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INTEGER-VALUED POLYNOMIALS OVER KRULL-TYPE DOMAINS AND PRÜFER v-MULTIPLICATION DOMAINS

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Abstract. Let D be a domain with quotient field K. The ring of integer-valued polynomials over D is $\operatorname{Int}(D) := \{f \in K[X]; f(D) \subseteq D\}$. We characterize the Krull-type domains D such that $\operatorname{Int}(D)$ is a Prüfer v-multiplication domain.

Introduction

One of the open questions about the ring of integer-valued polynomials, $\operatorname{Int}(D) := \{f \in K[X]; f(D) \subseteq D\}$, has been the characterization of the domains D such that $\operatorname{Int}(D)$ is a Prüfer domain. For Noetherian domains J.L. Chabert has stated that $\operatorname{Int}(D)$ is a Prüfer domain if and only if D is a Dedekind domain with finite residue fields [CC, Theorem VI.1.7]. Recently, A. Loper has completely solved this problem providing a description of all domains D such that $\operatorname{Int}(D)$ is Prüfer [L].

Back to J.L. Chabert's characterization, we can ask whether $\operatorname{Int}(D)$ still satisfies some multiplicative ideal properties when D is a Dedekind domain, not necessarily with all residue fields finite. This was exactly the starting point of this work. We obtain the answer to this question as a corollary of a more general statement.

In multiplicative ideal theory an important role is assumed by the set of the t-ideals of a domain. Many definitions of classical domains have been revisited and generalized by requiring that just t-ideals satisfy certain properties. For instance, *Prüfer v-multiplication domains* (PvMD) are domains for which the localization at each t-prime ideal is a valuation ring. Whence they generalize Prüfer domains, for which this last property holds for all prime ideals.

It is known that a domain D is a PvMD if and only if $D[X]$ is a PvMD. In this work we ask when $\operatorname{Int}(D)$ is a PvMD.

First, we characterize the valuation domains V such that $\operatorname{Int}(V)$ is a PvMD (Theorem 1.1). We also give a necessary condition for any domain D, showing that if $\operatorname{Int}(D)$ is a PvMD, then D is a PvMD (Proposition 3.1). From the analysis carried on $\operatorname{Int}(V)$, when V is a valuation domain, it easily follows that the condition of Proposition 3.1 is not, in general, sufficient.

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We consider *Krull-type domains*, that is, domains D which can be realized as an intersection of the type

$$D = \bigcap_{p \in \mathcal{P}} D_p,$$

where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, D_p is a valuation domain and the intersection is locally finite (that is, each nonzero element $x \in D$ belongs to finitely many $p \in \mathcal{P}$). For example, *Krull domains* or *generalized Krull domains* are domains of Krull-type [Gi1, p. 524]. We characterize the Krull-type domains D such that $\mathrm{Int}(D)$ is a PvMD (Theorem 3.2).

As a corollary we describe the Noetherian domains D such that $\mathrm{Int}(D)$ is a PvMD. Precisely, we prove that, if D is Noetherian, $\mathrm{Int}(D)$ is a PvMD if and only if D is a Krull domain, that is, D is integrally closed (Corollary 3.3).

When we restrict to one-dimensional Noetherian domains, we show that $\mathrm{Int}(D)$ is a PvMD if and only if D is Dedekind (Corollary 3.4), giving evidence to the fact that $\mathrm{Int}(D)$ verifies interesting multiplicative properties for any Dedekind domain D, also when the residue fields are not all finite.

1. When is Int(V) a Prüfer v-multiplication domain?

In this first section we give a characterization of the valuation domains V such that $\mathrm{Int}(V)$ is a PvMD. We briefly recall some definitions and general properties about v-ideals and t-ideals. We refer to [Gi1, § 32] or [J] for the general theory about the t-operation and the v-operation.

Let D be a domain and $\mathfrak{F}(D)$ the set of its fractional ideals. Let $I \in \mathfrak{F}(D)$.

The *v-closure* of I is the ideal $I_v := (I^{-1})^{-1} = (D\colon (D\colon I))$, and I is *v-finite* if there exists a finitely generated ideal $J \in \mathfrak{F}(D)$ such that $I_v = J_v$.

The *t-closure* of I is $I_t := \bigcup J_v$ where J ranges over the set of the finitely generated ideals contained in I.

The ideal I is a *t-ideal* if $I = I_t$ and I is *t-invertible* if there exists $J \in \mathfrak{F}(D)$ such that $(IJ)_t = D$.

If I is an integral ideal, then I is a *t-prime* if it is a prime and a t-ideal.

A *t-maximal* ideal is a maximal element among the integral t-ideals of D. A t-maximal ideal is prime and each integral t-ideal is contained in a t-maximal ideal. We denote by Θ the set of t-maximal ideals of D.

The *t-dimension* of D (t-dim(D)) is defined as the supremum of the lengths of chains of t-primes. If this supremum is not bounded, then the t-dimension of D is infinite.

In a Prüfer domain each ideal is a t-ideal. A domain D is a *Prüfer v-multiplication domain* (PvMD) if the following equivalent conditions hold:

(i) a fractional ideal of D is t-invertible if and only if it is v-finite;

(ii) D_p is a valuation domain for each t-prime ideal p of D.

(iii) D_m is a valuation domain for each t-maximal ideal m of D.

In particular, Prüfer domains are Prüfer v-multiplication.

For a more detailed account about PvMD's we refer the reader to [K1] and [K2].

Theorem 1.1. *Let V be a valuation domain with maximal ideal m. Then,* $\mathrm{Int}(V)$ *is a PvMD if and only if* $\mathrm{Int}(V) = V[X]$ *(that is, V has infinite residue field or nonprincipal maximal ideal* [CC, Proposition I.3.16]) *or V is a DVR with finite residue field.*

Proof. If $\text{Int}(V) = V[X]$, then $\text{Int}(V)$ is a PvMD by [K1, Theorem 3.7] and since V is a PvMD.

We suppose that $\text{Int}(V) \neq V[X]$. Thus $m := \pi V$ is principal and has finite residue field [CC, Proposition I.3.16].

If V is one-dimensional, then V is a DVR with finite residue field and $\text{Int}(V)$ is a Prüfer domain [CC, Theorem VI.1.7]. Hence $\text{Int}(V)$ is a PvMD.

We suppose that $\dim(V) > 1$ and show that $\text{Int}(V)$ is not a PvMD. From [CH] we have that all the prime ideals of $\text{Int}(V)$ above m are maximal and so they are minimal over $m\text{Int}(V)$. But, $m\text{Int}(V) = \pi\text{Int}(V)$ is a t-ideal (because it is principal) and its minimal primes are also t-ideals [J, Corollaire 3, p. 31]. Whence all the prime ideals of $\text{Int}(V)$ above m are t-ideals. The ideal $\mathfrak{M}_0 := \{f \in \text{Int}(V); f(0) \in m\}$ of $\text{Int}(V)$ is prime and it is a t-ideal (since it is above m). There exists, at least, one prime ideal q in V such that $(0) \subsetneq q \subsetneq m$. The ideals $q[X] = qV_q[X] = qV_q[X] \cap \text{Int}(V)$ and $\beta_X := XK[X] \cap \text{Int}(V) = \{f \in \text{Int}(V); f(0) = 0\}$ are primes and they are both contained in $\mathfrak{M}_0$. Thus they lift to $\text{Int}(V)_{\mathfrak{M}_0}$ and they are incomparable. In fact, $\beta_X \not\subseteq q[X]$ since $X \in \beta_X$ whence $X \notin q[X]$, and $q[X] \not\subseteq \beta_X$ since $q[X]$ contains nonzero constants. This implies that $\text{Int}(V)_{\mathfrak{M}_0}$ is not a valuation domain (since its prime spectrum is not linearly ordered). Therefore $\text{Int}(V)$ is not a PvMD (by definition). $\square$

2. Some results about t-ideals of Int(D)

In this section we give some results which relate the t-closure of the ideals of $\text{Int}(D)$ to the t-closure of their contraction in D.

Proposition 2.1. *Let D be a domain. If I is an ideal of D, then $(I\text{Int}(D))_t \cap D = I_t$. In particular, if $\mathfrak{A}$ is a t-ideal of* $\text{Int}(D)$*, then $\mathfrak{A} \cap D$ is a t-ideal of D.*

Proof. By definition, $(I\text{Int}(D))_t := \bigcup J_v$, where J ranges over the finitely generated ideals of $\text{Int}(D)$ contained in $I\text{Int}(D)$. Since each of these J is included in a finitely generated ideal $(a_1, \dots, a_r)\text{Int}(D)$, for $a_1, \dots, a_r \in I$, and $J_v \subseteq ((a_1, \dots, a_r)\text{Int}(D))_v$, the union above can be restricted to ideals of Int(D) generated by finitely many elements $a_1, \dots, a_r$ of I. Notice that

$$((a_1, \dots, a_r)\text{Int}(D))^{-1} = (\text{Int}(D) \colon (a_1, \dots, a_r)\text{Int}(D))$$

$$= (1/a_1)\text{Int}(D) \cap \dots \cap (1/a_r)\text{Int}(D)$$

and

$$((a_1, \dots, a_r)D)^{-1} = (D \colon (a_1, \dots, a_r)D) = (1/a_1)D \cap \dots \cap (1/a_r)D.$$

We will show that, if $a_1, \dots, a_r \in D$, then

$$((a_1, \dots, a_r)\text{Int}(D))_v \cap D = ((a_1, \dots, a_r)D)_v.$$

The inclusion "$\subseteq$" is easy to check. In fact, if $x \in ((a_1, \dots, a_r)\text{Int}(D))_v \cap D$, then $x((1/a_1)\text{Int}(D) \cap \dots \cap (1/a_r)\text{Int}(D)) \subseteq \text{Int}(D)$, so that $x((1/a_1)D \cap \dots \cap (1/a_r)D) \subseteq D$, that is, $x \in ((a_1, \dots, a_r)D)_v$.

The opposite inclusion "$\supseteq$" holds for the following argument.

If $x \in ((a_1, \dots, a_r)D)_v$, then $x((1/a_1)D \cap \dots \cap (1/a_r)D) \subseteq D$. Moreover, if $f \in (1/a_1)\text{Int}(D) \cap \dots \cap (1/a_r)\text{Int}(D)$, then $f(d) \in (1/a_1)D \cap \dots \cap (1/a_r)D$, for each $d \in D$. Thus $xf(d) \in x((1/a_1)D \cap \dots \cap (1/a_r)D) \subseteq D$ and $xf \in \text{Int}(D)$, that is, $x \in ((a_1, \dots, a_r)\text{Int}(D))_v \cap D$.

We have seen that $(I\mathrm{Int}(D))_t$ is the union of the ideals $((a_1, \dots, a_r)\mathrm{Int}(D))_v$, where $a_1, \dots, a_r \in I$. Therefore $(I\mathrm{Int}(D))_t \cap D = I_t$, since $((a_1, \dots, a_r)\mathrm{Int}(D))_v \cap D = ((a_1, \dots, a_r)D)_v$ and I_t is the union of the ideals $((a_1, \dots, a_r)D)_v$, for $a_1, \dots, a_r \in I$.

If $\mathfrak{A}$ is a t-ideal of $\mathrm{Int}(D)$, then $((\mathfrak{A}\cap D)\mathrm{Int}(D))_t = (\mathfrak{A}\cap D)_t$. But, $(\mathfrak{A}\cap D)\mathrm{Int}(D) \subseteq \mathfrak{A}$ and, since $\mathfrak{A}$ is a t-ideal $((\mathfrak{A}\cap D)\mathrm{Int}(D))_t \subseteq \mathfrak{A}$. By intersection with D, it follows that $\mathfrak{A}\cap D \subseteq ((\mathfrak{A}\cap D)\mathrm{Int}(D))_t \cap D \subseteq \mathfrak{A}\cap D$, that is, $\mathfrak{A}\cap D$ is a t-ideal. $\square$

Corollary 2.2. *Let D be a domain. Then*

(a) *if $\mathfrak{Q}$ is a t-prime ideal of* $\mathrm{Int}(D)$*, then $q := \mathfrak{Q}\cap D$ is a t-prime ideal of D;*

(b) *each t-maximal ideal of D is the contraction of a t-maximal ideal of* $\mathrm{Int}(D)$.

Proof. (a) directly follows from the fact that, if $\mathfrak{Q}$ is a prime ideal of $\mathrm{Int}(D)$, then $q := \mathfrak{Q}\cap D$ is a prime ideal of D and that, by Proposition 2.1, if $\mathfrak{Q}$ is a t-ideal, then q is a t-ideal too.

(b) We have that if p is a t-prime ideal of D, then $p\mathrm{Int}(D)$ is contained in a t-prime ideal of $\mathrm{Int}(D)$. In fact, if it was $(p\mathrm{Int}(D))_t = \mathrm{Int}(D)$, by Proposition 2.1 we would have that $p_t = (p\mathrm{Int}(D))_t \cap D = \mathrm{Int}(D)\cap D = D$, against the assumption that p is a t-ideal. Hence $p\mathrm{Int}(D)$ is contained in a t-maximal ideal $\mathfrak{P}$ of $\mathrm{Int}(D)$. From (a), $\mathfrak{P}\cap D$ is a t-prime ideal of D and, for the maximality of p, $\mathfrak{P}\cap D = p$. $\square$

In [CC, Propositon I.2.8] it is proved that if D is a Krull domain and p is a height-one prime ideal of D, then $\mathrm{Int}(D)_p = \mathrm{Int}(D_p)$. We show that if D is a domain representable as a locally finite intersection of overrings D_p, where p belongs to a subset $\mathcal{P} \subseteq \mathrm{Spec}(D)$, then the equality above holds for each $p \in \mathcal{P}$.

Proposition 2.3. *Let D be a domain and let us suppose that D has a locally finite representation, $D = \bigcap_{p\in\mathcal{P}} D_p$, where $\mathcal{P} \subseteq$* $\mathrm{Spec}(D)$. *Then,* $\mathrm{Int}(D)_p = \mathrm{Int}(D_p)$*, for each $p \in \mathcal{P}$.*

Proof. We fix $q \in \mathcal{P}$. We always have the containment $\mathrm{Int}(D)_q \subseteq \mathrm{Int}(D_q)$ [CC, Proposition I.2.2].

Now, let $f \in \mathrm{Int}(D_q)$ and let us see that there exists $s \in D$, $s \notin q$, such that $sf(D) \subseteq D_p$ for all $p \in \mathcal{P}$. In this case, we would have $sf(D) \subseteq \bigcap_{p\in\mathcal{P}} D_p = D$, and we would finally conclude that $f \in \mathrm{Int}(D)_q$.

From the locally finite character of the intersection $\bigcap_{p\in\mathcal{P}} D_p$, we have that $f \in D_p[X]$ for all $p \in \mathcal{P}$ but finitely many, namely $p_1, \cdots, p_n$, so that $f(D) \subseteq D_p$ for all $p \in \mathcal{P}$ but $p \neq p_1, \cdots, p_n$. Since $f \in \mathrm{Int}(D_q)$, then $f(D) \subseteq (D_q)_{p_i}$, for $i = 1, \cdots, n$.

If $p_i \subseteq q$, then $(D_q)_{p_i} = D_{p_i}$, whence $f(D) \subseteq D_{p_i}$.

If $p_i \not\subseteq q$, we consider the family $\mathcal{P}'$ of all the prime ideals p' of D such that $p' \subset p_i \cap q$. Then, $(D_q)_{p_i} = S_i^{-1}D = \bigcap_{p'\in\mathcal{P}'} D_{p'}$. For $p' \in \mathcal{P}'$, p' is distinct from q, whence p' is not maximal and $\mathrm{Int}(D_{p'}) = D_{p'}[X]$. Therefore, $f(D) \subseteq D_q \subseteq D_{p'}$. Hence $f \in \bigcap_{p'\in\mathcal{P}'} \mathrm{Int}(D_{p'}) = \bigcap_{p'\in\mathcal{P}'} D_{p'}[X] = S_i^{-1}D[X]$. There exists $s_i \in D$, $s_i \notin q$, such that $s_i f \in D_{p_i}[X]$, that is, $s_i f(D) \subseteq D_{p_i}$. If we take the product $s := s_1 \cdots s_n$, then $s \in D$, $s \notin q$, and $sf(D) \subseteq D$. $\square$

M. Griffin proved that a domain D can always be represented as follows:

$$D = \bigcap_{p\in\Theta} D_p \quad \text{[Gr, Proposition 4].} \tag{2.I}$$

Moreover, from [Ch, Corollaire 1], we have that $\mathrm{Int}(D) = \bigcap_{p\in\Theta} \mathrm{Int}(D_p)$.

Inside Θ we can consider the subset Θ_0 of the ideals p such that $\mathrm{Int}(D)_p \neq D_p[X]$ and the subset Θ_1 of the ideals p such that $\mathrm{Int}(D)_p = D_p[X]$. In this last case, we observe that the prime ideals of $\mathrm{Int}(D)$ above p are contractions of the prime ideals of $D_p[X]$ above pD_p. They are of the type $J_p := pD_p[X] \cap \mathrm{Int}(D)$ or they strictly contain J_p and they are upper to p.

If D is a PvMD (that is, D_p is a valuation domain for each $p \in \Theta$) and, further, the representation (2.I) is locally finite, then D is a Krull-type domain (that is, D is a locally finite intersection of valuation overrings D_p, where $p \in \mathrm{Spec}(D)$). More precisely, in [Gr, Theorem 7] the author proves that Krull-type domains are exactly PvMD's for which the representation (2.I) is locally finite (that is, PvMD's such that each nonzero element x belongs to finitely many t-maximal ideals).

Proposition 2.4. *Let D be a Krull-type domain and let us suppose that p has height one for each $p \in \Theta_0$. If p is a prime ideal of D such that $\mathrm{Int}(D)_p = D_p[X]$, then the prime ideals of $\mathrm{Int}(D)$ which are upper to p are not t-ideals.*

Proof. If p is not a t-prime ideal of D, from Corollary 2.2 (a) no prime ideal of $\mathrm{Int}(D)$ above p is a t-ideal.

Let us suppose that p is a t-prime ideal of D and let P be a prime ideal of $\mathrm{Int}(D)$ upper to p. Thus $P \cap D[X]$ is a prime ideal of $D[X]$ upper to p and $p[X] \subsetneq P \cap D[X]$. We claim that $(P \cap D[X])_t = D[X]$. Since D is integrally closed, by [HZ, Lemma 4.5] P is not a t-ideal. We want to see that P is not included in any other t-prime ideal of $D[X]$ (which would imply that $(P \cap D[X])_t = D[X]$).

If $\mathfrak{P}$ is a t-prime ideal of $D[X]$ containing P, then $\mathfrak{P} = q[X]$, for some t-prime ideal q of D. Thus $PD_q[X] \subset qD_q[X]$. But D_q is a valuation domain (because D is, in particular, a PvMD), and the extended prime ideals of $D_q[X]$ can only contain extended primes [Gi1, Theorem 19.5], so that we cannot have the inclusion $PD_q[X] \subset qD_q[X]$. Therefore, there exists a finitely generated ideal $I \subseteq P \cap D[X]$ such that $I_t = I_v = D[X]$. By the fact that $D = \bigcap_{p\in\Theta} D_p$ is a locally finite intersection, we can choose I such that $I \cap D \not\subseteq q$ for each t-prime ideal q of D that does not contain p. We set $J := I\mathrm{Int}(D)$. We have that $J_q = \mathrm{Int}(D)_q = \mathrm{Int}(D_q)$, for each $q \in \Theta$ and $q \not\supseteq p$.

Moreover we have that:

$$J_v = (\mathrm{Int}(D)\colon (\mathrm{Int}(D)\colon J)) = \Big(\bigcap_{q\in\Theta} \mathrm{Int}(D)_q\colon \bigcap_{q\in\Theta}(\mathrm{Int}(D)\colon J)_q\Big)$$

$$= \Big(\bigcap_{q\in\Theta} \mathrm{Int}(D_q)\colon \bigcap_{q\in\Theta}(\mathrm{Int}(D_q)\colon J_q)\Big) \supseteq \bigcap_{q\in\Theta}(\mathrm{Int}(D_q)\colon (\mathrm{Int}(D_q)\colon J_q)),$$

where $(\mathrm{Int}(D)\colon J)_q = (\mathrm{Int}(D)_q\colon J_q)$ since J is finitely generated and $\mathrm{Int}(D)_q = \mathrm{Int}(D_q)$ from Proposition 2.3.

If $q \not\supseteq p$, then

$$(\mathrm{Int}(D_q)\colon (\mathrm{Int}(D_q)\colon J_q)) = (\mathrm{Int}(D_q)\colon (\mathrm{Int}(D_q)\colon \mathrm{Int}(D_q))) = \mathrm{Int}(D_q).$$

If $q \supseteq p$, then $\mathrm{Int}(D_q) = D_q[X]$ (because either $q = p$ or q does not have height-one; hence $q \in \Theta_1$) and $J_q = I_q$. Thus we have:

$$(\mathrm{Int}(D_q)\colon (\mathrm{Int}(D_q)\colon J_q)) = (D_q[X]\colon (D_q[X]\colon J_q))$$

$$= (D_q[X]\colon (D_q[X]\colon I_q)) \supseteq (D[X]\colon (D[X]\colon I))_q = (I_v)_q = D_q[X] = \mathrm{Int}(D_q).$$

Therefore $J_v \supseteq \bigcap_{q \in \Theta} \mathrm{Int}(D_q) = \mathrm{Int}(D)$. Hence $J_v = \mathrm{Int}(D)$ and P is not a t-prime ideal. $\square$

As a corollary of these results we describe all the t-prime ideals of $\mathrm{Int}(D)$ when D is a Krull-type domain satisfying the hypotheses of Proposition 2.4. We briefly recall some results about the prime spectrum of $\mathrm{Int}(D)$.

• If D is a domain, all the prime ideals of $\mathrm{Int}(D)$ upper to (0) are of the type $\beta_q := qK[X] \cap \mathrm{Int}(D)$, where q is an irreducible polynomial of $K[X]$ [CC, Corollary V.1.2].

• If D is a domain, p is a prime ideal of D such that D_p is a DVR with finite residue field and such that $\mathrm{Int}(D)_p = \mathrm{Int}(D_p)$, then all the prime ideals of $\mathrm{Int}(D)$ above p are of the type $p_\alpha := \{f \in \mathrm{Int}(D); f(\alpha) \in \widehat{pD_p}\}$, where $\widehat{D_p}$ is the p-adic completion of D_p and $\alpha \in \widehat{D_p}$ [CC, Ch.V].

Corollary 2.5. *With the notation above, let D be a Krull-type domain and let us suppose that p has height one, for each $p \in \Theta_0$. Then, the t-prime ideals of* $\mathrm{Int}(D)$ *are the following:*

• *the primes upper to* (0)*, that is, the ideals of the type* $\beta_q := qK[X] \cap \mathrm{Int}(D)$*, where q is an irreducible polynomial of $K[X]$;*

• *the primes of the type* $J_p := pD_p[X] \cap \mathrm{Int}(D)$*, where p is a t-prime ideal of D and* $\mathrm{Int}(D)_p = D_p[X]$ *(that is, $p \in \Theta_1$);*

• *the primes above some prime ideal $p \in \Theta_0$, that is, the ideals of the type p_α, with $\alpha \in \widehat{D_p}$.*

Proof. From Corollary 2.2 (a) we only have to consider the prime ideals of $\mathrm{Int}(D)$ which are above some t-prime ideal of D.

By [J, Corollaire 3, p. 31] the primes upper to (0) are t-ideals (since they are minimal over the principal ideal (0)).

If p is a t-prime ideal of D such that $\mathrm{Int}(D)_p = D_p[X]$, by Proposition 2.4 the prime ideals of $\mathrm{Int}(D)$ which are upper to p are not t-ideals.

The prime ideals of the type $J_p := pD_p[X] \cap \mathrm{Int}(D)$ are t-ideals. In fact, pD_p is a t-ideal of D_p, since D_p is a Prüfer domain; then $pD_p[X]$ is a t-ideal in $D_p[X]$, by [K1, Corollary 2.3], and J_p is a t-ideal of $\mathrm{Int}(D)$ because it is the contraction of a t-ideal from a localization overring [K1, Lemma 3.17].

Finally, if $p \in \Theta_0$, applying Proposition 2.3 to the locally finite intersection $D = \bigcap_{p \in \Theta} D_p$, we have that $\mathrm{Int}(D)_p = \mathrm{Int}(D_p)$. Moreover, D_p is one-dimensional and, since $\mathrm{Int}(D)_p \neq D_p[X]$, D_p is a DVR with finite residue field [CC, Proposition I.3.16]. Thus all the prime ideals of $\mathrm{Int}(D)$ above p are of the type p_α, with $\alpha \in \widehat{D_p}$ and they are contractions of prime ideals of $\mathrm{Int}(D_p)$. Since $\mathrm{Int}(D_p)$ is a Prüfer domain [CC, Theorem VI.1.7], each ideal in $\mathrm{Int}(D_p)$ is a t-ideal and its contraction to $\mathrm{Int}(D)$ is still a t-ideal [K1, Lemma 3.17]. $\square$

3. Krull-type domains D such that Int(D) is a PvMD

We start this section giving a necessary condition on a domain D for $\mathrm{Int}(D)$ to be a PvMD.

Proposition 3.1. *Let D be a domain and let us suppose that* $\mathrm{Int}(D)$ *is a PvMD. Then, D is a PvMD.*

Proof. If p is a t-prime ideal of D, then p is contained in a t-maximal ideal p'. From Corollary 2.2 (b), there exists a t-maximal ideal $\mathfrak{P}$ of $\mathrm{Int}(D)$ such that $\mathfrak{P} \cap D = p'$.

Thus $p\mathrm{Int}(D) \subseteq p'\mathrm{Int}(D) \subseteq \mathfrak{P}$. Since $\mathrm{Int}(D)$ is a PvMD, $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain. Moreover, $\mathrm{Int}(D)_{\mathfrak{P}} \cap K = D_{\mathfrak{P}\cap D} = D_{p'}$ is also a valuation domain, so that D_p is a valuation domain, because $D_{p'} \subseteq D_p$. Therefore D is a PvMD by definition. □

The condition that we have given in Proposition 3.1 is not, in general, sufficient to have that $\mathrm{Int}(D)$ is a PvMD. In fact, for example, in Theorem 1.1 we have seen that there exist valuation domains (which are Prüfer domains, and so PvMDs) such that $\mathrm{Int}(V)$ is not a PvMD.

We observe that if D is a domain, p is a prime ideal of D and D_p is a valuation domain, then p is a t-prime ideal. In fact, pD_p is a t-ideal (since D_p is a Prüfer domain), and $p = pD_p \cap D$. By [K1, Lemma 3.17] p is a t-prime ideal. It directly follows that, if D is a PvMD and q is a prime ideal contained in a t-prime p, then q is a t-prime too. In fact, we have that $D_q \supseteq D_p$ and D_p is a valuation domain. Therefore D_q is also a valuation domain and q is a t-prime ideal. Thus $t\text{-}\dim(D) = \sup\{\dim(D_p); p \in \Theta\}$.

In the next theorem we characterize the Krull-type domains such that $\mathrm{Int}(D)$ is a PvMD.

Theorem 3.2. *Let D be a Krull-type domain. With the notation above,* $\mathrm{Int}(D)$ *is a PvMD if and only if D_p is one dimensional for each $p \in \Theta_0$. Moreover,*

$$t\text{-}\dim(\mathrm{Int}(D)) = \sup\{2, t\text{-}\dim(D)\} = \sup\{2, \sup_{p\in\Theta}\{\dim(D_p)\}\}.$$

Proof. Let us suppose that $\mathrm{Int}(D)$ is a PvMD and that there exists a prime ideal $p \in \Theta_0$ such that D_p has dimension n, with $n > 1$. Since $\mathrm{Int}(D)_p \neq D_p[X]$, then also $\mathrm{Int}(D_p) \neq D_p[X]$ (because, in general, $D_p[X] \subseteq \mathrm{Int}(D)_p \subseteq \mathrm{Int}(D_p)$). Thus the valuation domain D_p has principal maximal ideal and finite residue field [CC, Proposition I.3.16]. By Theorem 1.1, $\mathrm{Int}(D_p)$ is not a PvMD and, from Proposition 2.3, $\mathrm{Int}(D_p) = \mathrm{Int}(D)_p$. Hence $\mathrm{Int}(D_p)$ should be a PvMD by [K1, Theorem 3.7] and we have reached a contradiction.

On the contrary, let us suppose that

$$\dim(D_p) = 1, \quad \text{for each } p \in \Theta_0.$$

To prove that $\mathrm{Int}(D)$ is a PvMD it is sufficient to show that $\mathrm{Int}(D)$ localized at each t-prime ideal $\mathfrak{P}$ is a valuation domain.

Corollary 2.5 provides a complete description of the t-spectrum of $\mathrm{Int}(D)$. The localizations of $\mathrm{Int}(D)$ at its t-primes are the following:

- if $\mathfrak{P} = \beta_q$, then $\mathrm{Int}(D)_{\beta_q} = K[X]_q$ and it is a valuation domain;
- if $\mathfrak{P} = J_p$, for $p \in \Theta_1$, then $\mathrm{Int}(D)_{J_p} = D_p[X]_{(pD_p[X])}$ and it is a valuation domain since $pD_p[X]$ is a t-prime ideal of the PvMD $D_p[X]$;
- if $\mathfrak{P} = p_\alpha$, then $\mathrm{Int}(D)_{p_\alpha} = (\mathrm{Int}(D)_p)_{(p_\alpha \mathrm{Int}(D)_p)} = \mathrm{Int}(D_p)_{(p_\alpha \mathrm{Int}(D_p))}$ and it is a valuation domain since $\mathrm{Int}(D_p)$ is a Prüfer domain.

To compute the t-dimension of $\mathrm{Int}(D)$ we go through the following considerations. No prime ideals of the type β_q can be included in ideals of the type J_p. In fact,

$$\beta_q D_p[X] = qK[X] \cap D_p[X] \not\subseteq pD_p[X]$$

since D_p is a valuation domain [Gi1, Theorem 19.15].

The ideals of the type p_α are maximal and of heigth, at most, 2. They can only contain t-primes of the type β_q with $q(\alpha) = 0$ [CC, Proposition V.3.3]. Thus the only possible saturated chains of t-primes in $\mathrm{Int}(D)$ are the following:

(i) $(0) \subset \beta_q \subset p_\alpha$, where $p \in \Theta_0$ and $q(\alpha) = 0$.

(ii) $(0) \subset J_{p_1} \subset \cdots \subset J_{p_n}$, where $(0) \subset p_1 \subset \cdots \subset p_n$ is a saturated chain of t-primes of D. The length of such a chain can be, at most, t-$\dim(D)$.

The conclusion follows from the fact that D is a PvMD which implies that t-$\dim(D) = \sup_{p\in\Theta}\{\dim(D_p)\}$. $\square$

We observe that the finiteness-condition on the t-maximal ideals of D (that is, each nonzero element x of D belongs to finitely many t-maximal ideals and which holds in Krull-type domains) is not, in general, necessary. It is possible to find a PvMD such that $\mathrm{Card}(D/p) = \infty$, for all $p \in \Theta$ and which does not satisfy the above mentioned finiteness-conditon. Such a domain D can be obtained by considering an almost Dedekind domain, which is not Dedekind, with all residue fields infinite; we refer, for instance, to [Gi2] for this type of construction. In this case, $\mathrm{Int}(D) = \bigcap_{p\in\Theta} \mathrm{Int}(D_p) = \bigcap_{p\in\Theta} D_p[X] = D[X]$ is a PvMD [K1, Theorem 3.7].

This fact suggests that, for a general characterization, we do not have to ask that the intersection $\bigcap_{p\in\Theta} D_p$ is locally finite. More precisely, from [Ch, Corollaire 1] we can write

$$\mathrm{Int}(D) = \bigcap_{p\in\Theta} \mathrm{Int}(D_p) = (\bigcap_{p\in\Theta_0} \mathrm{Int}(D_p)) \cap (\bigcap_{p\in\Theta_1} \mathrm{Int}(D_p))$$

$$= \mathrm{Int}(\bigcap_{p\in\Theta_0} D_p) \cap (\bigcap_{p\in\Theta_1} D_p[X]) = \mathrm{Int}(D_0) \cap D_1[X],$$

where $D_0 := \bigcap_{p\in\Theta_0} D_p$ and $D_1 := \bigcap_{p\in\Theta_1} D_p$. We call $D_1[X]$ the *polynomial part* of $\mathrm{Int}(D)$, and we say that it is not *trivial* if $D_1 \neq K$. The existence of a nontrivial polynomial part, in the sense described above, represents a remarkable difference between the Prüfer and the PvMD characterization of $\mathrm{Int}(D)$. In fact, if $\mathrm{Int}(D)$ is a Prüfer domain, then $D_1[X]$ is also Prüfer (being an overring of $\mathrm{Int}(D)$). But $D_1[X]$ is a Prüfer domain if and only if D_1 is a field. Thus if we suppose $\mathrm{Int}(D)$ to be Prüfer, then $D_1 = K$. On the other hand, if $\mathrm{Int}(D)$ is a PvMD, then also $D_1[X]$ is PvMD (because D is a PvMD and D_1 is an intersection of localizations of D at t-prime ideals, so that D_1 is also a PvMD). But this is not a contradiction since the PvMD *property* transfers from any domain D to the polynomial ring $D[X]$, even if D is not a field.

Therefore for the PvMD's characterization we cannot exclude, a priori, the possibility that $\mathrm{Int}(D)$ has a nontrivial polynomial part, that is, $D_1 \neq K$.
We now consider the overring of D, $D_0 = \bigcap_{p\in\Theta_0} D_p$. We assume, as a necessary condition, that D is a PvMD (Proposition 3.1). If D is a Krull-type domain, by Theorem 3.2, $\dim(D_p) = 1$ and D_p has principal maximal ideal and finite residue field (since $\mathrm{Int}(D_p) \neq D_p[X]$ [CC, Proposition I.3.16]). Hence D_0 is a locally finite intersection of DVR's with finite residue fields, that is, D_0 is a Dedekind domain with finite residue fields. Therefore $\mathrm{Int}(D_0)$ is a Prüfer domain.

From this observation, in order to generalize this characterization to a generic domain D, we ask the following questions:

1) Should $\dim(D_p) = 1$, for each $p \in \Theta_0$?

2) Should $\mathrm{Int}(D_0)$ be a Prüfer domain?

The next result characterizes the Noetherian domains D for which $\mathrm{Int}(D)$ is a PvMD, giving an analogue, for PvMD's, of J.L. Chabert's characterization of Noetherian domains D such that $\mathrm{Int}(D)$ is Prüfer.

Corollary 3.3. *If D is a Noetherian domain, then* $\mathrm{Int}(D)$ *is a PvMD if and only if D is a Krull domain.*

Proof. From Theorem 3.2, if D is a Krull domain then $\mathrm{Int}(D)$ is a PvMD. If, on the contrary, $\mathrm{Int}(D)$ is a PvMD, then D is a PvMD by Proposition 3.1, so that D is integrally closed. Thus D is a Noetherian, integrally closed domain, that is, D is a Krull domain [Gi1, Theorem 43.4]. $\square$

A one-dimensional Krull domain is Dedekind and we easily have the following corollary:

Corollary 3.4. *Let D be a one-dimensional Noetherian domain. Then,* $\mathrm{Int}(D)$ *is a PvMD if and only if D is a Dedekind domain.*

References

[CC] P.J. Cahen - J.-L. Chabert, *Integer-Valued Polynomials*, Math. Surveys and Monographs (AMS) **48** (1997). MR **98a:**13002

[CH] P.-J. Cahen - Y. Haouat, *Polynômes à valeurs entières sur un anneau de pseudo-valuation*, Manuscripta Math. **61** (1988), 23–31. MR **89f:**13007

[Ch] J.-L. Chabert, *Anneaux de polynômes à valeurs entières et anneaux de Fatou*, Bull. Sc. Math. France **99** (1971), 273–283. MR **46:**1780

[Gi1] R. Gilmer, *Multiplicative Ideal Theory*, Marcel-Dekker, New York (1972). MR **55:**323

[Gi2] R. Gilmer, *Prüfer domains and rings of integer-valued polynomials*, J. Algebra **129, n.2** (1990), 502–517. MR **91b:**13023

[Gr] M. Griffin, *Some Results on v-multiplication Rings*, Canad. J. Math. **10** (1967), 710–722. MR **35:**6665

[HZ] E.G. Houston - M. Zafrullah, *Integral Domains in which each t-ideal is divisorial*, Michigan Math. J. **35** (1988), 291–300. MR **89i:**13027

[J] P. Jaffard, *Les Systèmes d'Idéaux*, Dunod, Paris (1960). MR **22:**5628

[K1] B.G. Kang, *Prüfer v-Multiplication Domains and the Ring* $R[X]_{N_v}$, J. of Algebra **123** (1989), 151–170. MR **90e:**13017

[K2] B.G. Kang, *Some questions about Prüfer v-Multiplication Domains*, Comm. Algebra **17(3)** (1989), 553–564. MR **90a:**13033

[L] A. Loper, *A classification of all D such that Int(D) is a Prüfer domain*, Proc. Amer. Math. Soc. (to appear).

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