. The reverse inclusion follows from the equality: $\varphi_{\mathcal{O}*} = h_{\mathcal{O}*}\cdot\varphi_*$. $\qquad\square$

The standard n-cell $\{(x_0,\cdots,x_{n-1}) : 0 \le x_i \le 1 \text{ for each } i\}$ is denoted by $\mathbb{I}^n$.

Lemma 2.4. *Let X be a connected, locally connected compact metric space. Then, there exists a continuous surjection $F : \mathbb{I}^{n+1} \to X$ which satisfies the following:*

(1) *$F(\partial\mathbb{I}^{n+1}) = \{x_*\}$ for some $x_* \in X$.*
(2) *F is null homotopic relative to $\partial\mathbb{I}^{n+1}$.*
(3) *For any open set U of X, there exist a point $y \in U$ and an open subset O of $\mathbb{I}^{n+1}$ such $F(O) = \{y\}$.*

Proof. The assumption on X implies that there exists a continuous surjection $f : [0,1] \to X$. Let C be the Cantor ternary set $\{\sum_{i=0}^{\infty}\delta_i/3^i : \delta_i = 0,2\}$. Let $g : [0,1] \to [0,1]$ be a continuous surjection such that g is constant on each component of $[0,1]\setminus C$. The existence of such a function is well known. For an explicit formula, see [6, pp. 74–76], for example. Let $h(x) = f(g(|2x-1|))$ for $0 \le x \le 1$. The map $h : \mathbb{I} \to X$ satisfies the condition of the lemma for $n = 0$.

For $x = (x_0,\cdots,x_{n-1}) \in \mathbb{I}^n$, let $\rho(x) = \max\{|2x_i - 1| : 0 \le i \le n-1\}$ and $F(x) = f(g(\rho(x)))$. Then, the map F takes the value $f\cdot g(1)$ on the boundary $\partial\mathbb{I}^{n+1}$ and factors through the interval $[0,1]$. Properties (1) and (2) follows easily from these. Further, since $[0,1]\setminus C$ is dense in $[0,1]$, we can see that F satisfies condition (3). $\qquad\square$

Proof of Theorem 1.1. Since $\check{H}_{n+1}(X)$ and $H_{n+1}(X)$ are finite direct sums of Čech and singular homology groups of the components of X respectively, we may assume that X is connected. Applying Lemma 2.1 and the remark after that lemma, we get a sequence of open covers $(\mathcal{P}_m : m < \omega)$ such that $\mathcal{P}_0 = \{X\}$, and the collection $\{\mathcal{P}_m : m < \omega\}$ is cofinal in the set of all open covers of X and $\mathcal{P}_{m+1}$ is a refinement of $\mathcal{P}_m$ satisfying the properties of Lemma 2.1 for $\mathcal{P}_m$. We may assume that each $P \in \mathcal{P}_m$ is path-connected and the diameter of each $P \in \mathcal{P}_m$ is less than $1/m$. Any element of $\check{H}_{n+1}(X)$ is given by a sequence $(c_m : m < \omega)$, where $c_m \in H_{n+1}(X_{\mathcal{P}_m})$ $(m < \omega)$ and $h_{\mathcal{P}_m\mathcal{P}_{m+1}*}(c_{m+1}) = c_m$. By Lemma 2.3, choose an element $z_0 \in Z_{n+1}(X)$ so that $\varphi_{\mathcal{P}_1*}([z_0]) = c_1$ and let $b_m = c_m - \varphi_{\mathcal{P}_m*}([z_0])$. Then, $h_{\mathcal{P}_m\mathcal{P}_{m+1}*}(b_{m+1}) = b_m$ for each m and $b_1 = 0$.

Next, we define a map $F_m \in C(\partial\Delta_{n+2}, X)$ for each $m < \omega$ by induction as follows. Let $F_1 \in C(\partial\Delta_{n+2}, X)$ be a null homotopic map which satisfies the conditions of Lemma 2.4. In particular, for any non-empty open set U of X there exist a point $y \in U$ and an open subset O of $\partial\Delta_{n+2}$ such that $F_1(O) = y$. Then, we obtain a countable disjoint collection of open disks $\{D_k : k < \omega\}$ of $\partial\Delta_{n+2}$ such that F_1 is constant on each D_k and $\{F(D_k) : k < \omega\}$ is dense in X. We construct maps $F_m \in C(\partial\Delta_{n+2}, X)$ and finite subcollections $\mathcal{U}_m$ of $\{D_k : k < \omega\}$ so that the following conditions $(\star)$ hold:

(1) $\varphi_{\mathcal{P}_m*}([\partial F_m]) = b_m$,
(2) $\bigcup\mathcal{U}_m \cap \bigcup\mathcal{U}_n = \emptyset$ for any distinct m and n,
(3) F_i is constant on each $D \in \mathcal{U}_m$ for $i < m$,
(4) $F_m \mid \partial\Delta_{n+2} - \bigcup_{l=1}^{m-1}\bigcup\mathcal{U}_l = F_1 \mid \partial\Delta_{n+2} - \bigcup_{l=1}^{m-1}\bigcup\mathcal{U}_l$, and
(5) $d(F_m(x), F_{m-1}(x)) < 1/m$.

Suppose that F_i $(1 \leq i < m)$ satisfy $(\star)$ and $\varphi_{\mathcal{P}_i*}([\partial F_i]) = b_i$. Choose $a_{m+1} \in Z_{n+1}(X_{\mathcal{P}_{m+1}})$ so that $b_{m+1} - \varphi_{\mathcal{P}_{m+1}*}([\partial F_{m-1}]) = [a_{m+1}]$.

Since we have chosen $\mathcal{P}_m$ so as to satisfy the conditions of Lemma 2.1 for $\mathcal{P}_{m-1}$, there exists an element $z \in Z_{n+1}(X) \cap S_{n+1}^{\mathcal{P}_m}(X)$ such that $\varphi_{\mathcal{P}_m}(z) = a_{m+1}$. On the other hand, in $H_{n+1}(X_{\mathcal{P}_{m-1}})$ we have

$$[a_{m+1}] = \varphi_{\mathcal{P}_{m-1}*}(z) = h_{\mathcal{P}_{m-1}\mathcal{P}_{m+1}*}(b_{m+1}) - \varphi_{\mathcal{P}_{m+1}*}([\partial F_{m-1}]) = b_{m-1} - b_{m-1} = 0,$$

and hence a_{m+1} belongs to $B_{n+1}(X_{\mathcal{P}_{m-1}})$. Since $m - 1 \geq 1$, we apply Lemma 2.2 and get $\mu_{mj} = \pm1$ and $u_{mj} \in C(\partial\Delta_{n+2}, X)$ such that $\text{Im}(u_{mj})$ is $\mathcal{P}_{m-2}$-small and $z = \sum_{j=0}^{k_m} \partial u_{mj}$.

Take $P_{mj} \in \mathcal{P}_{m-2}$ $(0 \leq j \leq k_m)$ so that $\text{Im}(u_{mj}) \subset P_{mj}$. For each $0 \leq j \leq k_m$, choose an open disk $U_{mj} \in \{D_k : k < \omega\} \setminus \bigcup_{l=0}^{m-1}\bigcup\mathcal{U}_l$ so that $F_1(U_{mj}) \in P_{mj}$ and $U_{mj} \cap U_{mj'} = \emptyset$ for $j \neq j'$, and let $\mathcal{U}_m = \{U_{mj} : 0 \leq j \leq k_m\}$. By condition (4), we have that $F_{m-1}|U_{mj} = F_1|U_{mj}$ is constant and takes a value in P_{mj}. Since P_{mj} is path-connected, we can choose $u'_{mj} \in C(\partial\Delta_{n+2}, P_{mj})$ such that u'_{mj} is homotopic to u_{mj} and u'_{mj} takes the constant value $F_1(U_{mj})$ on $\partial\Delta_{n+2} \setminus U_{mj}$. Define the map F_m by $F_m|U_{mj} = u'_{mj}|U_{mj}$ for $0 \leq j \leq k_m$ and $F_m \mid \partial\Delta_{n+2} \setminus \bigcup\mathcal{U}_m = F_{m-1} \mid \partial\Delta_{n+2} \setminus \bigcup\mathcal{U}_m$. Note u'_{mj} and F_{m-1} take the same constant value on ∂U_{mj}. Then, $F_m \in C(\partial\Delta_{n+2}, X)$ and

$$\partial F_m - (\partial F_{m-1} + z) = \partial F_m - \left(\partial F_{m-1} + \sum_{j=0}^{k_m} \partial u_{mj}\right) \in B_{n+1}(X).$$

Thus,

$$\varphi_{\mathcal{P}_m*}([\partial F_m]) = \varphi_{\mathcal{P}_m*}([\partial F_{m-1}] + [z]) = \varphi_{\mathcal{P}_m*}([\partial F_{m-1}]) + [a_{m+1}] = b_m.$$

Hence, $(\star)$ holds for F_i $(1 \leq i \leq m)$ and we have finished the induction step m.

Since $\{F_m : m < \omega\}$ forms a Cauchy sequence, it converges uniformly to a map $F_\infty \in C(\partial\Delta_{n+2}, X)$. For each point $x \in \partial\Delta_{n+2}$, there exists $P \in \mathcal{P}_{m-1}$ such that both $F_m(x)$ and $F_\infty(x)$ belong to P. Take a subdivision of $\partial\Delta_{n+2}$ so that the n-cycles ∂F_∞ and ∂F_m are homologous to sums of signed $\mathcal{P}_{m-1}$-small singular simplexes. By property (1) of Lemma 2.1, for such a singular simplex $u \in C(\Delta_{n+1}, X)$ which appears as a term for ∂F_∞ and for the corresponding singular simplex u_m for ∂F_m, we see that $\varphi_{m-2}(u)$ and $\varphi_{m-2}(u_m)$ are contained in a single simplex of $X_{\mathcal{P}_{m-2}}$. Thus $\varphi_{\mathcal{P}_{m-2}*}([\partial F_\infty]) = \varphi_{\mathcal{P}_{m-2}*}([\partial F_m]) = b_{m-2}$ for

each m and, therefore, $\varphi_{\mathcal{P}_k*}([z_0] + [\partial F_\infty]) = c_k$ for each $k < \omega$. This completes the proof. $\qquad\square$

Remark 2.5. Corollary 1.2 contrasts with the following result [3]: For a 1-dimensional locally connected compact metric space X, the canonical homomorphism from the fundamental group to the first Čech homotopy group is injective.

Remark 2.6. A compact Hausdorff space X is the inverse limit of an inverse system $(X_\alpha, p_{\alpha\beta} : \alpha \leq \beta, \alpha, \beta \in \Lambda)$ of compact polyhedra [9]. Then, $\check{H}_n(X) \simeq \varprojlim(H_n(X_\alpha), p_{\alpha\beta*} : \alpha \leq \beta, \alpha, \beta \in \Lambda)$ holds. Let $p_\alpha : X \to X_\alpha$ be the projection. Then, according to [1] the canonical homomorphism φ_* commutes with each $p_{\alpha*} : H_n(X) \to H_n(X_\alpha)$, that is, $\varphi_*(u)$ is the limit of $p_{\alpha*}(u)$ for $u \in H_n(X)$.

A factor of singular homology $H_n^T(X)$ is defined and a canonical surjection $\sigma_X : H_n(X) \to H_n^T(X)$ is constructed in [4]. Since $H_n^T(X_j) = H_n(X_j)$ holds by [4, Theorem 2.1], φ_* factors through σ_X and hence there exists a canonical homomorphism from $H_n^T(X)$ to $\check{H}_n(X)$. When X is LC^n, this canonical homomorphism from $H_{n+1}^T(X)$ to $\check{H}_{n+1}(X)$ is a surjection by Theorem 1.1.

3. APPLICATION TO COHOMOLOGY GROUPS

Mardešić [8] proved that for an LC^n compact Hausdorff space X, the canonical homomorphism $\varphi^* : \check{H}^{n+1}(X) \to H^{n+1}(X)$ is an injection. As an application of Theorem 1.1, here we prove that the injection is extended to an "injection" of the short exact sequence of the Universal Coefficient Theorem for cohomologies. No specific reference to open covers is necessary in this section, so we work on the inverse limit representation of compact metric spaces by compact polyhedra as in Remark 2.6.

Suppose that X is an LC^n compact metric space which is the limit of an inverse sequence $\varprojlim(X_j, p_{j\,j+1} : X_{j+1} \to X_j)$ of compact polyhedra. Let $p_j : X \to X_j$ be the projection. Applying the Universal Coefficient Theorem to each $H^{n+1}(X_j)$ and passing to the direct limit, we have the short exact sequence as follows:

$$0 \to \varinjlim \text{Ext}(H_n(X_j), \mathbb{Z}) \to \check{H}^{n+1}(X) \to \varinjlim \text{Hom}(H_{n+1}(X_j), \mathbb{Z}) \to 0.$$

We define homomorphisms $\lambda : \varinjlim \text{Ext}(H_n(X_j), \mathbb{Z}) \to \text{Ext}(H_n(X), \mathbb{Z})$ and $\mu : \varinjlim \text{Hom}(H_{n+1}(X_j), \mathbb{Z}) \to \text{Hom}(H_{n+1}(X), \mathbb{Z})$ as follows: For a homomorphism $f : A \to B$ between groups A and B, $\text{Hom}(f) : \text{Hom}(B, \mathbb{Z}) \to \text{Hom}(A, \mathbb{Z})$ denotes the $\mathbb{Z}$-dual homomorphism induced by f. Similarly, $\text{Ext}(f) : \text{Ext}(B, \mathbb{Z}) \to \text{Ext}(A, \mathbb{Z})$ denotes the induced homomorphism by f. The sequence $(\text{Hom}(p_{j*}) : j < \omega)$ induces the limit homomorphism

$$\mu = \varinjlim \text{Hom}(p_{j*}) : \varinjlim \text{Hom}(H_{n+1}(X_j), \mathbb{Z}) \to \text{Hom}(H_{n+1}(X), \mathbb{Z}).$$

Let $\check{p}_{j*} : \check{H}_{n+1}(X) \to H_{n+1}(X_j)$ be the projection homomorphism. By Remark 2.6 of the previous section, one can see that $\mu = \text{Hom}(\varphi_*) \cdot \varinjlim \text{Hom}(\check{p}_{j*})$, where $\varphi_* : H_{n+1}(X) \to \check{H}_{n+1}(X)$ is the homomorphism of Theorem 1.1.

Similarly, the sequence $(\text{Ext}(p_{j*}) : j < \omega)$ induces the limit homomorphism $\lambda = \varinjlim \text{Ext}(p_{j*}) : \varinjlim \text{Ext}(H_n(X_j), \mathbb{Z}) \to \text{Ext}(H_n(X), \mathbb{Z})$. We can now state our result as follows.

Theorem 3.1. *Let X be an LC^n compact metric space. Under the above notation, the homomorphisms λ, φ^* and μ form the commutative diagram below and they are*
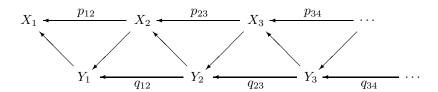
all injections.

$$0 \rightarrow \varinjlim \operatorname{Ext}(H_n(X_j), \mathbb{Z}) \rightarrow \check{H}^{n+1}(X) \rightarrow \varinjlim \operatorname{Hom}(H_{n+1}(X_j), \mathbb{Z}) \rightarrow 0$$
$$\qquad\qquad\quad \downarrow \lambda \qquad\qquad\qquad \downarrow \varphi^* \qquad\qquad\qquad\quad \downarrow \mu$$
$$0 \rightarrow \operatorname{Ext}(H_n(X), \mathbb{Z}) \rightarrow H^{n+1}(X) \rightarrow \operatorname{Hom}(H_{n+1}(X), \mathbb{Z}) \rightarrow 0$$

Proof. The commutativity of the above diagram follows easily from the definition of those homomorphisms and the naturality of the exact sequences involved. By Corollary 2 of [8], φ^* is an injection, which implies that λ is an injection as well (the commutativity of the diagram). It remains to prove the injectivity of the homomorphism μ. It follows from Theorem 1.1 that $\operatorname{Hom}(\varphi_*)$ is an injection. From the equality $\mu = \operatorname{Hom}(\varphi_*) \cdot \varinjlim \operatorname{Hom}(\check{p}_{j*})$, it suffices to prove that $\varinjlim \operatorname{Hom}(\check{p}_{j*}) : \varinjlim \operatorname{Hom}(H_{n+1}(X_j), \mathbb{Z}) \rightarrow \operatorname{Hom}(H_{n+1}(X), \mathbb{Z})$ is an injection.

In general $p_{j*} : H_{n+1}(X_{j+1}) \rightarrow H_{n+1}(X_j)$ need not be surjective. Applying the proof of [5, Theorem 4], we obtain an inverse sequence $(Y_j, q_{jj+1} : Y_{j+1} \rightarrow Y_j)$ of compact polyhedra such that each q_j induces an isomorphism of homotopy groups up to dimension n and a surjection up to dimension $n+1$, and, further (by taking a subsequence if necessary), there exists a homotopy commutative diagram as follows:

Let $Y = \varprojlim(Y_j, q_{jj+1})$ and let $q_j : Y \rightarrow Y_j$ be the projection to the j-th factor. Observe that $\check{H}_{n+1}(Y)$ is isomorphic to $\check{H}_{n+1}(X)$. As before, let $\check{q}_{j*} : \check{H}_{n+1}(Y) \rightarrow H_{n+1}(Y_j)$ be the projection homomorphism. By the Whitehead Theorem, $q_{jj+1*} : H_{n+1}(Y_{j+1}) \rightarrow H_{n+1}(Y_j)$ is a surjection and thus $\check{q}_{j*} : \check{H}_{n+1}(Y) \rightarrow H_{n+1}(Y_j)$ is surjective for each j. Taking the $\mathbb{Z}$-dual and passing to the limit, we see that $\varinjlim \operatorname{Hom}(\check{q}_{j*}) : \varinjlim \operatorname{Hom}(H_{n+1}(Y_j), \mathbb{Z}) \rightarrow \operatorname{Hom}(\check{H}_{n+1}(X), \mathbb{Z})$ is an injection. Finally the commutativity of the above diagram shows that $\varinjlim \operatorname{Hom}(\check{p}_{j*})$ is an injection as well. This completes the proof. $\qquad\square$

Notice that Dydak [2] proved that the group $\varinjlim \operatorname{Ext}(H_n(X_j), \mathbb{Z})$ is isomorphic to the group $\operatorname{Tor} \check{H}^{n+1}(X)$ which is finite, and $\varinjlim \operatorname{Hom}(H_{n+1}(X_j), \mathbb{Z})$ is isomorphic to $\check{H}^{n+1}(X)/\operatorname{Tor} \check{H}^{n+1}(X)$ which is free abelian.

References

[1] C. H. Dowker, *Homology groups of relations*, Ann. Math. **56** (1952), 84–95. MR **13:**967d
[2] J. Dydak, *On algebraic properties of continua*, Bull. de l'acad.Pol. Sci. **27** (1979), 717–721. MR **82e:**55020
[3] K. Eda and K. Kawamura, *The fundamental groups of 1-dimensional spaces*, Topology Appl. **87** (1998), 163–172. MR **99b:**55022
[4] K. Eda and K. Sakai, *A factor of singular homology*, Tsukuba J. Math. **15** (1991), 351–387. MR **93d:**55008
[5] S. Ferry, *A stable converse to the vietoris-smale theorem with applications to shape theory*, Trans. Amer. Math. Soc. **261** (1980), 369–386. MR **82c:**55018

[6] H. Sagan, *Space-filling curves*, Springer, 1994. MR **95h:**00001

[7] S. Mardešić, *Equivalence of singular and Čech homology for ANR-s application to unicoherence*, Fund. Math. **46** (1958), 29–45. MR **20:**5472

[8] ————, *Comparison of singular and Čech homology in locally connected spaces*, Michigan J. Math. **6** (1959), 151–166. MR **21:**4415

[9] S. Mardešić and J. Segal, *Shape theory* (*the inverse system approach*), North Holland, 1982. MR **84b:**55020

[10] E. H. Spanier, *Algebraic topology*, McGraw-Hill, 1966. MR **35:**1007

SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY, TOKYO 169-0072, JAPAN
E-mail address: eda@logic.info.waseda.ac.jp

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA 305, JAPAN
E-mail address: kawamura@math.tsukuba.ac.jp