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ON THE SCARCITY OF LATTICE-ORDERED MATRIX ALGEBRAS II

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ABSTRACT. We correct and complete Weinberg's classification of the lattice-orders of the matrix ring $\mathbb{Q}_2$ and show that this classification holds for the matrix algebra F_2 where F is any totally ordered field. In particular, the lattice-order of F_2 obtained by stipulating that a matrix is positive precisely when each of its entries is positive is, up to isomorphism, the only lattice-order of F_2 with $1 > 0$. It is also shown, assuming a certain maximum condition, that $(F^+)_n$ is essentially the only lattice-order of the algebra F_n in which the identity element is positive.

1. INTRODUCTION

Let F_n denote the $n \times n$ matrix ring over the ring F. If (F, F^+) is a lattice-ordered ring (ℓ-ring) with positive cone $F^+ = \{\alpha \in F : \alpha \geq 0\}$, then $(F_n, (F^+)_n)$ is also an ℓ-ring. In [12] Weinberg has conjectured that if $F = \mathbb{Q}$ is the field of rational numbers and if $(\mathbb{Q}_n, P)$ is an ℓ-ring with $1 \in P$, then $(\mathbb{Q}_n, P)$ is isomorphic to $(\mathbb{Q}_n, (\mathbb{Q}^+)_n)$. He proved this conjecture for $n = 2$ and also provided, for each $1 < \beta \in \mathbb{Q}$, the following additional non-isomorphic lattice-orders P_β of $\mathbb{Q}_2$: there are idempotents f_1, f_2, f_3, f_4 in $\mathbb{Q}_2$ with $1 = (1-\beta)(f_1+f_2)+\beta(f_3+f_4)$ and $(\mathbb{Q}_2, P_\beta)$ is the ℓ-group direct sum of its totally ordered subrings $\mathbb{Q}f_1, \mathbb{Q}f_2, \mathbb{Q}f_3, \mathbb{Q}f_4$.

The assertion in [12] that each lattice-order of $\mathbb{Q}_2$ in which 1 is not positive is isomorphic to one of the P_β's is not quite correct, however. There is one additional lattice-order P_1 of $\mathbb{Q}_2$ that must be added to the list to make this statement correct. We will describe P_1 below.

If R is an ℓ-ring and an algebra over the totally ordered field F, then R is an ℓ-algebra over F if it is a vector lattice, that is, if $F^+R^+ \subseteq R^+$. We show that the description of the lattice-orders of $\mathbb{Q}_2$ that is given above also holds for the lattice-orders of F_2 which make it into an ℓ-algebra over F and that Weinberg's conjecture is true for F_n provided that 1 has its maximum number n of nonzero components in the decomposition of the ℓ-algebra (F_n, P) as a direct sum of totally ordered vector lattices over F.

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2. Another lattice-order for $\mathbb{Q}_2$

We will first review a few definitions. A convex ℓ-subgroup of an ℓ-group is a subgroup C that is a sublattice and is also convex: if $a \leq x \leq b$ with $a, b \in C$, then $x \in C$. An ℓ-ideal of an ℓ-ring is an ideal that is also a convex ℓ-subgroup. A vector lattice over a totally ordered field F is called archimedean over F if it has no nonzero bounded subspaces.

If R is a finite dimensional ℓ-algebra over a totally ordered field F and R has no nonzero nilpotent ℓ-ideals, that is, R is ℓ-semiprime, then R is archimedean over F by [2, Corollary 1, p. 51]. So it is a vector lattice direct sum of totally ordered ℓ-simple subspaces (see [4, p. 3.27] or [8, Theorem 2.12]). Of course, if F is a subfield of the real numbers $\mathbb{R}$, these totally ordered subspaces are embeddable in $\mathbb{R}$. Weinberg's method of proof for $\mathbb{Q}_2$ is to consider and eliminate all but two (actually, three) of the twenty-nine cases that arise depending on the number of summands, the dimensions of the summands, and the number and signs of the coordinates of 1. The error occurs in the twenty-eighth case, which is case (7d) of [12].

In (7d) we have that $\mathbb{Q}_2 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ as an ℓ-group where each E_i is isomorphic to $\mathbb{Q}$, $1 = e_1 + e_2 + e_3$ where $0 \neq e_i \in E_i$, e_1 and e_2 have the same sign and it is opposite to that of e_3, and $0 < n \in E_4$. The calculation down to the last sentence of (ii) is correct. So we have that $E_1 \oplus E_4, E_2 \oplus E_4$ and $E_1 \oplus E_2 \oplus E_3$ are subalgebras of $\mathbb{Q}_2$, and

$$e_1^2 = k_1 e_1,\ e_2^2 = k_2 e_2,\ k_i \in \mathbb{Q} \text{ and } k_1 k_2 > 0,\ n^2 = n,$$
$$e_2 n = e_2,\ e_1 n = k_1 n,\ n e_2 = k_2 n,\ n e_1 = e_1,\ e_1 e_2 = 0.$$

But e_1 and e_2 do not enter symmetrically in these calculations and so the assertion that $e_2 e_1 = 0$ is not correct. We will complete the calculation and produce another lattice-order of $\mathbb{Q}_2$. Now, since $e_2 e_1$ is in the subalgebra $E_1 \oplus E_2 \oplus E_3$,

$$0 \leq e_2 e_1 = x e_1 + y e_2 + z e_3 \text{ with } x, y, z \in \mathbb{Q} \text{ and } x e_1,\ y e_2,\ z e_3\ \geq 0;$$

so

$$0 = e_2 e_1 e_2 = y k_2 e_2 + z(1 - e_1 - e_2) e_2 = (y k_2 + z - z k_2) e_2,$$

and $y = (k_2 - 1) k_2^{-1} z$. We have that $e_2 e_1 \notin E_1$, since otherwise $\mathbb{Q}_2 e_1 = E_1$; so $z \neq 0$. Suppose that $e_3 > 0 > e_1, e_2$. Then $z > 0$ and $(k_2 - 1) k_2^{-1} z \leq 0$; but this is nonsense since $k_2 < 0$. So we must have that $e_3 < 0 < e_1, e_2$, and hence $x \geq 0$, $z < 0$ and $0 < k_2 \leq 1$. Since

$$0 = e_1 e_2 e_1 = x k_1 e_1 + z e_1 e_3,$$

$x = 0$ and $e_1 e_3 = 0$; thus $e_1 = e_1^2 = k_1 e_1$ and $k_1 = 1$. Also,

$$0 \geq e_3 e_2 = (1 - e_1 - e_2) e_2 = (1 - k_2) e_2 \geq 0$$

gives that $k_2 = 1, e_3 e_2 = 0, y = 0$ and

$$e_1 = n e_2 e_1 = z n e_3 = z n (1 - e_1 - e_2) = -z e_1;$$

hence, $e_2 e_1 = -e_3$. It is now easy to complete the following multiplication table:

	e_1	e_2	e_3	n
e_1	e_1	0	0	n
e_2	$-e_3$	e_2	e_3	e_2
e_3	e_3	0	0	$-e_2$
n	e_1	n	$-e_1$	n

The matrices

$$e_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ n = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

satisfy this table and the equation $1 = e_1 + e_2 + e_3$. Thus, this lattice-order is given by $P_1 = \mathbb{Q}^+ e_1 + \mathbb{Q}^+ e_2 + \mathbb{Q}^+(-e_3) + \mathbb{Q}^+ n$, and, again, $\mathbb{Q}_2$ is the ℓ-group direct sum of four totally ordered subrings.

3. Lattice-orders of F_2

In this section we modify Weinberg's proof to obtain his theorem for any totally ordered field.

Theorem 1. *Let F be a totally ordered field.*

(a) For each $1 < \beta \in F$ there is a lattice-order P_β of F_2 for which $R = (F_2, P_\beta)$ is an ℓ-algebra, and there are four idempotents f_1, f_2, f_3, f_4 in P_β such that R is the vector lattice direct sum of the subalgebras Ff_1, Ff_2, Ff_3, Ff_4 with $1 = (1-\beta)(f_1 + f_2) + \beta(f_3 + f_4)$.

(b) There are idempotents e_1, e_2 and n in F_2 and a nilpotent element e_3 such that $1 = e_1 + e_2 + e_3$, and if P_1 is the positive cone of the vector lattice direct sum $F_2 = Fe_1 \oplus Fe_2 \oplus F(-e_3) \oplus Fn$, then (F_2, P_1) is an ℓ-algebra over F.

(c) If $R = (F_2, P)$ is an ℓ-algebra over F, then R is isomorphic to $(F_2, (F^+)_2)$ if $1 \in P$, and, otherwise, R is isomorphic to (F_2, P_β) for exactly one $\beta \geq 1$.

Before giving the proof we will review some definitions and facts that we will need. If M is a left module over the ring D, then M is an ℓ-module over D if D is a po-ring, M is an ℓ-group, and $D^+ M^+ \subseteq M^+$. The element $d \in D^+$ is an f-element (respectively, a d-element) on M if $dx \wedge y = 0$ (respectively, $dx \wedge dy = 0$) whenever x and y are elements of M with $x \wedge y = 0$. Each f-element is a d-element. The additive subgroup $T({}_D M)$ of D generated by the set of f-elements on M is a convex directed subring of D. If D is an ℓ-ring, then $T({}_D M) = \{d \in D : |d|$ is an f-element on $M\}$, and $T(D) = T({}_D D) \cap T(D_D)$ is the convex ℓ-subring of f-elements of D. M is an f-module over D if $D^+ \subseteq T({}_D M)$. For example, each vector lattice over a totally ordered division ring is an f-module. If D is an ℓ-ring, then $T(D)$ is an f-ring; that is, $T(D)$ is a right and left f-module over itself. If D is a commutative po-ring and R is an ℓ-ring and an algebra over D, then R is called an ℓ-algebra if R is an f-module over D.

Proof of Theorem 1. The proof of (a) is given by using the matrices f_1, f_2, f_3, and f_4 given below, and the proof of (b) is given by using the matrices presented in the previous section. As for (c), suppose that P is a positive cone for F_2 such that $R = (F_2, P)$ is an ℓ-algebra over F. As indicated at the beginning of section 2, R is the vector lattice direct sum of at most four totally ordered subspaces, and in [12], for $F = \mathbb{Q}$, Weinberg considers the various cases that arise depending on the number of summands, the dimensions of the summands and the coordinates of 1. It is only the three cases (1c), (7b) and (7d) that lead to the lattice-orders $(\mathbb{Q}^+)_2, P_\beta$, and P_1 of $\mathbb{Q}_2$, respectively; in each of the other twenty-six cases it is shown that there is no lattice-order of the type under consideration. As can partially be seen from section 2 all of the arguments that are used in [12] for $\mathbb{Q}_2$ with the modification given in section 2 and some other minor modifications are valid for R (although some of these arguments can be shortened) with the exception of those used in the three cases (3c), (3d), and (5). In each of these cases the fact that a totally ordered subspace of $\mathbb{Q}_2$ is embeddable in the complete field $\mathbb{R}$ is used to eliminate the possibility of a lattice-order of the type under consideration. We proceed to show that this fact can be avoided. Note that if A and B are totally ordered archimedean F-subspaces of an ℓ-algebra and C is a convex subspace, then $AB \subseteq C$ provided that $ab \in C$ for some $0 \neq a \in A$ and $0 \neq b \in B$.

Let $R = (F_2, P)$ be an ℓ-algebra. In the three cases to be considered we have that $R = E_1 \oplus E_2$ as vector lattices over F where E_1 and E_2 are totally ordered subspaces.

(3c) Here we have $1 \in E_1$ and $\dim_F E_2 = 3$. Since E_2 is not an ideal of R, it cannot be a subring of R. So if $0 \neq f \in E_2$, then, by the Cayley-Hamilton theorem, $f^2 = \alpha + \beta f$ with $\alpha, \beta \in F$ and $\alpha > 0$. If $\beta = 0$, then $E_2^2 \subseteq E_1$, and this is impossible since fE_2 is 3-dimensional over F. But then the trace function $\mathrm{tr} : E_2 \to F$ is monic.

(3d) In this case $\dim_F E_i = 2$ and $0 < 1 \in E_1$. Then $E_1 = T(R)$ is a division ring and hence is central by our extension of Tamhankar's extension of Albert's Theorem [10, Corollary 15]. This is absurd.

(5) Here, $1 = e_1 + e_2$ and $e_1 < 0 < e_2$. Since E_i is not an ideal it cannot be a subring of R. So if $0 \neq f \in E_i$, then $f^2 = \alpha + \beta f$ with $\alpha \neq 0$. If $\beta = 0$, then $f^2 = \alpha e_1 + \alpha e_2 \geq 0$ implies that $\alpha = 0$. Thus $\beta \neq 0$ and, again, tr is monic on E_i. This is impossible since E_1 or E_2 is at least 2-dimensional over F. □

If D is a totally ordered ring with the property that $DaD \neq 0$ if $0 \neq a \in D$, then it is easy to see that, for any n, $(D^+)_n$ is maximal among those ring partial orders P of D_n for which $D^+P + PD^+ \subseteq P$; this is also the case if n is infinite and D_n is the ring of column finite matrices over D. None of the P_β are maximal since in these lattice-orders 1 is not positive. Each P_β is, however, contained in the cardinality of F many lattice-orders each one of which is isomorphic to $(F^+)_2$. Specifically, let P_1 be given by the matrices in section 2, and take $b, d \in F$ with $0 \leq b \leq -d$ and $d < 0$. Then if $A_{b,d} = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$, it is easy to check that $A_{b,d}^{-1} P_1 A_{b,d} \subseteq (F^+)_2$, and hence P_1 is contained in the lattice-order $X_{b,d} = A_{b,d}(F^+)_2 A_{b,d}^{-1}$ of F_2. In particular, $P_1 \subseteq X_{0,-1} = \begin{pmatrix} F^+ & -F^+ \\ -F^+ & F^+ \end{pmatrix}$. Also, $X_{b,d} \neq X_{b,h}$ if $d \neq h$ and $b > 0$. This is a consequence of the fact that $C^{-1}(F^+)_2 C = (F^+)_2$ with $\det C > 0$ (respectively, $\det C < 0$) exactly when $C = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ (respectively, $C = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$) and $xy > 0$. So $A^{-1}(F^+)_2 A = B^{-1}(F^+)_2 B$ requires that the columns of A are multiples of the columns of B where the multipliers have the same sign. For $\beta > 1$ let P_β be given by the matrices from Weinberg's paper [12, p. 569]:

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} \beta(\beta-1)^{-1} & 1 \\ -\beta(\beta-1)^{-2} & -(\beta-1)^{-1} \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & (\beta-1)\beta^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 0 \\ (1-\beta)^{-1} & 0 \end{pmatrix}.$$

Then if $\beta(1-\beta)^{-1} \leq c \leq (1-\beta)^{-1}$ and $A_c = \begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix}$, one can verify that P_β is contained in the lattice-order $Y_c = A_c(F^+)_2 A_c^{-1}$ of F_2, and $Y_c \neq Y_t$ if $c \neq t$. It can also be shown that if $1 \leq \beta, \delta \in F$ and P_β and P_δ are the lattice-orders determined by the matrices listed above or in section 2, then $P_\beta \subseteq P_\delta$ only if $\beta = \delta$.

If F is a commutative totally ordered domain with totally ordered quotient field Q and $R = (F_n, P)$ is an ℓ-algebra over F, then $S = (Q_n, P^e)$ is an ℓ-algebra over Q and R is an F-ℓ-subalgebra of S, where $P^e = \{x \in S : \exists\, 0 < \alpha \in F \text{ with } \alpha x \in P\}$. Clearly, $T(S) \cap R = T(R)$ and $T(S) \neq 0$ if and only if $T(R) \neq 0$. It is easy to see, using Theorem 1, that for $n = 2$, $T(S) \neq 0$ precisely when $S \cong (Q_2, (Q^+)_2)$.

4. F_n

The element $0 < s$ in the po-ring D is a superunit if $sx \geq x$ and $xs \geq x$ for each $x \in D^+$. If D is a totally ordered domain, then D can be embedded in a (minimal) totally ordered unital domain D_u. Moreover, either D has a superunit or $D \leq 1$ (see [7, Theorem III.3.5 and Chapter III] and [6, Theorem 1.6]). The module ${}_DM$ is torsion-free if $dx = 0$ implies that $d = 0$ or $x = 0$, for all $d \in D$ and $x \in M$. If ${}_DM$ is a torsion-free f-module, then it is also a torsion-free f-module over D_u.

An ℓ-ring is ℓ-simple if it is not nilpotent and has no nonzero proper ℓ-ideals. If a totally ordered domain D is ℓ-simple, then so is D_u (and conversely); and D has a superunit since if $0 < \beta \in D$, then $\beta \leq \gamma\beta\gamma$ for some $\gamma \in D^+$ and $1 \leq \gamma$ in D_u. Also, an ℓ-simple f-ring has no proper nonzero one-sided ℓ-ideals [7, IV.1.3 and IV.2.1] or [5, p. 132].

Let ${}_DM$ be an ℓ-module over the directed po-ring D. If x and y are elements of M, then x is infinitely smaller than y with respect to D if $D|x| \leq |y|$; we indicate this relation by $x << y$. M is called archimedean over D if whenever $x << y$, then $x = 0$. If $Dx \neq 0$ for each $0 \neq x \in M$, then ${}_DM$ is archimedean if and only if each bounded submodule of M is trivially ordered.

Now, let F be a commutative totally ordered domain with quotient field Q. If ${}_FM$ is a torsion-free f-module, then its module of quotients (or injective hull) $E(M) = Q \otimes_F M$ is a vector lattice over Q with positive cone $E^+ = \{x \in E : \alpha x \in M^+ \text{ for some } 0 < \alpha \in F\}$. We are interested in deducing that ${}_FM$ is archimedean when ${}_QE$ is archimedean. For this purpose we have

Lemma 2. *Let F be a commutative totally ordered domain with quotient field Q. The following statements are equivalent.*

(a) *F is ℓ-simple.*
(b) *If ${}_FM$ is an ℓ-module and $x << y$ in M, then $x << \alpha y$ for each nonzero d-element α on M; and F has a superunit.*
(c) *If $x << y$ in F, then $x << \alpha y$ for each $0 \neq \alpha \in F$; and F has a superunit.*
(d) *${}_FF$ is archimedean.*
(e) *There is an archimedean f-module over F that is not torsion.*
(f) *If C is a convex ℓ-submodule of the torsion-free f-module ${}_FM$, then M/C is torsion-free.*
(g) *${}_FQ$ is archimedean.*
(h) *The following are equivalent for each torsion-free f-module ${}_FM$ with module of quotients $E = E(M)$.*
  (i) *${}_FM$ is archimedean.*
  (ii) *${}_FE$ is archimedean.*
  (iii) *${}_QE$ is archimedean.*

Proof. First note that the implications (b) $\Rightarrow$ (c), (f) $\Rightarrow$ (a), and (h) $\Rightarrow$ (g) $\Rightarrow$ (d) $\Rightarrow$ (e) are obvious.

(a) $\Rightarrow$ (b). Suppose that $x << y$ in the ℓ-module M and that $0 < \alpha \in F$ is a d-element on M. If $\beta \in F^+$, there exists $\gamma \in F^+$ with $\beta \leq \alpha\gamma$; so $\beta|x| \leq \alpha\gamma|x| \leq \alpha|y| = |\alpha y|$.

(c) $\Rightarrow$ (d). If $0 \neq \alpha, \beta \in F^+$ and $\alpha << \beta$, then the convex subgroup $C(F\alpha)$ of F generated by $F\alpha$ is a proper ideal of F. Consequently, $\alpha^2 << \alpha$, since if $\gamma\alpha^2 \geq \alpha$ for some $\gamma \in F$, then $\gamma\alpha$ is a superunit of F in $C(F\alpha)$. But then $\alpha^2 << \alpha^2$ and hence $F \leq 1$; this is impossible since F has a superunit.

(e) $\Rightarrow$ (a). If x is an element of the archimedean f-module ${}_FM$ and x is not torsion, then neither is $|x|$. Let I be a proper ℓ -ideal of F and suppose that $0 < \alpha \in I$ and $\beta \in F^+ \setminus I$. Then $\alpha << \beta$ and hence $\alpha x << \beta x$; so we have the contradiction $\alpha x = 0$.

(a) $\Rightarrow$ (f). Suppose that $0 < \beta \in F$, $x \in M$ and $\beta x \in C$. If δ is a superunit of F, then $\delta \leq \gamma\beta$ for some $\gamma \in F$. So $|x| \leq \gamma\beta|x| = \gamma|\beta x| \in C$ and M/C is torsion-free.

(a) and (d) $\Rightarrow$ (g). Let $\alpha, \beta, \gamma \in F^+$ with $\alpha > 0$ and suppose that $F\alpha^{-1}\beta \leq \alpha^{-1}\gamma$. If $\rho \in F^+$, then $\rho\alpha \leq \alpha\delta$ for some $\delta \in F^+$. So $\rho\beta \leq \alpha\delta\alpha^{-1}\beta \leq \gamma$ and hence $\beta = 0$.

(a) and (b) $\Rightarrow$ (h). For (i) $\Rightarrow$ (iii), if $Q|\alpha^{-1}x| \leq |\alpha^{-1}y|$ with $\alpha \in F^+$ and $x, y \in M$, then $Q|x| \leq |y|$; hence $x = 0$. For (iii) $\Rightarrow$ (ii), suppose that $F|x| \leq |y|$ where $x, y \in E$. Then for each $0 < \alpha, \beta \in F$, $\beta|x| \leq \alpha|y|$; so $x << y$ with respect to Q and thus $x = 0$. Clearly, (ii) $\Rightarrow$ (i). $\square$

From the proof just given it can be seen that Lemma 2 holds for other f-rings, also. For example, it holds if F is any totally ordered left Öre domain, the equivalences (a) through (f) hold for any totally ordered domain, and (d) and (g) are equivalent if Q is any quotient ring of the commutative f-ring F.

Recall that a module has finite Goldie dimension if it contains no infinite direct sum of nonzero submodules.

Lemma 3. *Let F be a commutative ℓ-simple totally ordered domain, and let R be an ℓ-semiprime torsion-free ℓ-algebra over F. Suppose that ${}_FR$ has finite Goldie dimension. Then ${}_FR$ is archimedean and is the direct sum of totally ordered ℓ-simple submodules.*

Proof. If Q is the quotient field of F, then $S = Q \otimes_F R$ is a finite dimensional ℓ-semiprime ℓ-algebra over Q. So S is archimedean over Q and ${}_FR$ is archimedean by Lemma 2; and, again, ${}_FR$ is the direct sum of totally ordered submodules. $\square$

Let x be an element of the f-module ${}_DM$ over the directed po-ring D. A convex ℓ-submodule of M that is maximal with respect to not containing x is called a D-value of x. For the theory of D-values see [3], [4] and [8]. We recall some facts about values. There is a bijection between the set of D-values of an element and its set of $\mathbb{Z}$-values. Also, if there is an integer m such that each disjoint subset of nonzero elements of M has cardinality at most m, then each element in M has at most m values. An element $x \in M$ has precisely k values if and only if $x = x_1 + \cdots + x_k$ where $\{x_i\}$ is a disjoint set and each x_i has just one value. If C is a convex ℓ-submodule of M and $x \in C$, then there is a bijection between its set of D-values in C and its set of D-values in M.

Theorem 4. *Let F be a commutative totally ordered domain. Suppose that $R = (F_n, P)$ is an ℓ-algebra with quotient ℓ-algebra $S = (Q_n, P^e)$. The following statements are equivalent.*

(a) *S is isomorphic to $(Q_n, (Q^+)_n)$.*
(b) *$T(R)$ has an element with exactly n values.*
(c) *$T(R)$ has an element with at least n values.*
(d) *$1 \in P^e$ and 1 has at least n values.*
(e) *S contains an F-ℓ-subalgebra that is isomorphic to $(F_n, (F^+)_n)$.*

Proof. (a) $\Rightarrow$ (b). $T(S)$ is isomorphic to the ℓ-subalgebra of diagonal matrices, so $1 = e_1 + \cdots + e_n$ where $\{e_1, \cdots, e_n\}$ are orthogonal primitive idempotents. If $0 < \alpha \in F$ with $\alpha e_i \in T(R)$ for each i, then $\alpha = \alpha e_1 + \cdots + \alpha e_n$ has exactly n values.

(c) $\Rightarrow$ (d). $T(S)$ is a finite dimensional f-subalgebra of ${}_QS$ and it is reduced [9, Lemma 2]. So $T(S)$ is the direct sum of totally ordered division algebras: $T(S) = D_1 \oplus \cdots \oplus D_k$. Since $T(S)$ contains a set of n disjoint elements, $k \geq n$. Now, if e is the identity of $T(S)$, then $e = e_1 + \cdots + e_k$ and $\{e_i\}$ is a set of orthogonal idempotents in S; so $k \leq n$ and $e = 1$.

(d) $\Rightarrow$ (a). Decompose S as the vector lattice direct sum of totally ordered ℓ-simple subspaces

$$S = \bigoplus_{i=1}^{N} G_i.$$

As in the previous paragraph, $T(S)$ is the sum of n of the G_i, $1 = e_1 + \cdots + e_n$, and $\{e_i\}$ is a complete set of primitive disjoint orthogonal idempotents of S. For each i, j, k, $e_i G_k$ and $G_k e_j$ are convex subspaces of G_k; so for each k there are unique i and j with $G_k = e_i G_k e_j \subseteq e_i S e_j \cong Q$. Thus, each $G_k = e_i S e_j$ is one-dimensional and $N = n^2$. We can now construct positive matrix units in the usual manner. For each j with $2 \leq j \leq n$ the modules $e_1 S$ and $e_j S$ are isomorphic, and so there exist elements $e_{1j} \in e_1 S^+ e_j$ and $e_{j1} \in e_j S^+ e_1$ with $e_{1j} e_{j1} = e_1$ and $e_{j1} e_{1j} = e_j$. Let $e_{ii} = e_i$ and $e_{ij} = e_{i1} e_{1j}$. Then $\{e_{ij} : 1 \leq i, j \leq n\}$ is a set of positive matrix units in S and $S = \bigoplus Q e_{ij} \cong (Q_n, (Q^+)_n)$.

Since the equivalence of (a) and (e) and the implication (b) $\Rightarrow$ (c) are obvious, the proof is complete. □

Recall that a unital ring is local if it has only one maximal right ideal.

Corollary 5. *Let F be a commutative unital ℓ-simple totally ordered local domain. If $R = (F_n, P)$ is an ℓ-algebra and $T(R)$ contains an element with at least n values, then R contains an ℓ-subalgebra that is isomorphic to $(F_n, (F^+)_n)$.*

Proof. Let $S = (Q_n, P^e)$ be the quotient ℓ-algebra of R. From Lemma 3 and the proof of Theorem 4

$$R = \bigoplus E_{ij}, \quad S = \bigoplus G_{ij}, \quad 1 \leq i, j \leq n,$$

where E_{ij} is a totally ordered F-ℓ-simple submodule of R and $E(E_{ij}) = G_{ij}$. Here, $1 = e_{11} + \cdots + e_{nn}$ with $e_{ii} \in R$, and $e_{11} R \cong e_{ii} R$ for each i since F is local; so the positive matrix units $\{e_{ij}\}$ may be taken in R. Then $U = \bigoplus F e_{ij}$ is the desired ℓ-subalgebra. □

If $1 \leq \beta \in Q$, then $(P_\beta)_n$ is a lattice-order of Q_{2n} in which 1 is not positive. Theorem 4 states that there is only one way (up to isomorphism) to lattice-order $S = Q_n$ so that 1 is positive and has its maximum number of nonzero components in the decomposition of S into totally ordered subspaces. Let us note that the obstruction to removing this latter condition comes from our inability to eliminate 1 having its minimum number of nonzero components. Specifically, we note that Weinberg's Conjecture can be rephrased as follows.

The following statements are equivalent for the totally ordered field Q.

(W1) $\forall n \geq 1$, if (Q_n, P) is an ℓ-algebra with $1 \in P$, then $(Q_n, P) \cong (Q_n, (Q^+)_n)$.
(W2) $\forall n \geq 2$, Q_n has no algebra lattice-orders for which $T(Q_n) = Q1$.
(W3) $\forall n \geq 2$, Q_n has no algebra lattice-orders with $1 \in T(Q_n)$ and $T(Q_n)$ is totally ordered.

The implications (W1) $\Rightarrow$ (W3) $\Rightarrow$ (W2) are obvious. For (W2) $\Rightarrow$ (W1), suppose that $S = (Q_n, P)$ is an ℓ-algebra with $1 \in P$. Then $1 = e_1 + \cdots + e_m$ where $\{e_i\}$ is a complete set of orthogonal idempotents in $T(S)$. Now, $m \geq 2$ since if $m = 1$, then $T(S)$ would be central by [10, Corollary 15], and hence $T(S) = Q1$. If some e_i is not primitive in S, then $e_iSe_i \cong Q_t$ with $2 \leq t < n$, and e_iSe_i is a convex ℓ-subalgebra of S. But then $e_i = f_1 + \cdots + f_k$ with $2 \leq k$ and $\{f_j\}$ is a set of nonzero orthogonal idempotents in $e_iSe_i \cap T(S)$; and this contradicts the completeness of $\{e_i\}$. □

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