PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1697–1701
S 0002-9939(99)05152-7
Article electronically published on September 30, 1999

A GENERALIZATION OF A THEOREM OF HEINS

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(Communicated by Steven R. Bell)

In memory of Professor M. Solveig Espelie

Abstract. Let $\mathcal{H}(\Delta, \Delta)$ be the family of holomorphic selfmaps of the unit disk Δ in the complex plane C. Heins established the continuity of the functional ψ which assigns to $f \in \overline{\mathcal{H}(\Delta, \Delta)} - \{id\}$ (id denotes the identity map) either (i) the fixed point of f or (ii) the limit of its iterations or (iii) $f(\Delta)$ if $f(\Delta) \cap \partial\Delta \neq \emptyset$ ($\partial\Delta$ represents the boundary of Δ). Using an Abate extension of the Denjoy-Wolff lemma to strongly convex domains, we extend this result of Heins to selfmaps of strongly convex domains in C^n with C^2 boundary.

Some of the most interesting properties of the group of selfmaps of the unit disk Δ in the complex plane C are reflected in the lemmas of Schwarz and Julia [2]. Using these lemmas, Denjoy and Wolff [2] proved the theorem: Let $f : \Delta \to \Delta$ be a holomorphic map. Then the sequence $\{f^n\}$ of iterates of f where f^n is defined inductively as $f^1 = f$ and $f^n = f \circ f^{n-1}$ does not converge iff f is an automorphism of Δ with exactly one fixed point. Moreover $\lim f^n$, when it exists, is a constant map $z_0 \in \overline{\Delta}$ unless f is the identity (id denotes the identity map) and if the limit z_0 is in Δ, it is the fixed point of f. Heins [3] proved the following theorem where $\mathcal{H}(X, Y)$ denotes the space of holomorphic maps from a complex space X to a complex space Y endowed with the compact-open topology. The notation ∂A will be used for the boundary of a subset A of a topological space.

Theorem of Heins. *The functional ψ defined on $\overline{\mathcal{H}(\Delta, \Delta)} - \{id\}$ by*

$$\psi(f) = \begin{cases} z_0 & \text{if } f(z_0) = z_0 \in \Delta, \\ \lim f^n(0) & \text{if } \lim f^n(0) \in \partial\Delta, \\ f(0) & \text{if } f(\Delta) \subset \partial\Delta \end{cases}$$

is continuous.

In this note we utilize recent work of Abate [1] which extended the Denjoy-Wolff lemma to selfmaps of strongly convex domains in C^n with C^2 boundary to prove the following theorem, an extension of the theorem of Heins to such maps. If $\{A_n\}$ is a sequence of subsets of a topological space X, recall

$$\limsup A_n = \{x \in X : x = \lim x_{n_k}, x_{n_k} \in A_{n_k}, \{A_{n_k}\} \text{ a subsequence of } \{A_n\}\}.$$

We denote the dimension of a complex manifold X by $\dim X$ and use the notation $Fix(f)$ for the set of fixed points of a selfmap $f \in \mathcal{H}(X, X)$.

Received by the editors February 18, 1998 and, in revised form, July 13, 1998.
1991 *Mathematics Subject Classification.* Primary 32H99; Secondary 30F99, 32H15.
Key words and phrases. Iterates, fixed points, strongly convex, horosphere.

Main Theorem. *Let D be a strongly convex domain in C^n with C^2 boundary and let ψ be the set-valued map on $\overline{\mathcal{H}(D,D)}$ given by*

$$\psi(f)=\begin{cases} Fix(f) & \text{if } Fix(f)\neq\emptyset,\\ \alpha(D) & \text{if } \alpha=\lim f^n \text{ and } \alpha(D)\subset\partial D,\\ f(D) & \text{if } f(D)\subset\partial D.\end{cases}$$

Let $\{f_n\}$ be a sequence in $\overline{\mathcal{H}(D,D)}$ converging to f. Then

(1) $\limsup\psi(f_n)=\psi(f)$ *if* $\dim\psi(f)=0$, *and*

(2) $D\cap\limsup\psi(f_n)=\psi(f)$ *if* $D\cap\limsup\psi(f_n)\neq\emptyset$ *and* $\limsup\dim\psi(f_n)=\dim\psi(f)$.

The following additional notation and four propositions on holomorphic selfmaps of convex domains will be utilized to facilitate the proof of Main Theorem. It is not difficult to see that these propositions verify that the function ψ is well-defined and that $\dim\psi(f)$ and $\dim\psi(f_n)$ are meaningful in the statement of the theorem. The first three are from Abate [1] and the fourth may be found in [1] and [5]. As is standard, k_D denotes the Kobayashi distance on D, $k_D(z,w)=\inf_\sigma \ell(\sigma)$ where σ is a chain of holomorphic maps $\{\varphi_i\}_{i=1,\dots,m}$ in $\mathcal{H}(\Delta,D)$ such that for $z_1,\cdots,z_m\in\Delta$, $\varphi_1(0)=z$, $\varphi_i(z_i)=\varphi_{i+1}(0)$ for $i=1,\dots,m-1$ and $\varphi_m(z_m)=w$ and $\ell(\sigma)=\sum_{i=1}^m\rho_\Delta(0,z_i)$, and ρ_Δ is the Poincaré distance on Δ.

The small horosphere $E_z(x,R)$ and the big horosphere $F_z(x,R)$ with center $x\in\partial D$, pole $z\in D$ and radius $R>0$ are defined by Abate in [1] as

$$E_z(x,R)=\{w\in D:\limsup_{z'\to x}[k_D(w,z')-k_D(z,z')]<\frac{1}{2}\log R\}\quad\text{and}$$

$$F_z(x,R)=\{w\in D:\liminf_{z'\to x}[k_D(w,z')-k_D(z,z')]<\frac{1}{2}\log R\},$$

and applied to prove Propositions 1 and 2.

Proposition 1. *Let D be a strongly pseudo-convex domain in C^n with C^2 boundary. Then $\overline{F_z(x,R)}\cap\partial D=\{x\}$ for every $z\in D, x\in\partial D$ and $R>0$.*

Proposition 2. *Let D be a strongly convex domain in C^n with C^2 boundary and let $f\in\mathcal{H}(D,D)$ be without fixed points. Then the sequence of iterates of f converges to a point of the boundary of D.*

Proposition 3. *Every strongly convex domain D in C^n with C^2 boundary has a simple boundary, i.e. every holomorphic map $f\in\mathcal{H}(\Delta,C^n)$ such that $f(\Delta)\subset\overline{D}$ and $f(\Delta)\cap\partial D\neq\emptyset$ is a constant.*

Proposition 4. *Let D be a bounded convex domain in C^n and let $f\in\mathcal{H}(D,D)$ with $Fix(f)\neq\emptyset$. Then $Fix(f)$ is a holomorphic retract of D, i.e. $Fix(f)=\alpha(D)$ where $\alpha\in\mathcal{H}(D,D)$ and $\alpha\circ\alpha=\alpha$. In particular $Fix(f)=\alpha(D)$ is a closed complex submanifold of D.*

Proof of (1) of Main Theorem. To establish the inclusion $\limsup\psi(f_n)\subset\psi(f)$ we suppose that $p_n\in\psi(f_n)$, $p_n\to p$ and show that $p\in\psi(f)$. The inclusion $\psi(f)\subset\limsup\psi(f_n)$ is then a consequence of that inclusion since $\limsup\psi(f_n)\neq\emptyset$. Let $f\in\mathcal{H}(D,D)$, $p_n\in D$. If $p\in D$, then p is a fixed point of f and obviously $p\in\psi(f)$. If $p\in\partial D$, we conclude from Theorem 4.3 in [6] that $Fix(f)=\emptyset$ since $\dim Fix(f)\geq 1$ otherwise. To show $p\in\psi(f)$, it is enough to show that

$f^n(E_z(p,R)) \subset F_z(p,R)$. Let $w \in E_z(p,R)$ and choose a positive integer N_0 and $\epsilon > 0$ such that , for $n \geq N_0$,

$$k_D(w,p_n) < k_D(z,p_n) + \frac{1}{2}\log R - \epsilon.$$

Since $p_n \in Fix(f_n)$, we have for positive integers m,

$$k_D((f_n)^m(w),p_n) \leq k_D(w,p_n).$$

Now, we have

$$|k_D((f_n)^m(w),p_n) - k_D(f^m(w),p_n)| \leq k_D((f_n)^m(w),f^m(w)) \to 0$$

as $n \to \infty$. Choose $N_1 \geq N_0$ such that, for $n \geq N_1$,

$$k_D(f^m(w),p_n) < k_D(z,p_n) + \frac{1}{2}\log R - \frac{\epsilon}{2}.$$

It follows that

$$\liminf_{x\to p}[k_D(f^m(w),x) - k_D(z,x)] \leq \liminf_{n\to\infty}[k_D(f^m(w),p_n) - k_D(z,p_n)] < \frac{1}{2}\log R$$

and $f^m(w) \in F_z(p,R)$. Consequently $p \in \psi(f)$.

Now suppose $f \notin \mathcal{H}(D,D)$ and $p_n \in D$. Then $f(D) \cap \partial D$ is a singleton, say, $\{q\}$. We claim that $p = q$. Otherwise let $z \in D$, $R > 0$ and $w \in E_z(p,R)$. Theorem 2.3.36 of Abate [1] provides N_1 and $K = K(p,q) > 0$ such that, for $n > N_1$,

$$k_D(f_n(w),p_n) > -\frac{1}{2}\log d(f_n(w),\partial D) - \frac{1}{2}\log d(p_n,\partial D) - K$$

where d is the Euclidean distance. On the other hand, Theorem 2.3.32 in [1] shows that there exists $c > 0$ depending only on the point z such that, for $w \in D$,

$$k_D(z,w) \leq c - \frac{1}{2}\log d(w,\partial D).$$

As in the previous case, there exist $\epsilon > 0$ and N_0 such that , for $n > N_0$,

$$k_D(w,p_n) < k_D(z,p_n) + \frac{1}{2}\log R - \epsilon.$$

Ultimately

$$\begin{aligned}\frac{1}{2}\log R - \epsilon &> k_D(w,p_n) - k_D(z,p_n)\\ &\geq k_D(f_n(w),p_n) - k_D(z,p_n)\\ &> -\frac{1}{2}\log d(f_n(w),\partial D) - \frac{1}{2}\log d(p_n,\partial D) - k_D(z,p_n) - K\\ &> -\frac{1}{2}\log d(f_n(w),\partial D) - c - K,\end{aligned}$$

a contradiction. Therefore $p = q \in f(D) \cap \partial D = \psi(f)$.

Finally suppose $p_n \in \partial D$ ultimately. We assume that the origin of C^n is in D and choose a positive real sequence $\{r_k\}$ such that $r_k \uparrow 1$. The sequence $g_{n,k} = r_k f_n$ in $\mathcal{H}(D,D)$ converges to f_n as $k \to \infty$. Each $g_{n,k}$ has a relatively compact image in D and thus a unique fixed point $q_{n,k}$ [4], and by the previous case for fixed n the sequence $\{q_{n,k}\}$ converges to p_n. We choose by induction a diagonal sequence $\{m_j\}$ such that $g_j = g_{j,m_j} \to f$ and $q_j = q_{j,m_j} \to p$. Then by the previous case we conclude that $p \in \psi(f)$. Let K_j be a sequence of relatively compact subsets of D such that $D = \bigcup_j K_j$ and $\overline{K_j} \subset K_{j+1}$. Suppose $m_1, ..., m_j$ have been chosen.

Choose m_{j+1} such that $m_{j+1} > m_j$, $d_D(g_{j+1,m_{j+1}}(w), f_{j+1}(w)) < \frac{1}{j+1}$ for $w \in \overline{K_{j+1}}$ and $d(q_{j+1,m_{j+1}}, p_{j+1}) < \frac{1}{j+1}$. The proof of part (1) is complete. $\square$

Prior to the proof of part (2) of Main Theorem, we present two examples. In connection with part (2), Example 5 shows that the hypothesis

$$\limsup \dim \psi(f_n) \leq \dim \psi(f) \text{ and } D \cap \limsup \psi(f_n) \neq \emptyset$$

is not enough to force $D \cap \limsup \psi(f_n) = \psi(f)$. In the course of the proof of part (2) below it is proved that

$$\limsup \dim \psi(f_n) \leq \dim \psi(f) \text{ if } D \cap \limsup \psi(f_n) \neq \emptyset.$$

Example 6 establishes that it is possible for the conditions $f, f_n \in \mathcal{H}(D,D), f_n \to f, D \cap \limsup \psi(f_n) = \emptyset,\ Fix(f_n) \neq \emptyset,\ Fix(f) \neq \emptyset$, and $\dim Fix(f_n) < \dim Fix(f)$ to hold simultaneously.

Example 5. Let $D = \{(z,w) \in \Delta^2 : |z|^2 + |w|^2 < 1\}$ and let $f_n \in \mathcal{H}(D,D)$ be defined by $f_n(z,w) = (ze^{i2\pi(1-\frac{1}{n})}, w^n)$. Then $f_n \to f \in \mathcal{H}(D,D)$ defined by $f(z,w) = (ze^{i2\pi}, 0)$ while $\psi(f) = Fix(f) = \{(z,0) : z \in \Delta\}$ and $\limsup \psi(f_n) = \{(0,0)\}$.

Example 6. Let D be as in Example 5 and let $f_n \in \mathcal{H}(D,D)$ be defined by $f_n(z,w) = ((1-\frac{1}{n})(z+\frac{1}{n}), 0)$. Then $Fix(f_n) = \{(1-\frac{1}{n}, 0)\},\ \limsup \psi(f_n) = \{(1,0)\},\ f_n \to f \in \mathcal{H}(D,D)$ defined by $f(z,w) = (z,0)$.

Proof of (2) of Main Theorem. The inclusion $D \cap \limsup \psi(f_n) \subset \psi(f)$ follows easily. To prove the other inclusion necessary for equality, we observe that $Fix(f) \neq \emptyset$, and $Fix(f_n) \neq \emptyset$ occurs frequently. Choose a subsequence of $\{f_n\}$, called again $\{f_n\}$, a sequence $\{\alpha_n\}$ in $\mathcal{H}(D,D)$ and a function α satisfying the conditions:

(a) $\psi(f) = Fix(f),\ \psi(f_n) = Fix(f_n) = \alpha_n(D),\ \dim \psi(f) = \dim \psi(f_n),\ \alpha_n \to \alpha,\ \alpha_n \circ \alpha_n = \alpha_n$ and

(b) $\emptyset \neq \alpha(D \cap \limsup \psi(f_n)) \subset D$.

By Propositions 3 and 4, $\alpha \in \mathcal{H}(D,D),\ \alpha \circ \alpha = \alpha$, and $\alpha(D)$ and $Fix(f)$ are closed complex submanifolds of D with $\alpha(D) \subset Fix(f)$ (note that $\alpha(D) = D \cap \limsup \psi(f_n)$ also holds for α). We claim $\dim \alpha(D) \geq \dim Fix(f)$. By Vigué [5] (see also 2.4.3 in [1]), for $q \in D$, $\dim \alpha_n(D)$ is the multiplicity of 1 as eigenvalues of $(d\alpha_n)_{\alpha_n(q)}$, the differential map of α_n at $\alpha_n(q)$. If v_n is an eigenvector of $(d\alpha_n)_{\alpha_n(q)}$ with eigenvalue 1 and $v_n \to v$, then v is an eigenvector of $(d\alpha)_{\alpha(q)}$ with eigenvalue 1. Hence $\dim \alpha_n(D) \leq \dim \alpha(D)$ and $\dim Fix(f) \leq \dim \alpha(D)$. Consequently $\psi(f) = \alpha(D)$ and the proof of (2) is complete. $\square$

In closing we point out that we have also proved the following extension of a theorem of Vigué [6] which showed that $\limsup \dim Fix(f_n) \leq \dim Fix(f)$ under the assumptions that $f_n \in \mathcal{H}(D,D),\ f_n \to f \in \mathcal{H}(D,D)$ and $K \cap Fix(f_n) \neq \emptyset$ for the sequence $\{f_n\}$ and some compact $K \subset D$.

Theorem. *Let D be a bounded convex domain in C^n. Let $\{f_n\}$ be a sequence in $\mathcal{H}(D,D)$ converging to $f \in \mathcal{H}(D,D)$ and satisfying $D \cap \limsup Fix(f_n) \neq \emptyset$. Then $D \cap \limsup Fix(f_n)$ is a complex submanifold of $Fix(f)$ and*

$$\limsup \dim Fix(f_n) \leq \dim D \cap \limsup Fix(f_n) \leq \dim Fix(f).$$

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