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A CHARACTERIZATION OF TOTAL REFLECTION ORDERS

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Abstract. Let (W, S) be a Coxeter system with set of reflections T. It is known that if $\prec$ is a total reflection order for W, then, for each $s \in S$, $\{t \in T \mid t \prec s\}$ and its complement are stable under conjugation by s. Moreover the upper and lower s-conjugates of $\prec$ are still total reflection orders. For any total order $\prec$ on T, say that $\prec$ is stable if $\{t \in T \mid t \prec s\}$ is stable under conjugation by s for each $s \in S$. We prove that if $\prec$ and all orders obtained from $\prec$ by successive lower or upper S-conjugations are stable, then $\prec$ is a total reflection order.

1. Introduction

The notion of *total reflection order* was introduced by Dyer [D2] in the context of his study of reflection subgroups in Coxeter systems and its further developments regarding Hecke algebras and the related Kazhdan-Lusztig theory. The original definition is introduced in terms of dihedral reflection subgroups and has the following equivalent geometric formulation [D2]. If (W, S) is a Coxeter system, T is the set of its reflections, and Φ is its root system in the geometric representation [H], then there is a natural bijection between the set Φ^+ of positive roots and T, which maps the root $\alpha \in \Phi^+$ to the reflection r_α in T. A *total reflection order* is a total order $\prec$ on T which satisfies the following convexity condition: for each α, $\beta \in \Phi^+$ and $c, d \in \mathbb{R}^+$ such that $c\alpha + d\beta \in \Phi^+$, either $r_\alpha \prec r_{c\alpha+d\beta} \prec r_\beta$ or $r_\beta \prec r_{c\alpha+d\beta} \prec r_\alpha$.

This notion has interesting applications: for instance, Dyer [D2] and Brenti [B1], [B2] give respectively formulas for the R-polynomials and the Kazhdan-Lusztig polynomials involving total reflection orders.

If $\prec$ is a total reflection order, then, for each $s \in S$, the sets $\{t \in T \mid t \prec s\}$ and $\{t \in T \mid t \succ s\}$ are stable under conjugation by s [D2]. We shall call *stable* any total order on T that satisfies this property. The upper s-conjugate of a stable order $\prec$ is the total order $\prec^s$ on T defined as follows: for each $t, t' \in T$, $t \neq t'$, we have $t \prec^s t'$ if and only if either $t \prec t' \prec s$, or $t \prec s \prec t'$, or $s \prec sts \prec st's$, or $t' = s$. The lower s-conjugate $\prec_s$ of $\prec$ is defined similarly. Dyer [D2] proves that if $\prec$ is a total reflection order, then $\prec_s$ and $\prec^s$ are total reflection orders too; thus they are stable. Stability and preservation of stability under successive upper or lower S-conjugations are exactly the properties of total reflection orders which are needed for the explicit computation of the R-polynomials and hence of Kazhdan-Lusztig polynomials (see [BB, 8]). We say that a total order on T is a *KL-order* if it is

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stable and each successive upper or lower S-conjugate of it is stable. Indeed, as is seen in the proof of Dyer's formulas for the R-polynomials given in [BB, 8.4], such formulas hold for any fixed KL-order. For this reason, we investigate these orders. By Dyer's results any total reflection order is KL. The main result of this paper is that the converse holds too.

Theorem 1.1. *Any KL-order is a total reflection order.*

Thus the KL-property characterizes the total reflection orders.

Theorem 1.1 is proved in section 3. In section 4 we prove that, if W is finite, for a total order on T to be KL it is sufficient to be stable under successive just upper or just lower S-conjugations. Moreover, we prove that if each proper parabolic subgroup of W is finite, then the stable orders on T which remain stable under successive upper (lower) S-conjugations have the same finite initial (final) sections as those of total reflection orders.

2. Notation and preliminaries

In this section we fix notation and recall some well known facts (see [De] or [H, Chapter 5]). Let (W, S) be a Coxeter system. Then $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$ is the set of reflections of W. It is well known that W can be realized as a real reflection group as follows. Let V be a real vector space of dimension $|S|$ with basis $\Pi = \{\alpha_s \mid s \in S\}$, and define the *standard bilinear form* $(\cdot\,,\cdot)$ on V as the unique symmetric bilinear form such that $(\alpha_r, \alpha_s) = -\cos\frac{\pi}{m(r,s)}$ for each r, $s \in S$, where $m(r, s)$ is the order of rs. For a non-isotropic $\alpha \in V$ let r_α denote the reflection in α, $r_\alpha(x) = x - \frac{2(\alpha,x)}{(\alpha,\alpha)}\,\alpha$, for each $x \in V$. Then the map $s \mapsto r_{\alpha_s}$, for $s \in S$, extends to a faithful representation of W in $GL(V)$, the *geometric representation*, so that W is isomorphic to the reflection group $\langle r_{\alpha_s} \mid s \in S\rangle$. We identify W with such a reflection group and s with r_{α_s}, for each $s \in S$. The standard bilinear form is W-invariant. The root system of W is defined as $\Phi = W\Pi = \{w(\beta) \mid w \in W,\ \beta \in \Pi\}$; a root $\alpha \in \Phi$ is called *positive* ($\alpha > 0$) if $\alpha = \sum_{\beta\in\Pi} c_\beta\beta$ with $c_\beta \geq 0$, and Φ^+ denotes the set of all positive roots. Then $\Phi = \Phi^+ \cup -\Phi^+$; if $\alpha \in -\Phi^+$, then α is called *negative* ($\alpha < 0$). We have $r_\beta = r_{-\beta}$ and $r_{w(\beta)} = w\, r_\beta\, w^{-1}$, for each $\beta \in \Phi$ and $w \in W$, so that $T = \{r_\alpha \mid \alpha \in \Phi^+\}$ and the map $\alpha \mapsto r_\alpha$ is a bijection between Φ^+ and T. Moreover each element of W which acts as a reflection on V belongs to T [H, 5.8, Exercise 3].

For each $w \in W$ set $N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$, where ℓ is the length function of (W, S). Then we have $|N(w)| = \ell(w)$; moreover if $t = r_\alpha$ with $\alpha \in \Phi^+$, then $\ell(tw) < \ell(w)$ if and only if $w^{-1}(\alpha) < 0$. Thus it is easy to see that, for each $v, w \in W$, we have $N(vw) = N(v) + vN(w)v^{-1}$, where $+$ denotes the symmetric difference of sets.

A subgroup G of W is called a reflection subgroup if $G = \langle G \cap T\rangle$. For any reflection subgroup G of W, the set $\chi(G) = \{t \in T \mid N(t) \cap G = \{t\}\}$ is a set of Coxeter generators for G [De], [D1]. Moreover, if $T_G = \{gtg^{-1} \mid g \in G, t \in \chi(G)\}$ is the set of reflections of $(G, \chi(G))$, ℓ_G is the length function, and $N_G(v) = \{t \in T_G \mid \ell_G(tv) < \ell_G(v)\}$ for each $v \in G$, then $T_G = T \cap G$ and $N_G(v) = N(v) \cap G$, for each $v \in G$ [D1, 3.3].

A reflection subgroup G is called *dihedral* if $|\chi(G)| = 2$. If D is a dihedral reflection subgroup with $\chi(D) = \{r, t\}$, we set $r_i = (rt)^i r$, and $t_i = (tr)^i t$, for $i \in \mathbb{N}$, so that $T_D = \{r_i, t_i \mid i \geq 0\}$.

Definition 2.1 ([D2])**.** A total reflection order of W is a total order $\prec$ on T which satisfies the following condition: for each dihedral reflection subgroup D of W, if $\chi(D) = \{r, t\}$ and $r \prec t$, then

$$r_{i-1} \prec r_i \preccurlyeq t_i \prec t_{i-1}$$

for $1 \leq i < \frac{m(r,t)}{2}$, where $m(r, t)$ is the order of rt.

The definition can be geometrically reformulated as follows [D2]. Let $\prec$ be a total order on T and let $\prec'$ be the order induced on Φ^+ by the natural bijection. Then $\prec$ is a total reflection order if and only if, for each α, $\beta \in \Phi^+$ and c, $d \in \mathbb{R}^+$ such that $c\alpha + d\beta \in \Phi^+$, if $\alpha \prec' \beta$, then $\alpha \prec' c\alpha + d\beta \prec' \beta$.

Henceforth we denote by $\cdot$ the standard left action of W on its subsets given by conjugation; thus for $A \subseteq W$ and $w \in W$, $w \cdot A = wAw^{-1}$.

Let $\prec$ be a total order on T. For each $t \in T$ we set $(\prec t) = \{t' \in T \mid t' \prec t\}$ and $(\succ t) = \{t' \in T \mid t' \succ t\}$. We say that $\prec$ is *stable* if, for each $s \in S$, $s \cdot (\prec s) = (\prec s)$; in such a case we also have $s \cdot (\succ s) = (\succ s)$. Let $\prec$ be stable. Then, for $s \in S$, the upper s-conjugate $\prec^s$ of $\prec$ is defined as follows:

$$t \preccurlyeq^s s \quad \text{for each } t \in T; \qquad t \prec^s t' \quad \text{if} \quad \begin{cases} t \prec t' \prec s, & \text{or} \\ t \prec s \prec t', & \text{or} \\ s \prec sts \prec st's. \end{cases}$$

Thus $\prec^s$ is obtained from $\prec$ by shifting s to the last position and conjugating by s each overtaken element. The lower s-conjugate $\prec_s$ of $\prec$ is defined similarly: $s \preccurlyeq_s t$ for each $t \in T$; $t \prec_s t'$ if either $sts \prec st's \prec s$, or $t \prec s \prec t'$, or $s \prec t \prec t'$. The following result holds:

Proposition ([D2, 2.4–2.5])**.** *If $\prec$ is a total reflection order, then $\prec$ is stable and, for each $s \in S$, $\prec^s$ and $\prec_s$ are total reflection orders.*

If $\prec$ and $\prec'$ are total orders on T, we set $\prec \rightarrow \prec'$ if $\prec$ is stable and either $\prec' = \prec_s$ or $\prec' = \prec^s$ for some $s \in S$. We define $\twoheadrightarrow$ as the transitive closure of $\rightarrow$. Thus $\prec \twoheadrightarrow \prec'$ if and only if $\prec'$ is obtained from $\prec$ by successive lower or upper S-conjugations, each conjugation being performed on a stable order.

Definition 2.2. Let $\prec$ be a total order on T. We say that $\prec$ is a *KL-order* if $\prec$ is stable and each total order $\prec'$ such that $\prec \twoheadrightarrow \prec'$ is stable.

By the above Proposition any total reflection order is KL. We remark that the explicit formulas for the R-polynomials and KL-polynomials of W given in [D2], [B1], [B2] in terms of a total reflection order hold with a KL-order. So it is natural to investigate whether the KL property is weaker than that of being a total reflection order. In the next section we prove that indeed the KL property characterizes the total reflection orders.

3. Proof of Theorem 1.1

Henceforth we denote by D a dihedral subgroup of W with $\chi(D) = \{r, t\}$, and by m the order of rt. Moreover we denote by $\prec$ a fixed KL-order. In this section we shall prove that $\prec$ satisfies the condition of Definition 2.1. For $m = 2$ the condition is void, so we assume $m > 2$, possibly $m = \infty$. We keep in force the notation fixed above Definition 2.1. Note that if D is finite, then we have $T_D = \{r_i \mid 0 \leq i < m\}$.

Lemma 3.1. *Let $x, y, z \in T_D$ and $x \prec y \prec z$. Then there exist a KL-order $\prec'$ on T and an element $w \in W$ such that $\chi(D^w) = \{r^w, t^w\}$, r^w is simple, and $x^w \prec' y^w \prec' z^w$.*

Proof. Let $r = s_1 \cdots s_k\, s\, s_k \cdots s_1$, with $s \in S$, $s_i \in S$ for $i = 1, \ldots, k$, and $\ell(r) = 2k+1$ (for the existence of such a reduced expression see [H, 5.8, Exercise 3]). Set $w_0 = 1$, and $w_i = s_1 \cdots s_i$, for $i = 1, \ldots, k$. First we notice that for each $u \in T_D$ and $0 \le i < k$ we have $u^{w_i} \neq s_{i+1}$. Otherwise, we would have $u = s_1 \cdots s_i s_{i+1} s_i \cdots s_1$ and thus $ur = s_1 \cdots \widehat{s_{i+1}} \cdots s_k s s_k \cdots s_1$, where '$\widehat{\ }$' denotes omission. Thus $\ell(ur) < \ell(r)$, but $N(r) \cap T_D = \{r\}$ and $u \neq r$: a contradiction. Therefore we have that $s_{i+1} \notin D^{w_i}$ for $i = 0, \ldots, k-1$. By [D1, 3.2(i)], it follows that $\chi(D^{s_1}) = \{r^{s_1}, t^{s_1}\}$ and, inductively, that $\chi(D^{w_i}) = \{r^{w_i}, t^{w_i}\}$, for $i = 0, \ldots, k$.

Now set $\prec_0 = \prec$, let $0 \le i < k$, and assume that we have defined a KL-order $\prec_i$ on T such that $x^{w_i} \prec_i y^{w_i} \prec_i z^{w_i}$. Then we define a KL-order $\prec_{i+1}$ as follows. If either $s_{i+1} \prec_i x^{w_i} \prec_i y^{w_i} \prec_i z^{w_i}$, or $x^{w_i} \prec_i s_{i+1} \prec_i y^{w_i} \prec_i z^{w_i}$, then we define $\prec_{i+1}$ as the upper conjugate $(\prec_i)^{s_{i+1}}$. If either $x^{w_i} \prec_i y^{w_i} \prec_i z^{w_i} \prec_i s_{i+1}$, or $x^{w_i} \prec_i y^{w_i} \prec_i s_{i+1} \prec_i z^{w_i}$, then we define $\prec_{i+1}$ as the lower conjugate $(\prec_i)_{s_{i+1}}$. Then it is easily seen that $x^{w_{i+1}} \prec_{i+1} y^{w_{i+1}} \prec_{i+1} z^{w_{i+1}}$. Thus $w = w_k$ and $\prec' = \prec_k$ have the claimed properties. □

Proposition 3.2. *If $r \prec t$, then $r \prec rtr$ and $trt \prec t$.*

Proof. Assume by contradiction that $rtr \prec r \prec t$. By Lemma 3.1 we can find a dihedral subgroup D', with $\chi(D') = \{r', t'\}$, r' simple, and a KL-order $\prec'$ such that $r't'r' \prec' r' \prec' t'$. This is a contradiction since $(r' \prec')$ must be stable under conjugation by r'. Hence $r \prec rtr$ as claimed. Similarly we have $trt \prec t$. □

Lemma 3.3. *If $r \prec t$, then $r_{i-1} \prec r_i$ and $t_i \prec t_{i-1}$ for $1 \le i < m$.*

Proof. We prove the claim by induction. Let $1 \le k < m-1$ and assume that for each KL-order $\prec'$ and for each dihedral subgroup D' with $\chi(D') = \{r', t'\}$, if $r' \prec' t'$, then we have $r'_{i-1} \prec' r'_i$ and $t'_i \prec' t'_{i-1}$ for $1 \le i \le k$. For $k = 1$ this follows from Proposition 3.2. Assume by contradiction that $r_{k+1} \prec r_k$. By Lemma 3.1 we can find an element $w \in W$ and a KL-order $\prec'$ such that $r' = r^w$ is simple, $r' \prec' r'_k$, and $r'_{k+1} \prec' r'_k$, where $u' = u^w$ for each u. Since $r' \prec' r'_k$, by the inductive assumption we have $r' \prec' t'$ and $r' \prec' r'_1 \preccurlyeq' \cdots \preccurlyeq' r'_k$. Now we consider the upper conjugate $\prec'' = (\prec')^{r'}$. Since $(r'_i)^{r'} = t'_{i-1}$, we obtain $t' \preccurlyeq'' t'_{k-1}$; hence, by the inductive assumption, $t' \preccurlyeq'' t'_{k-1} \prec'' t'_k \prec'' r'$. But we have $r'_{k+1} \prec' r'_k$, $r' \prec' r'_k$, and $r' \neq r'_{k+1}$; hence $t'_k \prec'' t'_{k-1}$, a contradiction. Therefore the claim is proved. □

Proposition 3.4. *If D is finite and $r \prec t$, then $r \prec r_1 \prec \cdots \prec r_{m-1} = t$. If D is infinite and $r \prec t$, then $r_i \prec r_{i+1} \prec t_{j+1} \prec t_j$ for each i and $j \ge 0$.*

Proof. If D is finite, the claim follows directly from Lemma 3.3. Assume that D is infinite. By Lemma 3.3 we have only to prove that $r_i \prec t_j$ for each $i, j \ge 0$.

First we prove that, for each $i, j \ge 0$, the subgroup $D_{ij} = \langle r_i, t_j \rangle$ is dihedral and $\chi(D_{ij}) = \{r_i, t_j\}$. For $k \ge 1$, we have $N_D(r_{i+1}) = N_D(r_i t\, r) = N_D(r_i) + r_i \cdot N_D(t) + r_i t \cdot N_D(r) = N_D(r_i) + \{r_i t\, r_i\} + \{r_i t\, r\, t\, r_i\} = N_D(r_i) + \{r_{2i+1}, r_{2i+2}\}$. Since $r_0 = r$, we have $N_D(r_0) = \{r_0\}$ and hence, by induction, $N_D(r_i) = \{r_h \mid 0 \le h \le 2i\}$, for each $i \ge 0$. Similarly we have $N_D(t_j) = \{t_k \mid 0 \le k \le 2j\}$, for each $j \ge 0$. Now we

have $r_i t_j r_i = r_{2i+j+1}$; therefore, if $r_h \in D_{ij}$ and $r_h \neq r_i$, then $h > 2i$. It follows that $N(r_i) \cap D_{ij} = N_D(r_i) \cap D_{ij} = \{r_i\}$. Similarly, we have $N(t_j) \cap D_{ij} = \{t_j\}$.

Now if, by contradiction, $t_j \prec r_i$ for some i and j, then, applying Lemma 3.3 to the dihedral subgroup D_{ij}, we would obtain $t_j \prec t_j r_i t_j = t_{2j+i+1}$. But by Lemma 3.3 applied to D, we have $t_{2j+i+1} \prec t_j$. Thus we conclude that $r_i \prec t_j$ for all $i, j \geq 0$. □

Proposition 3.4 shows that $\prec$ satisfies the conditions of Definition 2.1; thus the proof is complete.

4. One side stability

In this section we consider stability under successive one side S-conjugations.

Definition 4.1. Let $\prec$ be a total order on T. If $\prec$ and $\prec'$ are total orders on T, we set $\prec \xrightarrow{u} \prec'$ if $\prec$ is stable and $\prec' = \prec^s$ for some $s \in S$. We define $\overset{u}{\twoheadrightarrow}$ as the transitive closure of $\xrightarrow{u}$. We say that a total order $\prec$ on T is *u-stable* if $\prec$ and each $\prec'$ such that $\prec \overset{u}{\twoheadrightarrow} \prec'$ are stable.

Similarly, we define the l-stable orders of T as the stable orders which remain stable under successive lower conjugations. Clearly, a total order $\prec$ on T is l-stable if and only if the reverse order is u-stable.

We recall that if $w = s_1 \cdots s_k$ is reduced, then we have $N(w) = \{t_1, \ldots, t_k\}$, where $t_1 = s_1$, $t_i = s_1 \ldots s_{i-1} s_i s_{i-1} \cdots s_1$; moreover $w = t_k \cdots t_1$. With this notation, we say that the order $t_1 \prec \cdots \prec t_k$ is induced by the reduced expression $s_1 \cdots s_k$ of w. By [D2, 2.11], $A \subseteq T$ is a finite initial section of some total reflection order if and only if $A = N(w)$ for some $w \in W$. Moreover, $t_1 \prec \cdots \prec t_k$ is a finite initial section of $(T, \prec)$, with $\prec$ a total reflection order, if and only if the order $t_1 \prec \cdots \prec t_k$ is induced by some reduced expression of $w = t_k \cdots t_1$. In particular, if W is finite, then we have $T = N(\omega)$, where ω is the longest element of W, and the total reflection orders are exactly the orders induced by the reduced expressions of ω. In the following lemma we state a simple fact which will be useful in the following.

Lemma 4.1. *Let $\{t_1, \ldots, t_k\} \subseteq T$, $s_1 = t_1$, $s_i = t_1 \cdots t_{i-1} t_i t_{i-1} \cdots t_1$ for $i = 2, \ldots, k$. Assume that $s_1, \ldots, s_k \in S$. Then $w = s_1 \cdots s_k$ is a reduced expression and $N(w) = \{t_1, \ldots, t_k\}$.*

Proof. Put $w_i = s_1 \cdots s_i$, for $i = 1, \ldots, k$; from the definitions it follows that $w_i = t_i \cdots t_1$. We prove by induction that $s_1 \cdots s_i$ is reduced and $N(w_i) = \{t_1, \ldots, t_i\}$. The case $i = 1$ is obvious, so let $2 \leq i \leq k$. We have $N(w_i) = N(w_{i-1}) + w_{i-1} N(s_i) w_{i-1}^{-1} = \{t_1, \ldots, t_{i-1}\} + \{t_{i-1} \cdots t_1 s_i t_1 \cdots t_{i-1}\} = \{t_1, \ldots, t_i\}$; in particular $s_1 \cdots s_i$ is reduced. □

We are now able to prove our result on finite Coxeter systems.

Theorem 4.2. *Assume that W is finite and let $\prec$ be a total order on T. If $\prec$ is u-stable, then it is a total reflection order.*

Proof. We define inductively a sequence $\prec_1 \xrightarrow{u} \cdots \prec_i \xrightarrow{u} \cdots$ setting

$$\prec_1 = \prec, \quad s_i = \min(S, \prec_i), \quad \prec_{i+1} = \prec_i^{s_i} .$$

Then we set $n_i = |(s_i \preccurlyeq_i)| = |\{t \in T \mid s_i \preccurlyeq_i t\}|$. The sequence $(n_i)_{i \geq 1}$ is non-increasing and positive; therefore for some $k \geq 1$ we have $n_k = n_{k+1} = \cdots$. For such

a k, let $\{t_1, \dots, t_{n_k}\} = (s_k \preccurlyeq_k)$ and $s_k = t_1 \prec_k \cdots \prec_k t_{n_k}$. Then, by the choice of k, $\{t_1, \dots, t_{n_k}\}$ satisfies the assumptions of Lemma 4.1, so that $\{t_1, \dots, t_{n_k}\} = N(w)$ for $w = t_{n_k} \cdots t_1$ and the order $t_1 \prec_k \cdots \prec_k t_{n_k}$ is induced by a reduced expression of w. But, by our assumptions, $S \subseteq \{t_1, \dots, t_{n_k}\}$; therefore w is the longest element of W and $N(w) = T$. It follows that $|T| = n_1 = n_2 = \cdots$, $s_1 = \min(T, \prec_1)$, and $\prec_1 = \prec$ is induced by a reduced expression of the longest element. □

Now we study some consequences of u-stability for Coxeter groups in which each proper parabolic subgroup is finite.

Lemma 4.3. *Let $\prec$ be a u-stable order on T and $s \in S$ be the least simple root in $(T, \prec)$. Then, if W is infinite, $(s \prec)$ is infinite.*

Proof. Assume by contradiction that $(s \prec)$ is finite. We perform the same construction as that in the proof of Theorem 4.2; we consider the same sequence of u-stable orders $\prec_i$, the same sequence of non-negative integers (n_i), and take k such that $n_k = n_{k+1} = \cdots$. Then we have $S \subset (s \preccurlyeq) = N(w)$ for some $w \in W$, which is impossible since W is infinite. □

Theorem 4.4. *Assume that each proper parabolic subgroup of W is finite and let $\prec$ be a u-stable order on T. Then $(T, \prec)$ has a minimal element, which is a simple reflection. Moreover each finite initial section of $(T, \prec)$ is an initial section of some total reflection order.*

Proof. We first prove that $\min(S, \prec) = \min(T, \prec)$. If W is finite, the claim follows by Theorem 4.2, so we assume that W is infinite. We define inductively $\prec_1 = \prec$, $s_i = \min(S, \prec_i)$, and $\prec_{i+1} = \prec_i^{s_i}$. Then we consider the sequence $t_1 = s_1$, $t_{i+1} = s_1 \cdots s_i s_{i+1} s_i \cdots s_1$, for $i \geq 1$. We shall prove that for each $i \geq 1$ we have $t_i \prec t_{i+1}$. In fact, by the definition of s_2, we have $t_1 = s_1 \prec s_1 s_2 s_1 = t_2$. For $k \geq 2$, we assume inductively that $t_1 \prec \cdots \prec t_k$. Then we have $s_k \prec_k s_k s_{k+1} s_k$. From the definitions $(s_k \prec_k)$ is equal to $(s_{k-1} \cdots s_1) \cdot (t_k \prec)$ plus a finite final section; if, by contradiction, $s_k s_{k+1} s_k \notin (s_{k-1} \cdots s_1) \cdot (t_k \prec)$, then $s_k s_{k+1} s_k$ is included in a finite final section of $(s_k \prec_k)$, which implies that $(s_{k+1} \prec_{k+1})$ is finite, against Lemma 4.3. Thus $s_k s_{k+1} s_k \in (s_1 \cdots s_{k-1}) \cdot (t_k \prec)$, which implies $t_{k+1} \in (t_k \prec)$, as claimed.

In particular, for each $k \geq 1$ the elements $t_1, \dots, t_k$ are distinct and the sequence $t_1, \dots, t_k$ satisfies the assumptions of Lemma 4.1, so that $s_1 \cdots s_k$ is a reduced expression. Since each proper parabolic subgroup of W is finite, this implies that for each $s \in S$ and each $h \geq 1$ there exists a $k \geq h$ such that $s_k = s$.

Now we assume by contradiction that $(\prec_1 s_1) \neq \emptyset$; then clearly $(\prec_i s_i) \neq \emptyset$ for each $i \geq 1$. Put $\ell_i = \min\{\ell(t) \mid t \prec_i s_i\}$; the sequence $(\ell_i)_{i \geq 1}$ is non increasing and greater than 2, so that, for some $h \geq 1$, $\ell_h = \ell_{h+1} = \cdots$. Let $t \prec_h s_h$ be such that $\ell(t) = \ell_h$ and $s \in S$ be such that $\ell(sts) < \ell(t)$. Then there exists a $k \geq h$ such that $s = s_k$; since $(\prec_h s_h) \subseteq (\prec_k s_k)$ we have $t \prec_k s_k$ and, by stability, $sts = s_k t s_k \prec_k s_k$: thus we have a contradiction, since $\ell(sts) < \ell(t) = \ell_k$. Therefore we conclude that $s_1 = \min(S, \prec) = \min(T, \prec)$, as claimed.

The second statement of the theorem follows easily from the above result. In fact, let $\{t_1, \dots, t_k\}$ be an initial section of T and set $s_1 = t_1$, $s_i = t_1 \cdots t_{i-1} t_i t_{i-1} \cdots t_1$ for each $i \in \{2, \dots, k\}$. Then s_1 is simple, $s_2 = t_1 t_2 t_1 = \min(T, \prec^{t_1})$ is simple and, inductively, s_i is simple for each $i \in \{1, \dots, k\}$. It follows, by Lemma 4.1, that $s_1 \cdots s_k$ is reduced, and $\{t_1, \dots, t_k\} = N(s_1 \cdots s_k)$. Moreover the order $t_1 \prec \cdots \prec t_k$ is induced by the reduced expression $s_1 \cdots s_k$. By [D2, 2.11] we get the thesis. □

The results of this section have an analogous formulation for l-stable orders. Since a total order $\prec$ on T is l-stable if and only if the reverse order is u-stable and since the reverse of a total reflection order is a total reflection order, we have that, for a finite Coxeter group, l-stable orders are total reflection orders. Moreover, if each proper parabolic subgroup of W is finite, then the l-orders on T have a maximum, which is a simple reflection, and the finite final sections of l-orders are the finite final sections of the total reflection orders.

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References

[BB] A. Björner, F. Brenti, *Combinatorics of Coxeter groups*, to appear.

[B1] F. Brenti, *Combinatorial expansion of Kazhdan-Lusztig polynomials*, J. London Math Soc. **55** (1997), 448–472. MR **99a:**05143

[B2] F. Brenti, *Lattice paths and Kazhdan-Lusztig polynomials*, J. Amer. Math. Soc. **11** (1998), 229–259. MR **98h:**20072

[De] V.V. Deodhar, *On the root system of a Coxeter group*, Eins. Math. **32** (1986), 611–630. MR **83j:**20052a

[D1] Dyer, M., *Reflection subgroups of Coxeter systems*, J. of Alg. **135** (1991), 57–73. MR **91j:**20100

[D2] Dyer, M., *Hecke algebras and shellings of Bruhat intervals*, Comp. Math. **89** (1993), 91–115. MR **95c:**20053

[H] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990. MR **92h:**20002

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