

INTERPOLATING SEQUENCES IN THE SPECTRUM OF H^∞ I

RAYMOND MORTINI

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ABSTRACT. We show that a sequence of trivial points in $M(H^\infty)$ is interpolating if and only if it is discrete. This answers a question of K. Izuchi. We also give a sufficient topological condition for a sequence of nontrivial points to be interpolating.

Let H^∞ be the uniform algebra of all bounded analytic functions in the open unit disk \mathbb{D} . Its spectrum, or maximal ideal space, is the space $M(H^\infty)$ of all nonzero multiplicative linear functionals on H^∞ endowed with the weak- $*$ -topology. As usual, we identify a function f in H^∞ with its Gelfand transform, \hat{f} , defined by $\hat{f}(m) = m(f)$ for $m \in M(H^\infty)$. Moreover, we look upon \mathbb{D} as a subset of $M(H^\infty)$. A sequence $(x_n)_{n \in \mathbb{N}}$ of points in $M(H^\infty)$ is said to be *interpolating* if for any bounded sequence $(w_n)_{n \in \mathbb{N}}$ of complex numbers there exists a function $f \in H^\infty$ so that $f(x_n) = w_n$ for every n . Carleson's well known interpolation theorem describes the interpolating sequences in the unit disk (see the book [4] of Garnett for an extensive presentation of this beautiful theory). In fact, a sequence (a_n) in \mathbb{D} is interpolating if and only if

$$(C) \quad \inf_{k \in \mathbb{N}} \prod_{j: j \neq k} \left| \frac{a_j - a_k}{1 - \bar{a}_j a_k} \right| \geq \delta > 0.$$

It is the aim of this paper to continue the study of the interpolating sequences in the whole spectrum of H^∞ . For previous results in this direction we refer the reader to [1], [2], [5], [6], [7], [9], [10], [11], [14], [15].

To proceed, we need to present a few definitions.

Let x and y be two points in $M(H^\infty)$. We define the pseudohyperbolic distance of x to y by

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, \|f\|_\infty \leq 1, f(y) = 0\}.$$

Then for $a, b \in \mathbb{D}$ we have $\rho(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right|$.

It is well known that the relation defined on $M(H^\infty)$ by

$$x \sim y \quad \Leftrightarrow \quad \rho(x, y) < 1$$

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point $m \in M(H^\infty)$ is called the Gleason part of m and is denoted by $P(m)$. If the

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part, $P(m)$, consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman's theory [8], [4] shows that for every Gleason part $P(m)$ there is a continuous map L_m of \mathbb{D} onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m$ is analytic on \mathbb{D} for all $f \in H^\infty$. When the Gleason part of m is trivial, L_m is just a constant map. When $P(m)$ is nontrivial, the map L_m is a bijection. The set of all nontrivial points in $M(H^\infty)$ is denoted by G , and the set of all trivial points is denoted by Γ . It is worth noting that there do exist nontrivial parts which are not homeomorphic to \mathbb{D} (see [8]). A description of the homeomorphic parts has been given in [5]. Moreover, it is well known that there exists trivial points not belonging to the Shilov boundary (see e.g. [3], p. 162, [4], p. 438 or [8], p. 78).

Let us call a sequence of different points $(x_n)_{n \in \mathbb{N}}$ in a topological space (X, \mathcal{T}) *discrete*, if the restriction of the topology \mathcal{T} to the set $\{x_n : n \in \mathbb{N}\}$ is the discrete topology. We have the following elementary and well known lemma.

Lemma 0.1. *Let X be a compact Hausdorff space and let (x_n) be a sequence of different points in X . Then the following assertions are equivalent:*

- (1) (x_n) is discrete.
- (2) There exists a sequence of open sets U_n with $x_n \in U_n$ whose closures are pairwise disjoint.
- (3) $x_j \notin \overline{\{x_n : n \neq j\}}$.

Remark. If the topology is merely assumed to be a T_1 topology, then (1) and (3) above are equivalent.

As an example of a T_1 -topology, we mention the hull-kernel topology in $M(H^\infty)$. Recall that a set in $M(H^\infty)$ is hull-kernel closed, if the hull of the ideal $I(E) = \{f \in H^\infty : f \equiv 0 \text{ on } E\}$ coincides with E (see [3], p. 13).

We are now able to briefly review some of the results known so far on interpolating sequences in $M(H^\infty)$. Most of them have been obtained by S. Axler and P. Gorkin in [1], by K. Izuchi in [9], [10], [11] and by Gorkin, Lingenberg and the author of this paper in [5]. Some results have been reproven or/and slightly generalized by Sun Wei [14], [15].

First we note that every interpolating sequence is discrete (in the weak- $*$ -topology). However, since the closure E of an interpolating sequence is obviously an interpolation set (that is, $H^\infty|_E = C(E)$), hence hull-kernel closed, it is also clear that an interpolating sequence is discrete in the hull-kernel topology of $M(H^\infty)$. Of course, not every hull-kernel discrete sequence is interpolating (take any non-interpolating Blaschke sequence e.g.). It is useful to notice that a sequence (x_n) is hull-kernel discrete if and only if the following condition (B_2) holds:

$$(B_2) \quad \text{There exist functions } f_n \text{ in } H^\infty \text{ such that} \\ \|f_n\| \leq 1, f_n(x_k) = 0 \text{ for } k \neq n \text{ and } f_n(x_n) \neq 0.$$

In view of a generalization of Carleson's theorem above, Izuchi [11] introduced the following conditions for a sequence (x_n) of points in $M(H^\infty)$.

$$(C_1) \quad \inf_k \prod_{j:j \neq k} \rho(x_j, x_k) \geq \delta > 0.$$

(C₂) There exist functions f_n in H^∞ such that $\|f_n\| \leq 1, f_n(x_k) = 0$ for $k \neq n$ and $\inf_n |f_n(x_n)| \geq \delta > 0$.

(C₃) There exist functions f_n in H^∞ such that

$$\|f_n\| \leq 1, f_n(x_n) = 0, \prod_{n=1}^\infty f_n \in H^\infty \text{ and } \inf_k \left| \left(\prod_{n:n \neq k} f_n \right)(x_k) \right| \geq \delta > 0.$$

(D₂) For every $\sigma \in]0, 1[$, there exist $f_n \in H^\infty$ such that $\|f_n\| \leq 1, f_n(x_k) = 0$ for $k \neq n$ and $\inf_n |f_n(x_n)| \geq \sigma > 0$.

In the case of points in the unit disk, all the conditions (C₁), (C₂) and (C₃) are equivalent to Carleson’s condition (C). Izuchi showed in [11] that (C₃) \implies (C₂) \implies (C₁). Obviously (D₂) \implies (C₂) \implies (B₂).

Moreover, if (x_n) is interpolating, then, by the open mapping theorem, (x_n) satisfies (C₂). If (x_n) is a sequence of trivial points, then (x_n) is interpolating if and only if (x_n) satisfies (C₂) (see [11], p. 224). Finally, if (x_n) satisfies condition (D₂), then (x_n) is interpolating (see [11], p. 227).

In [5] it is shown that whenever a sequence (x_n) is entirely contained in a Gleason part which is homeomorphic to \mathbb{D} , then (x_n) is interpolating if and only if (x_n) satisfies condition (C₁), the direct analogue of Carleson’s condition (C) for the disk. On the other hand, condition (C₁) is far from being sufficient for an arbitrary sequence to be interpolating (take for example a nondiscrete sequence of trivial points). In [11] Izuchi even gave an example of discrete sequence in $M(H^\infty)$, entirely contained in a nontrivial Gleason part, satisfying (C₁), but which is not interpolating. (See also [14] for another example.) To get more insight into the structure of the interpolating sequences in $M(H^\infty)$, the following questions were asked by Izuchi ([11], p. 227 and 231):

(1) *Suppose that $x_n \in G$ for all n . Is condition (C₂) equivalent to condition (C₃)?*

We note that one of the principal results of Izuchi in [11] was that whenever $x_n \in G$ satisfies condition (C₃), then (x_n) is interpolating.

(2) *Suppose that $x_n \in \Gamma$ for all n . Is condition (C₂) in that case equivalent to (C₃)?*

(3) *Is every discrete sequence of trivial points interpolating?*

In this paper we shall give positive answers to questions (2) and (3). Moreover, we shall present another, more topological, sufficient condition for a sequence of nontrivial points to be interpolating.

1. TRIVIAL POINTS

In this section we give a complete topological characterization of the interpolating sequences consisting of trivial points. The proof is based on the following result of D. Suarez. To state it, we need to recall that a Blaschke product

$B(z) = \prod_{n=0}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$ is said to be interpolating if its zero sequence (a_n) is an interpolating sequence for H^∞ . Finite Blaschke products with distinct zeros are considered as interpolating. Moreover, we note that by Hoffman a point $m \in M(H^\infty)$

has a nontrivial Gleason part if and only if it lies in the closure of an interpolating sequence in \mathbb{D} .

Lemma 1.1 ([13], p. 888). *Let $E \subseteq M(H^\infty)$ be a closed set and B a Blaschke product such that $|B| > 0$ on E . Let $0 < \sigma < 1$. Then there exists a finite factorization $B = B_0 B_1 \cdots B_N$, so that B_0 is a finite product of interpolating Blaschke products and $|B_j| > \sigma$ on E for $1 \leq j \leq N$.*

Theorem 1.2. *Let $(x_n)_n$ be a sequence of trivial points in $M(H^\infty)$. Then $(x_n)_n$ is interpolating if and only if $(x_n)_n$ is discrete.*

Proof. We have already seen that any interpolating sequence is discrete. So, conversely, let (x_n) be a discrete sequence of trivial points. By Izuchi's result mentioned in the introduction, it is sufficient to show that (x_n) satisfies (D_2) ; that is, for every $\varepsilon \in]0, 1[$ there exists a sequence of functions f_n in the unit ball of H^∞ so that $f_n(x_k) = 0$ for every $k \neq n$ and such that $\inf_{n \in \mathbb{N}} |f_n(x_n)| > \varepsilon$.

So, let $\varepsilon \in]0, 1[$. Choose $\varepsilon_n \in]0, 1[$, so that $\prod_{n=1}^\infty \varepsilon_n > \varepsilon$. According to Lemma 0.1 there exist open sets U_j with $x_j \in U_j$ such that the closures of the U_j are pairwise disjoint. Obviously, this implies that $x_n \notin \overline{\{x_k : k \neq n\}}$. So we may assume that $U_n \cap \overline{\bigcup_{k \neq n} U_k} = \emptyset$. Fix n . Using the fact proven by Suarez ([12], p. 242) that H^∞ is separating, there exists a function $g_n \in H^\infty$ of norm less than one such that $g_n(x_n) = 0$, but $|g_n| > 0$ on $\overline{\bigcup_{k \neq n} U_k}$. Let B be the Blaschke factor of g_n . Then $g_n = BG$ for some function $G \in H^\infty$ which has no zeros in \mathbb{D} . By Lemma 1.1, there exist factors B_j of B ($j = 1, \dots, N$), such that $|B_j| > \sqrt{\varepsilon_n}$ on $\overline{\bigcup_{k \neq n} U_k}$ and $B = CB_1 \cdots B_N$, where C is a finite product of interpolating Blaschke products. Since interpolating Blaschke products never vanish at a trivial point (see [8]), we see that at least one of the B_j or G vanishes at x_n . By taking roots of G , we may also assume that $|G| > \sqrt{\varepsilon_n}$ on $\overline{\bigcup_{k \neq n} U_k}$. We conclude that there exists a function \tilde{g}_n of norm less than one vanishing at x_n and which is bigger than ε_n on $\overline{\bigcup_{k \neq n} U_k}$. Let $\tilde{g}_n = \tilde{B}_n \tilde{G}_n$, where \tilde{B}_n is the Blaschke factor of \tilde{g}_n . By deleting, if necessary, a finite number of zeros of each of the Blaschke products \tilde{B}_n , we may assume that the product $\prod_{n=1}^\infty \tilde{B}_n$ converges locally uniformly on \mathbb{D} . Since for any bounded holomorphic function with no zeros in \mathbb{D} , the sequence of its n -th roots (main branches) converges in \mathbb{D} locally uniformly to 1, we may assume that the series $\sum_{n=1}^\infty (|1 - \tilde{G}_n^{1/N_n}|)$ converges locally uniformly on \mathbb{D} for suitable chosen integers N_n . Hence the infinite product $\prod_{n=1}^\infty \tilde{G}_n^{1/N_n}$ converges to a bounded analytic function of norm less than one. Thus, we have constructed functions $\tilde{B}_n \tilde{G}_n^{1/N_n}$ which vanish at x_n , are bigger than ε_n on $\overline{\bigcup_{k \neq n} U_k}$ and whose product

$$\prod_{n=1}^\infty \tilde{B}_n \tilde{G}_n^{1/N_n}$$

converges locally uniformly in \mathbb{D} . Let

$$f_n = \prod_{k:k \neq n} \tilde{B}_k \tilde{G}_k^{1/N_k}.$$

Then, of course, these products converge and $\|f_n\| \leq 1$. Moreover, $f_n(x_k) = 0$ whenever $k \neq n$. Note also that for $k \neq n$, we have $|\tilde{B}_k \tilde{G}_k^{1/N_k}| > \varepsilon_k$ on $U_n \cap \mathbb{D}$.

Hence for every $z \in U_n \cap \mathbb{D}$

$$|f_n(z)| = \prod_{k \neq n} \left| \tilde{B}_k \tilde{G}_k^{1/N_k} \right|(z) \geq \prod_{k \neq n} \varepsilon_k > \varepsilon.$$

Since x_n lies in the closure of $U_n \cap \mathbb{D}$, we conclude that $|f_n(x_n)| > \varepsilon$. □

As a corollary we obtain Hoffman’s unpublished result that a sequence of points in the Shilov boundary of H^∞ is interpolating if and only if it is a discrete sequence. His proof was entirely different and valid for a large class of strongly logmodular algebras with extremely disconnected Shilov boundary.

Corollary 1.3. *A sequence of trivial points in $M(H^\infty)$ is discrete if and only if it is hull-kernel discrete.*

Proof. Since the hull-kernel topology is weaker than the weak- $*$ -topology, every hull-kernel discrete sequence obviously is discrete. Conversely, let (x_n) be discrete. Then, by Theorem 1.2, (x_n) is an interpolating sequence. Those, however, are, by the discussion in the introduction, hull-kernel discrete. □

The proof of Theorem 1.2 shows that the given discrete sequence actually satisfies (C_3) . Hence, in view of the aforementioned facts that $(C_3) \Rightarrow (C_2) \Rightarrow$ discrete, we obtain the following corollary:

Corollary 1.4. *If (x_n) is a sequence of trivial points in $M(H^\infty)$, then the conditions (C_2) and (C_3) are equivalent.*

2. NONTRIVIAL POINTS

In this section we give a topological condition (in terms of the pseudohyperbolic metric) for a sequence of nontrivial points to be interpolating in $M(H^\infty)$. For the reader’s convenience, we first recall two results of K. Hoffman.

Lemma 2.1 ([8], [4], sec. 10). *Let m be a nontrivial point and U a neighborhood of m in $M(H^\infty)$. Then for every $0 < \delta < 1$ there exists an interpolating Blaschke product B with zeros (z_n) in $\mathbb{D} \cap U$ such that B vanishes at m and for which the uniform separating constant*

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2) |B'(z_n)| > \delta.$$

Let $Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$ be the zero set for $f \in H^\infty$ and let $\rho(m, E) = \inf\{\rho(m, x) : x \in E\}$ denote the pseudohyperbolic distance of a point $m \in M(H^\infty)$ to a subset E of $M(H^\infty)$.

Hoffman’s Lemma 2.2 ([8], p. 89, 86, 106 and [4], p. 379, 404). *Let $0 < \delta < 1$, $0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$ (that is, $0 < \eta < \rho(\delta, \eta)$) and let*

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

Then any interpolating Blaschke product B with zeros $\{z_n\}$ such that $\delta(B) > \delta$ satisfies:

- (1) $Z(B)$ is the closure of the zero set of B in \mathbb{D} ,
- (2) $\rho(x, y) \geq \delta$ for any $x, y \in Z(B)$, $x \neq y$, and
- (3) $\{m \in M(H^\infty) : |B(m)| < \varepsilon\} \subseteq \{m \in M(H^\infty) : \rho(m, Z(B)) < \eta\}$
 $\subseteq \{m \in M(H^\infty) : |B(m)| < \eta\}.$

In particular, by choosing $\varepsilon > 0$ so that $\eta^2 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta \eta}$, we obtain that $|B(m)| > \eta^2$ whenever $\rho(m, Z(B)) > \eta$.

With the exception of (2), these results are stated in Hoffman’s paper. Although (2) is not explicitly stated, one can prove it as follows.

Let $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ be disjoint subsets of the zeros of B in \mathbb{D} such that $x \in \overline{\{u_n\}}$ and $y \in \overline{\{v_n\}}$. Let B_1 denote the subproduct of B associated with $\{u_n\}$, and let B_2 denote the subproduct of B associated with $\{v_n\}$. Then, for each n , we see that

$$|B_2(u_n)| \geq \prod_{k: z_k \neq u_n} \rho(z_k, u_n) \geq \delta.$$

Since B_2 is continuous on the maximal ideal space, $|B_2(x)| \geq \delta$ and $B_2(y) = 0$. Thus

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, \|f\|_\infty \leq 1, f(y) = 0\} \geq |B_2(x)| \geq \delta.$$

Theorem 2.3. *Let (x_n) be a discrete sequence of nontrivial points in $M(H^\infty)$. Suppose that there exist pairwise disjoint open sets U_n in $M(H^\infty)$ containing x_n such that*

$$\delta = \inf_j \prod_{n: n \neq j} \rho(U_j, U_n) > 0.$$

Then (x_n) is an interpolating sequence satisfying (C_3) .

Proof. Let $\varepsilon_k, 0 < \varepsilon_k < 1$, satisfy $\prod_{k=1}^\infty \varepsilon_k > \delta$. Choose $\varepsilon_{j,k}$ with $0 < \varepsilon_{j,k} < 1$ and $\inf_j \prod_{k: k \neq j} \varepsilon_{j,k} = \delta_1 > 0$ such that

$$\rho(U_j, U_k) > \varepsilon_{j,k} \rho(x_j, x_k).$$

(For example $\varepsilon_{j,k} = \rho(U_j, U_k)^2 / \rho(x_j, x_k)$.) Fix k . Since $\rho(U_k, U_j) \rightarrow 1$ for $j \rightarrow \infty$, there exists $\theta(k) \in \mathbb{N}$ such that $\rho(U_k, U_j) > \varepsilon_k$ for every $j \geq \theta(k)$. Without loss of generality $k < \theta(k) < \theta(k + 1)$. Put $\theta(0) = 1$.

We are now going to show that (x_n) satisfies (C_3) .

Let

$$\eta_k = \max\{\varepsilon_k, \varepsilon_{1,k} \rho(x_1, x_k), \varepsilon_{2,k} \rho(x_2, x_k), \dots, \varepsilon_{\theta(k),k} \rho(x_{\theta(k)}, x_k)\}.$$

Choose $\delta_k \in]0, 1[$ so that $\eta_k < \frac{1 - \sqrt{1 - \delta_k^2}}{\delta_k}$. By Lemma 2.1 there exists an interpolating Blaschke product b_k such that $b_k(x_k) = 0, \delta(b_k) > \delta_k$ and $Z(b_k) \subseteq U_k$. Let $j \geq \theta(k)$ and let $x \in U_j$. Since $\rho(U_k, U_j) > \varepsilon_k$, we have by Hoffman’s Lemma that $|b_k(x)| > \varepsilon_k^2$. Now let $1 \leq j < \theta(k)$. Then $Z(b_k) \subseteq U_k, x \in U_j$ and $\rho(U_j, U_k) > \varepsilon_{j,k} \rho(x_j, x_k)$ imply by Hoffman’s Lemma that $|b_k(x)| > \varepsilon_{j,k}^2 \rho^2(x_j, x_k)$.

By cutting off, if necessary, for each k a finite number of zeros (in \mathbb{D}) of b_k , we may assume that the infinite product $\prod_{k=1}^\infty b_k$ converges locally uniformly in \mathbb{D} to a Blaschke product. Of course we should not remove the zero x_k if x_k happens to lie in the unit disk.

Now fix j . Choose $k \in \mathbb{N} \cup \{0\}$ so that $\theta(k) \leq j < \theta(k + 1)$. For every $z \in U_j \cap \mathbb{D}$ write

$$\left| \left(\prod_{n: n \neq j} b_n \right) (z) \right| = \prod_{n: n \neq j} |b_n(z)| = \prod_{n=1}^k |b_n(z)| \prod_{k+1 \leq n < j} |b_n(z)| \prod_{n: n > j} |b_n(z)|.$$

To estimate the first product, we note that $n \leq k < \theta(k) \leq j$. Since $n \leq k$ implies $\theta(n) \leq \theta(k) \leq j$, we obtain

$$\prod_{n=1}^k |b_n(z)| > \prod_{n=1}^k \varepsilon_n^2.$$

For the second product, we see that $k + 1 \leq n < j < \theta(k + 1) \leq \theta(n)$. Hence

$$\prod_{k+1 \leq n < j} |b_n(z)| > \prod_{k+1 \leq n < j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).$$

Finally, for the last product we have $j < n < \theta(n)$. Hence

$$\prod_{n>j} |b_n(z)| > \prod_{n>j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).$$

Therefore

$$\left| \left(\prod_{n:n \neq j} b_n \right) \right| (z) > \prod_{n=1}^\infty \varepsilon_n^2 \prod_{n:n \neq j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n) > \delta^4 \delta_1^2.$$

Since $x_j \in \overline{U_j \cap \mathbb{D}}$, we see that $\left| \left(\prod_{n:n \neq j} b_n \right) \right| (x_j) \geq \delta^4 \delta_1^2$. This shows that (x_n) satisfies (C_3) . Since $x_n \in G$, we get by Izuchi's result [11] cited above that (x_n) is an interpolating sequence. \square

Remark. It is worth noting that the condition $\delta = \inf_j \prod_{n:n \neq j} \rho(U_j, U_n) > 0$ is in fact necessary and sufficient for (x_n) to be interpolating whenever $x_n \in \mathbb{D}$ for every n . We doubt, however, whether it will be necessary in the general case. Evidence for the failure of necessity comes from the following facts.

Let (x_j) be a sequence in a nontrivial Gleason part $P(m)$ such that for an open set U in $M(H^\infty)$ with $m \in U$ we have $\rho(U, x_j) \rightarrow 1$ for $j \rightarrow \infty$; then (x_j) cannot cluster back into the part itself, that is,

$$(*) \quad \overline{\{x_j : j \in \mathbb{N}\}} \setminus \{x_j : j \in \mathbb{N}\} \cap P(m) = \emptyset$$

and $x_n \notin \overline{\{x_j : j \neq n\}}$ for every n . This can be seen in the following way: Assume that there exists a cluster point y of (x_n) such that $\eta := \rho(m, y) < 1$. Choose $\delta \in]0, 1[$ so that

$$0 \leq \sqrt[4]{\eta} < (1 - \sqrt{1 - \delta^2})/\delta.$$

Let b be any interpolating Blaschke product with $\delta(b) > \delta$, $Z(b) \subseteq U$ and $b(m) = 0$. Then by Schwarz's Lemma ([4], p. 401)

$$|b(y)| = \rho(b(y), b(m)) \leq \rho(y, m) = \eta.$$

But $\rho(Z(b), x_n) > \sqrt[4]{\eta}$ for all $n \geq n_0$. Hence, by Hoffman's Lemma, $|b(x_n)| > \sqrt{\eta}$ for $n \geq n_0$. In particular, $|b(y)| \geq \sqrt{\eta}$, an obvious contradiction.

On the other hand, if B is an interpolating Blaschke product which vanishes at a point $m \in G$ whose Gleason part is not homeomorphic to \mathbb{D} , then it is known from [5] that the sequence (x_n) of its zeros in $P(m)$ does not contain any isolated point within the topological space $P(m)$; in other words $x_n \in \overline{\{x_j : j \neq n\}}$ for every n . Now choose an interpolating subsequence of (x_n) —the existence of which can be deduced from ([1], Theorem 3). We think that this sequence may have cluster points within the part itself, so violating $(*)$.

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DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE METZ, ILE DU SAULCY, F-57045 METZ, FRANCE

E-mail address: mortini@poncelet.univ-metz.fr