ON THE SCARCITY
OF LATTICE-ORDERED MATRIX ALGEBRAS II

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Abstract. We correct and complete Weinberg’s classification of the lattice-orders of the matrix ring \( \mathbb{Q}^2 \) and show that this classification holds for the matrix algebra \( \mathbb{F}^2 \) where \( \mathbb{F} \) is any totally ordered field. In particular, the lattice-order of \( \mathbb{F}^2 \) obtained by stipulating that a matrix is positive precisely when each of its entries is positive is, up to isomorphism, the only lattice-order of \( \mathbb{F}^2 \) with \( 1 \geq 0 \). It is also shown, assuming a certain maximum condition, that \( (\mathbb{F}^n, P) \) is essentially the only lattice-order of the algebra \( \mathbb{F}^n \) in which the identity element is positive.

1. Introduction

Let \( F_n \) denote the \( n \times n \) matrix ring over the ring \( F \). If \( (F, F^+) \) is a lattice-ordered ring (\( \ell \)-ring) with positive cone \( F^+ = \{ \alpha \in F : \alpha \geq 0 \} \), then \( (F_n, (F^+_n)) \) is also an \( \ell \)-ring. In [12] Weinberg has conjectured that if \( F = \mathbb{Q} \) is the field of rational numbers and if \( (\mathbb{Q}_n, P) \) is an \( \ell \)-ring with \( 1 \in P \), then \( (\mathbb{Q}_n, P) \) is isomorphic to \( (\mathbb{Q}_n, (\mathbb{Q}^+_n)) \). He proved this conjecture for \( n = 2 \) and also provided, for each \( 1 < \beta \in \mathbb{Q} \), the following additional non-isomorphic lattice-orders \( P_\beta \) of \( \mathbb{Q}^2 \): there are idempotents \( f_1, f_2, f_3, f_4 \) in \( \mathbb{Q}^2 \) with \( 1 = (1 - \beta)(f_1 + f_2) + \beta(f_3 + f_4) \) and \( (\mathbb{Q}_2, P_\beta) \) is the \( \ell \)-group direct sum of its totally ordered subrings \( \mathbb{Q}f_1, \mathbb{Q}f_2, \mathbb{Q}f_3, \mathbb{Q}f_4 \).

The assertion in [12] that each lattice-order of \( \mathbb{Q}^2 \) in which \( 1 \) is not positive is isomorphic to one of the \( P_\beta \)'s is not quite correct, however. There is one additional lattice-order \( P_1 \) of \( \mathbb{Q}^2 \) that must be added to the list to make this statement correct. We will describe \( P_1 \) below.

If \( R \) is an \( \ell \)-ring and an algebra over the totally ordered field \( F \), then \( R \) is an \( \ell \)-algebra over \( F \) if it is a vector lattice, that is, if \( F^+ R^+ \subseteq R^+ \). We show that the description of the lattice-orders of \( \mathbb{Q}^2 \) that is given above also holds for the lattice-orders of \( \mathbb{F}^2 \) which make it into an \( \ell \)-algebra over \( F \) and that Weinberg’s conjecture is true for \( F_n \) provided that \( 1 \) has its maximum number \( n \) of nonzero components in the decomposition of the \( \ell \)-algebra \( (F_n, P) \) as a direct sum of totally ordered vector lattices over \( F \).
2. Another lattice-order for $\mathbb{Q}_2$

We will first review a few definitions. A convex $\ell$-subgroup of an $\ell$-group is a subgroup $C$ that is a sublattice and is also convex: if $a \leq x \leq b$ with $a, b \in C$, then $x \in C$. An $\ell$-ideal of an $\ell$-ring is an ideal that is also a convex $\ell$-subgroup. A vector lattice over a totally ordered field $F$ is called archimedean over $F$ if it has no nonzero bounded subspaces.

If $R$ is a finite dimensional $\ell$-algebra over a totally ordered field $F$ and $R$ has no nonzero nilpotent $\ell$-ideals, that is, $R$ is $\ell$-semiprime, then $R$ is archimedean over $F$ by [2, Corollary 1, p. 51]. So it is a vector lattice direct sum of totally ordered $\ell$-simple subspaces (see [4, p. 3.27] or [8, Theorem 2.12]). Of course, if $F$ is a subfield of the real numbers $\mathbb{R}$, these totally ordered subspaces are embeddable in $\mathbb{R}$. Weinberg’s method of proof for $\mathbb{Q}_2$ is to consider and eliminate all but two (actually, three) of the twenty-nine cases that arise depending on the number of summands, the dimensions of the summands, and the number and signs of the coordinates of $1$. The error occurs in the twenty-eighth case, which is case (7d) of [12].

In (7d) we have that $\mathbb{Q}_2 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ as an $\ell$-group where each $E_i$ is isomorphic to $\mathbb{Q}$, $1 = e_1 + e_2 + e_3$ where $0 \neq e_i \in E_i$, $e_1$ and $e_2$ have the same sign and it is opposite to that of $e_3$, and $0 < n \in E_4$. The calculation down to the last sentence of (ii) is correct. So we have that $E_1 \oplus E_4, E_2 \oplus E_4$ and $E_1 \oplus E_2 \oplus E_3$ are subalgebras of $\mathbb{Q}_2$, and

$$
e_1^2 = k_1 e_1, \quad n^2 = n,$$

which gives that $k_1 e_1 = 0$. Hence, $e_1 e_2 = 0$. Suppose that $e_2 e_1 = 0$. Then $e_2$ does not enter symmetrically in these calculations and so the assertion that $e_2 e_1 = 0$ is not correct. We will complete the calculation and produce another lattice-order of $\mathbb{Q}_2$. Now, since $e_2 e_1$ is in the subalgebra $E_1 \oplus E_2 \oplus E_3$,

$$0 \leq e_2 e_1 = xe_1 + ye_2 + ze_3$$

with $x, y, z \in \mathbb{Q}$ and $xe_1$, $ye_2$, $ze_3 \geq 0$; so

$$0 = e_2 e_1 e_2 = y k_2 e_2 + z(1 - e_1 - e_2) e_2 = (yk_2 + z - zk_2) e_2,$$

and $y = (k_2 - 1)k_2^{-1} z$. We have that $e_2 e_1 \not\in E_1$, since otherwise $\mathbb{Q}_2 e_1 = E_1$; so $z \neq 0$. Suppose that $e_3 > 0 > e_1, e_2$. Then $z > 0$ and $(k_2 - 1)k_2^{-1} z \leq 0$; but this is nonsense since $k_2 < 0$. So we must have that $e_3 < 0 < e_1, e_2$, and hence $x \geq 0, z < 0$ and $0 < k_2 \leq 1$. Since

$$0 = x_1 e_2 e_1 = xe_1 e_2 + ye_2 e_1 + ze_3 e_1,$$

$x = 0$ and $e_1 e_3 = 0$; thus $e_1 = e_1^2 = k_1 e_1$ and $k_1 = 1$. Also,

$$0 \geq e_3 e_2 = (1 - e_1 - e_2) e_2 = (1 - k_2) e_2 \geq 0$$

gives that $k_2 = 1, e_3 e_2 = 0, y = 0$ and

$$e_1 = ne_2 e_1 = zn e_3 = zn(1 - e_1 - e_2) = -ze_1;$$

hence, $e_2 e_1 = -e_3$. It is now easy to complete the following multiplication table:

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<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
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<td>$e_1$</td>
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<td>$n$</td>
<td>$e_1$</td>
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<td>$-e_1$</td>
<td>$n$</td>
</tr>
</tbody>
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The matrices
\[ e_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \]
satisfy this table and the equation \( 1 = e_1 + e_2 + e_3 \). Thus, this lattice-order is given by \( P_1 = \mathbb{Q}^+ e_1 + \mathbb{Q}^+ e_2 + \mathbb{Q}^+ (-e_3) + \mathbb{Q}^+ n \), and, again, \( Q_2 \) is the \( \ell \)-group direct sum of four totally ordered subrings.

3. Lattice-orders of \( F_2 \)

In this section we modify Weinberg’s proof to obtain his theorem for any totally ordered field.

**Theorem 1.** Let \( F \) be a totally ordered field.

(a) For each \( 1 < \beta \in F \) there is a lattice-order \( P_\beta \) of \( F_2 \) for which \( R = (F_2, P_\beta) \) is an \( \ell \)-algebra, and there are four idempotents \( f_1, f_2, f_3, f_4 \) in \( P_\beta \) such that \( R \) is the vector lattice direct sum of the subalgebras \( Ff_1, Ff_2, Ff_3, Ff_4 \) with \( 1 = (1 - \beta)(f_1 + f_2) + \beta(f_3 + f_4) \).

(b) There are idempotents \( e_1, e_2 \) and a nilpotent element \( e_3 \) such that \( 1 = e_1 + e_2 + e_3 \), and if \( P_1 \) is the positive cone of the vector lattice direct sum \( F_2 = Fe_1 + Fe_2 + F(-e_3) + Fn \), then \( (F_2, P_1) \) is an \( \ell \)-algebra over \( F \).

(c) If \( R = (F_2, P) \) is an \( \ell \)-algebra over \( F \), then \( R \) is isomorphic to \( (F_2, (F^+)_2) \) if \( 1 \in P \), and, otherwise, \( R \) is isomorphic to \( (F_2, P_\beta) \) for exactly one \( \beta \geq 1 \).

Before giving the proof we will review some definitions and facts that we will need. If \( M \) is a left module over the ring \( D \), then \( M \) is an \( \ell \)-module over \( D \) if \( D \) is a po-ring, \( M \) is an \( \ell \)-group, and \( D^+ M^+ \subseteq M^+ \). The element \( d \in D^+ \) is an \( f \)-element (respectively, a \( d \)-element) on \( M \) if \( dx \wedge y = 0 \) (respectively, \( dx \wedge dy = 0 \)) whenever \( x \) and \( y \) are elements of \( M \) with \( x \wedge y = 0 \). Each \( f \)-element is a \( d \)-element. The additive subgroup \( T(DM) \) of \( D \) generated by the set of \( f \)-elements on \( M \) is a convex directed subring of \( D \). If \( D \) is an \( \ell \)-ring, then \( T(DM) = \{ d \in D : |d| \} \) is an \( f \)-element on \( M \}, \) and \( T(D) = T(DD) \cap T(DP) \) is the convex \( \ell \)-subring of \( f \)-elements of \( D \). \( M \) is an \( f \)-module over \( D \) if \( D^+ \subseteq T(DM) \). For example, each vector lattice over a totally ordered division ring is an \( f \)-module. If \( D \) is an \( \ell \)-ring, then \( T(D) \) is an \( f \)-ring: that is, \( T(D) \) is a right and left \( f \)-module over itself. If \( D \) is a commutative po-ring and \( R \) is an \( \ell \)-ring and an algebra over \( D \), then \( R \) is called an \( \ell \)-algebra if \( R \) is an \( f \)-module over \( D \).

**Proof of Theorem 1.** The proof of (a) is given by using the matrices \( f_1, f_2, f_3, \) and \( f_4 \) given below, and the proof of (b) is given by using the matrices presented in the previous section. As for (c), suppose that \( P \) is a positive cone for \( F_2 \) such that \( R = (F_2, P) \) is an \( \ell \)-algebra over \( F \). As indicated at the beginning of section 2, \( R \) is the vector lattice direct sum of at most four totally ordered subspaces, and in \([12]\), for \( F = \mathbb{Q} \), Weinberg considers the various cases that arise depending on the number of summands, the dimensions of the summands and the coordinates of 1. It is only the three cases (1c), (7b) and (7d) that lead to the lattice-orders \( (\mathbb{Q}^+)_2, P_\beta \), and \( P_1 \) of \( \mathbb{Q}_2 \), respectively; in each of the other twenty-six cases it is shown that there is no lattice-order of the type under consideration. As can partially be seen from section 2 all of the arguments that are used in \([12]\) for \( \mathbb{Q}_2 \) with the modification given in section 2 and some other minor modifications are valid for \( R \) (although some of these arguments can be shortened) with the exception of those used in the three cases (3c), (3d), and (5). In each of these cases the fact that a totally ordered
subspace of \( \mathbb{Q}_2 \) is embeddable in the complete field \( \mathbb{R} \) is used to eliminate the possibility of a lattice-order of the type under consideration. We proceed to show that this fact can be avoided. Note that if \( A \) and \( B \) are totally ordered archimedean \( F \)-subspaces of an \( \ell \)-algebra and \( C \) is a convex subspace, then \( AB \subseteq C \) provided that \( ab \in C \) for some \( 0 \neq a \in A \) and \( 0 \neq b \in B \).

Let \( R = (F_2, P) \) be an \( \ell \)-algebra. In the three cases to be considered we have that \( R = E_1 \oplus E_2 \) as vector lattices over \( F \) where \( E_1 \) and \( E_2 \) are totally ordered subspaces.

(3c) Here we have \( 1 \in E_1 \) and \( \dim_F E_2 = 3 \). Since \( E_2 \) is not an ideal of \( R \), it cannot be a subring of \( R \). So if \( 0 \neq f \in E_2 \), then, by the Cayley-Hamilton theorem, \( f^2 = \alpha + \beta f \) with \( \alpha, \beta \in F \) and \( \alpha > 0 \). If \( \beta = 0 \), then \( E_2^2 \subseteq E_1 \), and this is impossible since \( fE_2 \) is 3-dimensional over \( F \). But then the trace function \( \text{tr} : E_2 \to F \) is monic.

(3d) In this case \( \dim_F E_1 = 2 \) and \( 0 < 1 \in E_1 \). Then \( E_1 = T(R) \) is a division ring and hence is central by our extension of Tamhankar’s extension of Albert’s Theorem [10, Corollary 15]. This is absurd.

(5) Here, \( 1 = e_1 + e_2 \) and \( e_1 < 0 < e_2 \). Since \( E_1 \) is not an ideal it cannot be a subring of \( R \). So if \( 0 \neq f \in E_1 \), then \( f^2 = \alpha + \beta f \) with \( \alpha \neq 0 \). If \( \beta = 0 \), then \( f^2 = \alpha e_1 + \alpha e_2 \geq 0 \) implies that \( \alpha = 0 \). Thus \( \beta \neq 0 \) and, again, \( \text{tr} \) is monic on \( E_1 \).

This is impossible since \( E_1 \) or \( E_2 \) is at least 2-dimensional over \( F \).

If \( D \) is a totally ordered ring with the property that \( DaD \neq 0 \) if \( 0 \neq a \in D \), then it is easy to see that, for any \( n, (D^+)_n \) is maximal among those ring partial orders \( P \) of \( D_n \) for which \( D^+P + PD^+ \subseteq P \); this is also the case if \( n \) is infinite and \( D_n \) is the ring of column finite matrices over \( D \). None of the \( P_\beta \) are maximal since in these lattice-orders 1 is not positive. Each \( P_\beta \) is, however, contained in the cardinality of \( F \) many lattice-orders each one of which is isomorphic to \( (F^+)_2 \). Specifically, let \( P_1 \) be given by the matrices in section 2, and take \( b, d \in F \) with \( 0 \leq b \leq -d \) and \( d < 0 \). Then if \( A_{b,d} = \begin{pmatrix} b & 1 \\ d & 0 \end{pmatrix} \), it is easy to check that \( A_{b,d}^{-1}P_1A_{b,d} \subseteq (F^+)_2 \), and hence \( P_1 \) is contained in the lattice-order \( X_{b,d} = A_{b,d}(F^+)_2A_{b,d}^{-1} \) of \( F_2 \). In particular, \( P_1 \subseteq X_{0,-1} = \begin{pmatrix} F^+ & -F^+ \\ -F^+ & F^+ \end{pmatrix} \). Also, \( X_{b,d} \neq X_{b,h} \) if \( d \neq h \) and \( b > 0 \). This is a consequence of the fact that \( C^{-1}(F^+)_2C = (F^+)_2 \) with det \( C > 0 \) (respectively, det \( C < 0 \)) exactly when \( C = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \) (respectively, \( C = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \)) and \( xy > 0 \).

So \( A^{-1}(F^+)_2A = B^{-1}(F^+)_2B \) requires that the columns of \( A \) are multiples of the columns of \( B \) where the multipliers have the same sign. For \( \beta > 1 \) let \( P_\beta \) be given by the matrices from Weinberg’s paper [12, p. 569]:

\[
f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} \beta(\beta-1)^{-1} & 1 \\ -\beta(\beta-1)^{-2} & -(\beta-1)^{-1} \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & (\beta-1)^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 0 \\ (1-\beta)^{-1} & 0 \end{pmatrix}.
\]

Then if \( (1-\beta)^{-1} \leq c \leq (1-\beta)^{-1} \) and \( A_c = \begin{pmatrix} 1 & 1 \\ c & 0 \end{pmatrix} \), one can verify that \( P_\beta \) is contained in the lattice-order \( Y_c = A_c(F^+)_2A_c^{-1} \) of \( F_2 \), and \( Y_c \neq Y_t \) if \( c \neq t \). It can
also be shown that if $1 \leq \beta, \delta \in F$ and $P_{\beta}$ and $P_{\delta}$ are the lattice-orders determined by the matrices listed above or in section 2, then $P_{\beta} \subseteq P_{\delta}$ only if $\beta = \delta$.

If $F$ is a commutative totally ordered domain with totally ordered quotient field $Q$ and $R = (F_n, P)$ is an $\ell$-algebra over $F$, then $S = (Q_n, P^\ell)$ is an $\ell$-algebra over $Q$ and $R$ is an $F-\ell$-subalgebra of $S$, where $P^\ell = \{x \in S : \exists 0 < \alpha \in F \text{ with } \alpha x \in P\}$. Clearly, $T(S) \cap R = T(R)$ and $T(S) \neq 0$ if and only if $T(R) \neq 0$. It is easy to see, using Theorem 1, that for $n = 2$, $T(S) \neq 0$ precisely when $S \cong (Q_2, (Q^+)_2)$.

4. $F_n$

The element $0 < s$ in the po-ring $D$ is a superunit if $sx \geq x$ and $xs \geq x$ for each $x \in D^+$. If $D$ is a totally ordered domain, then $D$ can be embedded in a (minimal) totally ordered unital domain $D_u$. Moreover, either $D$ has a superunit or $D \leq 1$ (see [7, Theorem III.3.5 and Chapter III] and [6, Theorem 1.6]). The module $D M$ is torsion-free if $dx = 0$ implies that $d = 0$ or $x = 0$, for all $d \in D$ and $x \in M$. If $D M$ is a torsion-free $f$-module, then it is also a torsion-free $f$-module over $D_u$.

An $\ell$-ring is $\ell$-simple if it is not nilpotent and has no nonzero proper $\ell$-ideals. If a totally ordered domain $D$ is $\ell$-simple, then so is $D_u$ (and conversely); and $D$ has a superunit since if $0 < \beta \in D$, then $\beta \leq \gamma \beta \gamma$ for some $\gamma \in D^+$ and $1 \leq \gamma$ in $D_u$. Also, an $\ell$-simple $f$-ring has no proper nonzero one-sided $\ell$-ideals [7, IV.1.3 and IV.2.1] or [5, p. 132].

Let $D M$ be an $\ell$-module over the directed po-ring $D$. If $x$ and $y$ are elements of $M$, then $x$ is infinitely smaller than $y$ with respect to $D$ if $D|x| \leq |y|$; we indicate this relation by $x << y$. $M$ is called archimedean over $D$ if whenever $x << y$, then $x = 0$. If $Dx \neq 0$ for each nonzero $0 \neq x \in M$, then $D M$ is archimedean if and only if each bounded submodule of $M$ is trivially ordered.

Now, let $F$ be a commutative totally ordered domain with quotient field $Q$. If $F M$ is a torsion-free $f$-module, then its module of quotients (or injective hull) $E(M) = Q \otimes_F M$ is a vector lattice over $Q$ with positive cone $E^+ = \{x \in E : \alpha x \in M^+ \text{ for some } 0 < \alpha \in F\}$. We are interested in deducing that $F M$ is archimedean when $QE$ is archimedean. For this purpose we have

**Lemma 2.** Let $F$ be a commutative totally ordered domain with quotient field $Q$. The following statements are equivalent.

(a) $F$ is $\ell$-simple.

(b) If $F M$ is an $\ell$-module and $x << y$ in $M$, then $x << \alpha y$ for each nonzero $\alpha$-element $\alpha$ on $M$; and $F$ has a superunit.

(c) If $x << y$ in $F$, then $x << \alpha y$ for each $0 \neq \alpha \in F$; and $F$ has a superunit.

(d) $F F$ is archimedean.

(e) There is an archimedean $f$-module over $F$ that is not torsion.

(f) If $C$ is a convex $\ell$-submodule of the torsion-free $f$-module $F M$, then $M/C$ is torsion-free.

(g) $F Q$ is archimedean.

(h) The following are equivalent for each torsion-free $f$-module $F M$ with module of quotients $E = E(M)$.

(i) $F M$ is archimedean.

(ii) $F E$ is archimedean.

(iii) $Q E$ is archimedean.
Proof. First note that the implications (b) ⇒ (c), (f) ⇒ (a), and (h) ⇒ (g) ⇒ (d) ⇒ (e) are obvious.

(a) ⇒ (b). Suppose that \( x << y \) in the \( \ell \)-module \( M \) and that \( 0 < \alpha / F \) is a \( d \)-element on \( M \). If \( \beta / F \in F^+ \), there exists \( \gamma / F \in F^+ \) with \( \beta / F \leq \alpha \gamma / F \); so \( \beta / F \leq \alpha \gamma / F \leq \alpha (\gamma / F) = |\alpha y| \).

(c) ⇒ (d). If \( 0 \neq \alpha, \beta / F \in F^+ \) and \( \alpha << \beta \), then the convex subgroup \( C(F\alpha) \) of \( F \) generated by \( F\alpha \) is a proper ideal of \( F \). Consequently, \( \alpha^2 << \alpha \), since if \( \gamma \alpha^2 \geq \alpha \) for some \( \gamma / F \in F \), then \( \gamma \alpha \) is a superunit of \( F \) in \( C(F\alpha) \). But then \( \alpha^2 << \alpha^2 \) and hence \( F \leq 1 \); this is impossible since \( F \) has a superunit.

(e) ⇒ (a). If \( x \) is an element of the archimedean \( \ell \)-module \( \ell M \) and \( x \) is not torsion, then neither is \( M \). Then \( \alpha \) is a superunit of \( F \), then \( \alpha << \beta \) and hence \( \alpha x << \beta x \); so we have the contradiction \( \alpha x = 0 \).

(a) ⇒ (f). Suppose that \( 0 < \beta / F \in \ell M \) and \( \beta x \in C \). If \( \delta / F \) is a superunit of \( F \), then \( \delta / F \leq \gamma / F \) for some \( \gamma / F \in F^+ \). So \( |x| \leq |\gamma \beta / F| / F = |\gamma \beta / F| \in C \) and \( M / C \) is torsion-free.

(a) and (d) ⇒ (g). Let \( \alpha, \beta, \gamma / F \in F^+ \) with \( \alpha > 0 \) and suppose that \( F\alpha^{-1} \beta \leq \alpha^{-1} \gamma \). If \( \rho / F \in F^+ \), then \( \rho \alpha \leq \rho \delta \) for some \( \delta / F \in F^+ \). So \( \rho \beta \leq \rho \delta \alpha^{-1} \beta \leq \gamma \) and hence \( \beta = 0 \).

(a) and (b) ⇒ (h). For (i) ⇒ (iii), if \( Q|\alpha^{-1} x| \leq |\alpha^{-1} y| \) with \( \alpha / F \in F^+ \) and \( x, y / F \in M \), then \( Q|x| \leq |y| \); hence \( x = 0 \). For (iii) ⇒ (ii), suppose that \( F|x| \leq |y| \) where \( x, y / F \in E \). Then for each \( 0 < \alpha, \beta / F \in F, \beta|x| \leq \alpha|y| \); so \( x << y \) with respect to \( Q \) and thus \( x = 0 \). Clearly, (ii) ⇒ (i). \( \square \)

From the proof just given it can be seen that Lemma 2 holds for other \( \ell \)-rings, also. For example, it holds if \( F \) is any totally ordered left \( \ell \)-domain, the equivalences (a) through (f) hold for any totally ordered domain, and (d) and (g) are equivalent if \( Q \) is any quotient ring of the commutative \( \ell \)-ring \( F \).

Recall that a module has finite Goldie dimension if it contains no infinite direct sum of nonzero submodules.

Lemma 3. Let \( F \) be a commutative \( \ell \)-simple totally ordered domain, and let \( R \) be an \( \ell \)-semiprime torsion-free \( \ell \)-algebra over \( F \). Suppose that \( \ell R \) has finite Goldie dimension. Then \( \ell R \) is archimedean and is the direct sum of totally ordered \( \ell \)-simple submodules.

Proof. If \( Q \) is the quotient field of \( F \), then \( S = Q \otimes F R \) is a finite dimensional \( \ell \)-semiprime \( \ell \)-algebra over \( Q \). So \( S \) is archimedean over \( Q \) and \( \ell R \) is archimedean by Lemma 2; and, again, \( \ell R \) is the direct sum of totally ordered submodules. \( \square \)

Let \( x \) be an element of the \( \ell \)-module \( D M \) over the directed po-ring \( D \). A convex \( \ell \)-submodule of \( M \) that is maximal with respect to not containing \( x \) is called a \( D \)-value of \( x \). For the theory of \( D \)-values see [3], [4] and [8]. We recall some facts about \( D \)-values. There is a bijection between the set of \( D \)-values of an element and its set of \( \mathbb{Z} \)-values. Also, if there is an integer \( m \) such that each disjoint subset of nonzero elements of \( M \) has cardinality at most \( m \), then each element in \( M \) has at most \( m \) \( D \)-values. An element \( x / F \in M \) has precisely \( k \) \( D \)-values if and only if \( x = x_1 + \cdots + x_k \) where \( \{x_i\} \) is a disjoint set and each \( x_i \) has just one value. If \( C \) is a convex \( \ell \)-submodule of \( M \) and \( x \) \( D \)-values in \( C \) and its set of \( D \)-values in \( M \).
Theorem 4. Let $F$ be a commutative totally ordered domain. Suppose that $R = (F_n, P)$ is an $\ell$-algebra with quotient $\ell$-algebra $S = (Q_n, P^e)$. The following statements are equivalent.

(a) $S$ is isomorphic to $(Q_n, (Q^+)_n)$.
(b) $T(R)$ has an element with exactly $n$ values.
(c) $T(R)$ has an element with at least $n$ values.
(d) $1 \in P^e$ and $1$ has at least $n$ values.
(e) $S$ contains an $F$-$\ell$-subalgebra that is isomorphic to $(F_n, (F^+)_n)$.

Proof. (a) $\Rightarrow$ (b). $T(S)$ is isomorphic to the $\ell$-subalgebra of diagonal matrices, so $1 = e_1 + \cdots + e_n$ where $\{e_1, \cdots, e_n\}$ are orthogonal primitive idempotents. If $0 < \alpha \in F$ with $\alpha e_i \in T(R)$ for each $i$, then $\alpha = \alpha e_1 + \cdots + \alpha e_n$ has exactly $n$ values.

(c) $\Rightarrow$ (d). $T(S)$ is a finite dimensional $f$-subalgebra of $Q S$ and it is reduced [9, Lemma 2]. So $T(S)$ is the direct sum of totally ordered division algebras: $T(S) = D_1 \oplus \cdots \oplus D_k$. Since $T(S)$ contains a set of $n$ disjoint elements, $k \geq n$. Now, if $e$ is the identity of $T(S)$, then $e = e_1 + \cdots + e_k$ and $\{e_i\}$ is a set of orthogonal idempotents in $S$; so $k \leq n$ and $e = 1$.

(d) $\Rightarrow$ (a). Decompose $S$ as the vector lattice direct sum of totally ordered $\ell$-simple subspaces

$$S = \bigoplus_{i=1}^N G_i.$$ 

As in the previous paragraph, $T(S)$ is the sum of $n$ of the $G_i$, $1 = e_1 + \cdots + e_n$, and $\{e_i\}$ is a complete set of primitive disjoint orthogonal idempotents of $S$. For each $i, j, k$, $e_i G_k$ and $G_k e_j$ are convex subspaces of $G_k$; so for each $k$ there are unique $i$ and $j$ with $G_k = e_i G_k e_j \subseteq e_i S e_j \cong Q$. Thus, each $G_k = e_i S e_j$ is one-dimensional and $N = n^2$. We can now construct positive matrix units in the usual manner. For each $j$ with $2 \leq j \leq n$ the modules $e_j S$ and $e_j S$ are isomorphic, and so there exist elements $e_{ij} \in e_j S^+ e_j$ and $e_{j1} \in e_j S^+ e_1$ with $e_{ij} e_{j1} = e_1$ and $e_{j1} e_{ij} = e_j$. Let $e_{ii} = e_i$ and $e_{ij} = e_{i1} e_{ij}$. Then $\{e_{ij} : 1 \leq i, j \leq n\}$ is a set of positive matrix units in $S$ and $S = \bigoplus Q e_{ij} \cong (Q_n, (Q^+)_n)$.

Since the equivalence of (a) and (e) and the implication (b) $\Rightarrow$ (c) are obvious, the proof is complete.

Recall that a unital ring is local if it has only one maximal right ideal.

Corollary 5. Let $F$ be a commutative unital $\ell$-simple totally ordered local domain. If $R = (F_n, P)$ is an $\ell$-algebra and $T(R)$ contains an element with at least $n$ values, then $R$ contains an $\ell$-subalgebra that is isomorphic to $(F_n, (F^+)_n)$.

Proof. Let $S = (Q_n, P^e)$ be the quotient $\ell$-algebra of $R$. From Lemma 3 and the proof of Theorem 4

$$R = \bigoplus E_{ij}, \quad S = \bigoplus G_{ij}, \quad 1 \leq i, j \leq n,$$

where $E_{ij}$ is a totally ordered $F$-$\ell$-simple submodule of $R$ and $E(E_{ij}) = G_{ij}$. Here, $1 = e_{i1} + \cdots + e_{in}$ with $e_{ii} \in R$, and $e_{ii} R \cong e_{ii} R$ for each $i$ since $F$ is local; so the positive matrix units $\{e_{ij}\}$ may be taken in $R$. Then $U = \bigoplus F e_{ij}$ is the desired $\ell$-subalgebra.

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If \( 1 \leq \beta \in Q \), then \((P_3)_n\) is a lattice-order of \(Q_{2n}\) in which 1 is not positive. Theorem 4 states that there is only one way (up to isomorphism) to lattice-order \(S = Q_n\) so that 1 is positive and has its maximum number of nonzero components in the decomposition of \(S\) into totally ordered subspaces. Let us note that the obstruction to removing this latter condition comes from our inability to eliminate 1 having its minimum number of nonzero components. Specifically, we note that Weinberg’s Conjecture can be rephrased as follows.

The following statements are equivalent for the totally ordered field \(Q\).

(W1) \(\forall n \geq 1, \text{ if } (Q_n, P) \text{ is an } \ell\text{-algebra with } 1 \in P, \text{ then } (Q_n, P) \cong (Q_n, (Q^+)_n)\).

(W2) \(\forall n \geq 2, Q_n\) has no algebra lattice-orders for which \(T(Q_n) = Q_1\).

(W3) \(\forall n \geq 2, Q_n\) has no algebra lattice-orders with 1 in \(T(Q_n)\) and \(T(Q_n)\) is totally ordered.

The implications \((W1) \Rightarrow (W3) \Rightarrow (W2)\) are obvious. For \((W2) \Rightarrow (W1)\), suppose that \(S = (Q_n, P)\) is an \(\ell\)-algebra with 1 in \(P\). Then 1 = \(e_1 + \cdots + e_m\) where \(\{e_i\}\) is a complete set of orthogonal idempotents in \(T(S)\). Now, \(m \geq 2\) since if \(m = 1\), then \(T(S)\) would be central by [10, Corollary 15], and hence \(T(S) = Q_1\). If some \(e_i\) is not primitive in \(S\), then \(e_i Se_i \cong Q_t\) with \(2 \leq t < n\), and \(e_i Se_i\) is a convex \(\ell\)-subalgebra of \(S\). But then \(e_i = f_1 + \cdots + f_k\) with \(2 \leq k\) and \(\{f_j\}\) is a set of nonzero orthogonal idempotents in \(e_i Se_i \cap T(S)\); and this contradicts the completeness of \(\{e_i\}\).

References


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