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DENDRITES AND LIGHT OPEN MAPPINGS

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Abstract. It is shown that a metric continuum X is a dendrite if and only if for every compact space Y and for every light open mapping $f : Y \to f(Y)$ such that $X \subset f(Y)$ there is a copy X' of X in Y for which the restriction $f|X' : X' \to X$ is a homeomorphism. Another characterization of dendrites in terms of continuous selections of multivalued functions is also obtained.

All spaces considered in this paper are assumed to be metric, and all mappings are continuous.

In [3] J. Mioduszewski proved the following result.

1. Theorem. *Let X be a dendrite. Then the following condition (μ_0) is satisfied.*

(μ_0) *For every compact space Y, for every continuous 0-dimensional multifunction $F : X \to 2^Y$ and for every point $(x_0, y_0) \in X \times Y$ with $y_0 \in F(x_0)$ there exists a continuous selection $f : X \to Y$ of F such that $f(x_0) = y_0$.*

If we delete a part of condition (μ_0) related to the point (x_0, y_0), we get a weaker condition (μ).

2. Corollary. *Every dendrite X satisfies the following condition.*

(μ) *For every compact space Y and for every continuous 0-dimensional multifunction $F : X \to 2^Y$ there exists a continuous selection $f : X \to Y$ of F.*

In [4, (2.4), p. 188] G. T. Whyburn proved the following theorem.

3. Theorem. *Let X be a dendrite. Then the following condition (ω_0) is satisfied.*

(ω_0) *For every compact space Y, for every light open mapping $f : Y \to f(Y)$ with $X \subset f(Y)$ and for every point $y_0 \in f^{-1}(X) \subset Y$ there exists a homeomorphic copy X' of X in Y with $y_0 \in X'$ such that the restriction $f|X' : X' \to f(X') = X$ is a homeomorphism.*

If, similarly as for (μ_0) we delete a part of (ω_0) related to the point y_0, we get a weaker condition (ω).

4. Corollary. *Every dendrite X satisfies the following condition.*

(ω) *For every compact space Y and for every light open mapping $f : Y \to f(Y)$ with $X \subset f(Y)$ there exists a homeomorphic copy X' of X in Y such that the restriction $f|X' : X' \to f(X') = X$ is a homeomorphism.*

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5. Proposition. *For every continuum X the following implications hold:*

$$\begin{array}{ccc} (\mu_0) & \Rightarrow & (\omega_0) \\ \Downarrow & & \Downarrow \\ (\mu) & \Rightarrow & (\omega) \end{array}$$

Proof. Since the vertical implications are obvious, only the horizontal ones need a proof. Assume (μ_0). Let Y and Z be compact spaces and let $f : Y \to Z$ be a light open mapping with $X \subset f(Y)$. Choose a point $y_0 \in f^{-1}(X)$. Since the restriction $g = f|f^{-1}(X) : f^{-1}(X) \to X$ is an open mapping [4, (7.2), p. 147], there is no loss of generality in assuming $f(Y) = X$. Define a multifunction $F : X \to 2^Y$ by $F(x) = f^{-1}(x)$ for each $x \in X$. Then F is continuous by the openness of g, and it is 0-dimensional by its lightness. Then the selection $s : X \to Y$ for F such that $s(f(y_0)) = y_0$ which exists by (μ_0) embeds X in Y. So $X' = s(X) \subset Y$ is the needed continuum, and therefore the implication from (μ_0) to (ω_0) is shown. The argument for the implication from (μ) to (ω) is the same. Thus the proof is complete.

In this paper we show that if an arbitrary continuum X (not necessarily locally connected) satisfies condition (ω), then X is a dendrite. So, not only (μ_0) but even the weakest condition of the four specified in Proposition 5 characterizes dendrites in the class of all continua. We start with the following proposition.

6. Proposition. *Each continuum X that satisfies condition (ω) is unicoherent.*

Proof. Suppose on the contrary that a continuum X which satisfies (ω) is not unicoherent. Then there are continua P and Q and nonempty closed sets A and B such that

$$X = P \cup Q, \qquad P \cap Q = A \cup B, \qquad A \cap B = \emptyset.$$

Let P_0 and P_1 be two disjoint copies of P, and let Q_0 and Q_1 be two disjoint copies of Q. Assume moreover that

$$P_0 \cap Q_0 = A_0, \quad Q_0 \cap P_1 = B_0, \quad P_1 \cap Q_1 = A_1, \quad Q_1 \cap P_0 = B_1,$$

where A_0, A_1 are copies of A, and B_0, B_1 are copies of B. Then

$$Y = P_0 \cup Q_0 \cup P_1 \cup Q_1$$

is a continuum. Let $f : Y \to X$ be the natural projection. Then f is two-to-one and open. Since X satisfies (ω), there is a copy X' of X in Y such that $h = f|X' : X' \to f(X') = X$ is a homeomorphism. Let $x_0 \in P \setminus Q \subset X$, and take a point $y_0 \in X'$ with $f(y_0) = x_0$. Then either $y_0 \in P_0$ or $y_0 \in P_1$. By symmetry let $y_0 \in P_0$. Since $h^{-1}(P)$ is a continuum which is homeomorphic to P, intersecting P_0 (at y_0) and contained in $P_0 \cup P_1$, we infer that $h^{-1}(P) = P_0$. Similarly $h^{-1}(Q) = Q_0$ or $h^{-1}(Q) = Q_1$. Thus either $X' = P_0 \cup Q_0$ or $X' = P_0 \cup Q_1$. In the former case we have $B_0 \cup B_1 \subset X'$; in the latter one $A_0 \cup A_1 \subset X'$. So, in both cases, h is not one-to-one on X', a contradiction. The proof is complete.

7. Proposition. *Each continuum X that satisfies condition (ω) is locally connected.*

Proof. Suppose X is not locally connected at a point $p \in X$. Then there is an open set U containing p and such that each open neighborhood of p contained in U is not connected. Let C be the component of U containing p. Let $\{K_n\}$ be

a sequence of components of U such that the sequence $\{\operatorname{cl} K_n\}$ of closures of K_n tends to such a subcontinuum K of $\operatorname{cl} C$ that $p \in K$. For each $n \in \mathbb{N}$ choose a point $q_n \in \operatorname{cl} K_n \cap \operatorname{bd} U$. By compactness we may assume, taking a subsequence if necessary, that $\{q_n\}$ converges to a limit point $q \in K \cap \operatorname{bd} U$. Let A and B be two open subsets of $U \setminus C$ such that $U = A \cup C \cup B$, $A \cap B = \emptyset$, with $K_{2n-1} \subset A$ and $K_{2n} \subset B$ for each $n \in \mathbb{N}$. Thus every component of U different from C is contained either in A or in B. Let V be a neighborhood of p such that $\operatorname{cl} V \subset U$. We will define a continuum T. To this aim consider two disjoint copies of X, that is, the product $X \times \{0,1\}$. On the set $((X \setminus B) \times \{0\}) \cup ((X \setminus A) \times \{1\})$ we introduce an equivalence relation $\sim$ as follows:

$$(x, t_1) \sim (y, t_2) \quad \Leftrightarrow \quad x = y \text{ and } (x \notin V \text{ or } t_1 = t_2).$$

Define

$$T = (((X \setminus B) \times \{0\}) \cup ((X \setminus A) \times \{1\}))\,/\sim,$$

and note that T is compact. We will use the notation (x,t), where $x \in X$ and $t \in \{0,1\}$ for points of T. Observe that T looks like the continuum X except $C \cap V$ is "doubled"; therefore the copies of $\operatorname{cl} K_{2n-1}$ tend to one copy of K, while the copies of $\operatorname{cl} K_{2n}$ tend to the other one.

We will show that T is connected. Assume on the contrary that there are compact subsets P and Q of T such that $T = P \cup Q$, and $P \cap Q = \emptyset$. Let $s : T \to X$ be defined by $s((x,t)) = x$. Then $s(P)$ and $s(Q)$ are compact subsets of X such that $s(P) \cup s(Q) = X$, so $s(P) \cap s(Q) \neq \emptyset$. Let $x \in s(P) \cap s(Q)$. Because of symmetry we can assume that $(x,0) \in P$ and $(x,1) \in Q$. Denote by L the component of V containing x. Then $L \times \{0\} \subset P$ and $L \times \{1\} \subset Q$. Since the closure of a component of a proper subset of a continuum meets the boundary of the subset [2, §47, III, Theorem 2, p. 172], there is a point $y \in \operatorname{cl} L \cap \operatorname{bd} V$. Then $(y,0) \in P$ and $(y,1) \in Q$, but $(y,0) = (y,1)$, so $P \cap Q \neq \emptyset$, a contradiction. Thus connectedness of T is shown.

To construct the continuum Y take two disjoint copies of T, that is, the product $T \times \{0,1\}$. In $T \times \{0\}$ consider the (closed) subset $Z = ((C \cap \operatorname{cl} V) \times \{0,1\}) \times \{0\}$, and let a mapping $g : Z \to T \times \{1\}$ be defined by

$$g\,(((x,0),0)) = ((x,1),1) \quad \text{and} \quad g\,(((x,1),0)) = ((x,0),1).$$

In the product $T \times \{0,1\}$ generate an equivalence relation $\approx$ by $z \approx g(z)$ for each element $z \in Z \subset T \times \{0\}$. Define

$$Y = (T \times \{0,1\})/\approx .$$

Note that the space Y equals $T \times \{0\}$ attached to $T \times \{1\}$ by g; in symbols $Y = T \times \{0\} \cup_g T \times \{1\}$, according to Definition 6.1 of Dugundji's book [1, p. 127].

Let $f : Y \to X$ be the natural projection. Then $f^{-1}(x)$ is a two point set unless $x \in C \cap \operatorname{bd} V$, when it is a singleton. Thus f is light. To show the openness of f recall that a mapping is open if and only if it is interior at each point y of its domain, that is, for each open neighborhood W of y the point $f(y)$ is an interior point of $f(W)$ [4, p. 149]. So, take a point $y = ((x,i),j) \in Y$ and consider five cases.

1) If $x \notin \operatorname{cl} V$, then $(s^{-1}(X \setminus \operatorname{cl} V)) \times \{j\}$ is an open neighborhood of y that goes homeomorphically onto $X \setminus \operatorname{cl} V$ under f, so f is interior at y.

2) If $x \in A \cup B$, then $(s^{-1}(A \cup B)) \times \{j\}$ is an open neighborhood of y with the same property.

3) If $x \in C \cap V$ and $y = ((x,0),0)$, then

$$(((A \cup C) \cap V) \times \{0\} \times \{0\}) \cup_g (((B \cup C) \cap V) \times \{1\} \times \{1\})$$

is again an open neighborhood of y that is mapped homeomorphically onto V under f.

4) If $x \in C \cap V$ and $y = ((x,1),0)$, then the needed neighborhood of y is $(((B \cup C) \cap V) \times \{1\} \times \{0\}) \cup_g (((A \cup C) \cap V) \times \{0\} \times \{1\})$.

5) If $x \in C \cap \operatorname{bd} V$, then $f^{-1}(x)$ is a singleton, i.e., $((x,i),j) = ((x,i'),j') = y$ for every $i, i', j, j' \in \{0,1\}$. Let $W \subset f^{-1}(U)$ be an open subset of Y that contains the point y. Define

$$W_1 = (W \cap (A \cup C)) \times \{0\} \times \{0\} \quad \text{and} \quad W_2 = (W \cap (B \cup C)) \times \{1\} \times \{0\}.$$

Observe that $f|W_1$ and $f|W_2$ are homeomorphisms, and therefore $f(W_1)$ is open in $A \cup C$, and $f(W_2)$ is open in $B \cup C$. Hence the set $f(W_1) \cup f(W_2) \subset f(W)$ contains x in its interior, so f is interior at y. This finishes the proof of openness of f.

Now we will show that there is no copy X' of X in Y such that $f|X' : X' \to X$ is a homeomorphism. Suppose the contrary. Then either $((p,0),0) = ((p,1),1) \in X'$, or $((p,1),0) = ((p,0),1) \in X'$. Assume the former case. Hence for almost all $n \in \mathbb{N}$ we have $\operatorname{cl} K_{2n-1} \times \{0\} \times \{0\} \subset X'$. Thus, since $q \notin V$, we infer $((q,0),0) = ((q,1),0) \in X'$. On the other hand, $\operatorname{cl} K_{2n} \times \{1\} \times \{1\} \subset X'$ for almost all $n \in \mathbb{N}$, so $((q,1),1) = ((q,0),1) \in X'$, and consequently $f|X'$ is not one-to-one. The proof is then complete.

8. Proposition. *If a nondegenerate continuum X satisfies condition (ω), then* $\dim X = 1$.

Proof. By Proposition 7 the continuum X is locally connected. Let M stand for the Menger universal curve. By Theorem 2 of [5, p. 497] there exists a light open surjection $f : M \to X$. By (ω) there is a homeomorphic copy X' of X in M. So, $X' \subset M$, which implies $1 \le \dim X = \dim X' \le \dim M = 1$. Thus the conclusion follows.

9. Theorem. *Each continuum X that satisfies condition (ω) is a dendrite.*

Proof. It is known that every locally connected 1-dimensional unicoherent continuum is a dendrite [2, §57, III, Corollary 8, p. 442]. Thus the conclusion follows from Propositions 6, 7 and 8.

10. Corollary. *For each continuum X the following conditions are equivalent.*

(11) *X is a dendrite;*
(12) *X satisfies condition (μ_0);*
(13) *X satisfies condition (μ);*
(14) *X satisfies condition (ω_0);*
(15) *X satisfies condition (ω).*

Proof. The implication from (11) to (12) is Mioduszewski's Theorem 1. Proposition 5 provides the implications (12) to each of (13) and (14), and each of (13) and (14) to (15). Finally (15) implies (11) by Theorem 9.

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