RIGIDITY OF AUTOMORPHISMS AND SPHERICAL CR STRUCTURES

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Abstract. We establish Bochner-type formulas for operators related to CR automorphisms and spherical CR structures. From such formulas, we draw conclusions about rigidity by making assumptions on the Tanaka-Webster curvature and torsion.

1. Statement of results

It is now clear (e.g. [CL1], [CL2], [CT], [Rum]) that certain distinguished second-order partial differential operators and their fourth-order “Laplacians” play important roles in the study of three-dimensional CR geometry. In this paper we will establish Bochner-type formulas for operators related to CR automorphisms and spherical CR structures. From such formulas, we can draw conclusions about rigidity by making assumptions on the so-called Tanaka-Webster curvature and torsion.

To be precise, let \((M; J; \theta)\) be a smooth, closed (compact without boundary) 3-dimensional strictly pseudoconvex pseudohermitian manifold (see, e.g. [Web], [Le1]). Here \(J\) denotes a CR structure and \(\theta\) is a contact form, i.e. a nonvanishing real 1-form defining the underlying contact structure. Associated to \((M; J; \theta)\), we have a canonical affine connection and notions of Tanaka-Webster scalar curvature and torsion, denoted \(R\) and \(A\) (a tensor with coefficient \(A_{11}\) or \(A_{11}^1\)) respectively ([Le1], [Tan], [Web]).

Let \(T\) be the characteristic vector field of \(\theta\) defined by \(\theta(T) = 1, L_T \theta = 0\). (Some authors call \(T\) the Reeb vector field.) By choosing suitable complex vector fields \(Z_1, Z_1\) such that \(JZ_1 = iZ_1, JZ_1 = -iZ_1\), we form a (unitary) frame \(\{Z_1, Z_1, T\}\) \((h_{11} = 1\). The covariant derivatives are taken with respect to this frame and indicated by \(\overline{1}, \overline{1}, 0\) and so on. Let \(\text{Aut}_0(J)\) denote the identity component of the CR automorphism group with respect to \(J\).

Now we can state our first result.

Theorem A. Let \((M; J; \theta)\) be a smooth, connected, closed 3-dimensional strictly pseudoconvex pseudohermitian manifold.

(a) Suppose \(R < 0, \sqrt{3}R_0 - 2\text{Im}(A_{11,11}) > 0\).
Then $\text{Aut}_0(J)$ consists of only the identity diffeomorphism.

(b) Suppose $R < 0$, $\sqrt{3}R_{0,0} - 2\text{Im}(A_{11,11}) = 0$. Then $\dim \text{Aut}_0(J) \leq 1$.

We remark that the torsion $A = 0$ implies the second condition in (b) holds due to the Bianchi identity: $R_{0,0} = A_{11,11} + A_{11,11}$. And the result is compatible with Proposition 4.8(a) in [CT]. On the other hand, we do not know any examples satisfying curvature conditions in (a).

The idea of proving Theorem A goes as follows. Consider a certain second-order linear partial differential operator $\mathcal{D}_J$ and its “Laplacean” $\mathcal{D}_J^* \mathcal{D}_J$ acting on functions. Here $\mathcal{D}_J^*$ denotes the adjoint of $\mathcal{D}_J$. The kernel of $\mathcal{D}_J$ parametrizes infinitesimal $CR$ automorphisms (see [CL1] where $\mathcal{D}_J$ was denoted $B^*_J$). Establish a suitable Bochner-type formula for $(\mathcal{D}_J^* \mathcal{D}_J f, f)$ by using commutation relations and integration by parts repeatedly (see section 2 for details). Theorem A then follows easily from the final formula. Note that the kernel of $\mathcal{D}_J^*$ parametrizes the infinitesimal slice in the study of $CR$ moduli spaces ([CL2]).

Next we consider deformation of spherical $CR$ structures. A $CR$ structure or $CR$ manifold is called spherical if it is locally $CR$-equivalent to the unit sphere with the standard $CR$ structure (e.g. [BS], [CL1]). In dimension 3 it can be characterized quantitatively by the vanishing of a certain fourth-order partial differential operator, the so-called Cartan (curvature) tensor, denoted $Q_J$ (while in higher dimensions, Chern’s curvature tensor plays the similar role ([CM]), which is of second order).

A spherical $CR$ structure $J$ is called rigid if there is no infinitesimal deformation up to diffeomorphisms, i.e. for any smooth family of spherical $CR$ structures $J(t)$ on the base manifold $M$ with $J(0) = J$, $d/dt|_{t=0}J(t)$ equals $\mathcal{L}_X J$ for some vector field $X$ of $M$.

To study the rigidity of spherical $CR$ structures, we consider the linearization of $Q_J$ plus a symmetry-breaking term provided by $\mathcal{D}_J \mathcal{D}_J^*$. In section 3 we work out a Bochner-type formula and analyze it to obtain pointwise conditions for $J$ to be rigid.

**Theorem B.** Let $(M, J)$ be a smooth, closed, spherical $CR$ 3-manifold. Suppose there is a contact form $\theta$ such that $R > 0$, (3.11) and (3.12) hold. Then $J$ is rigid.

Note that conditions (a),(b) in Theorem A and (3.11),(3.12) in Theorem B are independent of positive constant multiples of $\theta$. When the torsion vanishes, we have the simplified expression as follows.

**Corollary C.** Let $(M, J)$ be a smooth, closed, spherical $CR$ 3-manifold. Suppose there is a contact form $\theta$ such that the torsion $A = 0$ and

$$R > 0, 4R(5R^2 + 3\Delta_0 R) - 3|\nabla_b R|_{b}^2 > 0.$$  

Then $J$ is rigid.

The sublaplacian and subgradient operators $\Delta_b, \nabla_b$ acting on (smooth) functions are defined by $\Delta_b f = -f_1^1 - f_1^1$ and $\nabla_b f = f_1^1 Z_1 + f_1^1 Z_1$ respectively (cf. [Le1] or [Cia]). Also we define $|\nabla_b f|^2 = 2f_1^1 f_1^1$ for real $f$.

Observe that $A = 0$ and $R$ being a positive constant satisfy conditions in Corollary C. In this case the universal cover of $(M, J)$ must be compact by Rumin’s pseudohermitian version of Myers’ theorem ([Rum]), and hence $CR$-equivalent to
the standard $S^3$. It follows that the fundamental group $\Gamma$ of $M$ is finite. Hence the
fundamental group cohomology $H^1(\Gamma, \mathbb{Z})$ in deformation theory (p.232 in [BS]) vanishes. So in
this special case our result is compatible with the result obtained by “Lie theoretical” argument. Note that a small perturbation of $A = 0$ and $R$ being a positive
can still satisfies the conditions in Theorem B.

2. Proof of Theorem A

Let $\{\theta^1, \theta^\bar{1}, \theta\}$ be the coframe dual to the “unitary” frame $\{Z_1, Z_{\bar{1}}, T\}$ (with $h^{1\bar{1}} = h_{1\bar{1}} = 1$ in mind, hereafter, we’ll write tensors with only lower indices).
Recall ([CL2] or [CL1]) that $D_J f = 2\text{Re}(f_{11} + i A_{11} f)\theta^1 \otimes Z_1$ and the adjoint operator $D_J^* \$ acts on a deformation tensor $E = 2\text{Re}(E_{11} \theta^1 \otimes Z_1)$ by

\begin{equation}
D_J^* E = E_{11, \bar{1}1} + i A_{11} E_{1\bar{1}} + \text{conjugate}.
\end{equation}

Also the generalized Folland-Stein operator $L_\alpha$ is defined by $L_\alpha f = \Delta_b f + i \alpha f$, for a function $f$. By Lemma 2.1 in [CL2], we have

\begin{equation}
D_J^* D_J f = (1/2)L_\alpha^* L_\alpha + O_2
\end{equation}

with $\alpha = i\sqrt{3}$, where $O_2$ is an operator of weight $\leq 2$. In the rest of this section, we’ll look into the details of $O_2$. (Note that $L_\alpha$ in the leading term of (2.2) is subelliptic. This implies the existence of the “infinitesimal slice decompositions” in the study of $CR$ moduli spaces without knowing details of the lower-weight term ([CL2]).)

A direct computation shows that for a real-valued function $f$, we have

\begin{equation}
D_J^* D_J f = f_{11, \bar{1}1} + f_{1\bar{1}11} + 2\text{Re}(2i A_{11} f_{1\bar{1}} + 2i A_{11,1} f_{1\bar{1}} + i A_{11,\bar{1}} f + |A_{11}|^2 f).
\end{equation}

We’ll frequently use the following commutation relations.

**Lemma 2.1** (Ricci identities in pseudohermitian geometry). Let $c_I$ be a coefficient
of some tensor with multi-indices $I$. Suppose $I$ consists of only 1 and $\bar{1}$, and $\alpha = (\# \text{ of } 1 \text{ in } I) - (\# \text{ of } \bar{1} \text{ in } I)$. Then

\begin{equation}
c_{I,1\bar{1}} - c_{I,\bar{1}1} = ic_{I,0} + \alpha c_I R,
\end{equation}

\begin{equation}
c_{I,01} - c_{I,10} = c_{I,1} A_{11} - \alpha c_I A_{11,\bar{1}},
\end{equation}

\begin{equation}
c_{I,01} - c_{I,10} = c_{I,1} A_{1\bar{1}} + \alpha c_I A_{1\bar{1},1},
\end{equation}

(this lemma generalizes Lemma 2.3 in [Le2] for the three-dimensional case).

By using (2.4),(2.5),(2.6) repeatedly, we obtain

\begin{equation}
\frac{1}{2}L_\alpha^* L_\alpha f = f_{11, \bar{1}1} + f_{1\bar{1}11} + 2\text{Re}[(\sqrt{3} - i)(A_{1\bar{1}} f_{1,1}) - (R f_{1,1}, \bar{1})]
\end{equation}

with the choice of $\alpha = i\sqrt{3}$ eliminating terms having covariant derivative in the
direction $T$. Comparing (2.3) with (2.7) and taking the $L^2$-inner product with $f$
gives

\begin{equation}
\|D_J f\|^2 = \frac{1}{2} \|L_{i\sqrt{3}} f\|^2 - \int_M R|\nabla_b f|^2 d\nu
+ 2 \int_M [\text{Re}(\sqrt{3} + iA_{1\overline{1}}\overline{i}) + |A_{1\overline{1}}|^2] f^2 d\nu
+ 2 \int_M \text{Re}(-i - \sqrt{3}) A_{i\overline{1}} f d\nu.
\end{equation}

Here the volume form \( d\nu = \theta \wedge d\theta \). With respect to \( d\nu \), we have the divergence theorem and hence integration by parts in calculus of pseudohermitian geometry ([Le2], [Che]). For instance,

\begin{equation}
A_{i\overline{1}} f,_{i\overline{1}} = -i|A_{1\overline{1}}|^2 f.
\end{equation}

Substituting (2.9), (2.10) in (2.8), we finally obtain

\begin{equation}
0 = \|D_J f\|^2 - 2 \int_M R|\nabla_b f|^2 d\nu + \int_M [\sqrt{3} R_{i\overline{1}0} + i(A_{1\overline{1}},_{i\overline{1}} - A_{i\overline{1}},_{1\overline{1}})] f^2 d\nu.
\end{equation}

Now it is easy to see from (2.11) that the condition in (a) of Theorem A implies \( f = 0 \). Therefore \( X = X_J = 0 \). For (b), the condition implies \( \nabla_b f = 0 \). So \( f,0 = 0 \) by (2.4). Thus \( f \) is constant since \( M \) is connected. It follows that \( \text{dim}(\text{Aut}_0(J)) = \text{dim}(\text{Lie Aut}_0(J)) \leq 1 \).

3. PROOF OF THEOREM B

Recall ([CL1]) that the Cartan tensor \( Q_J = i Q_{1\overline{1}} \theta^1 \otimes Z_1 - iQ_{i\overline{1}} \theta^i \otimes Z_1 \) where

\[ Q_{1\overline{1}} = \frac{1}{6} R_{1\overline{1}} + \frac{i}{2} RA_{1\overline{1}} - A_{1\overline{1},0} - \frac{2i}{3} A_{1\overline{1},i\overline{1}} \]

and \( J \) is spherical if and only if \( Q_J = 0 \). (Note that we have lowered indices using \( h_{i\overline{1}} = 1 \); also \( Q_J \) changes “tensorially” when we make a different choice of contact form.) Let \( J(t) \) be a smooth family of spherical \( CR \) structures with \( J(0) = J \). By a theorem of Gray ([Gra] or [Ham]), there exists a smooth family of diffeomorphisms
\( \phi_t \) with \( \phi_0 = \text{identity} \) so that, for all \( t \), \( J(t) = \phi_t^* \tilde{J}(t) \) has the same underlying contact structure as \( J \) does. Write the infinitesimal deformation

\[
\frac{d}{dt} |_{t=0} J(t) = 2E = 4\text{Re}(E_{11} \theta^1 \otimes Z_1)
\]

and compute \( DQ_j(2E) = \partial_t Q_{J(t)} |_{t=0} \) as we did in [CL1]. There appears a “bad” term \( E_{11,1111} \) in the formula, so we add a “symmetry-breaking” term \( D_j D_j^* E \) to cancel it. The final formula including terms of lower weights reads

\[
(3.1)
\]

\[
- DQ_j(2E) + \frac{1}{6} D_j D_j^* E
\]

\[
= 2\text{Re}(\frac{1}{3} E_{11,1111} E_{11,00} - \frac{2i}{3} E_{11,0111}) + \frac{i}{3} \langle A_{11} E_{11} \rangle_{11}
\]

\[
- \frac{1}{6} E_{11} R_{11} + \frac{1}{6} E_{11,1} R_{11} - \frac{1}{6} (E_{11} R_{11})_{11}
\]

\[
+ \frac{1}{2} A_{11} (i E_{11,11} - i E_{11,111} - A_{11} E_{11} - A_{11} E_{111}) + \frac{i}{2} RE_{11,0}
\]

\[
+ 2 A_{11} (A_{11} E_{11} + A_{11} E_{111}) - \frac{2i}{3} E_{11,111} A_{11,11} - \frac{2i}{3} E_{11,111} A_{11,11} - \frac{2i}{3} (E_{11} A_{11,111})_{11}
\]

\[
- \frac{4i}{3} (E_{11,11} A_{11,11} + i E_{11,111}, E_{11,11} + i A_{11} E_{11} - i A_{11} E_{111}) \theta^1 \otimes Z_1.
\]

The right-hand side of (3.1) can be written as \( \frac{1}{12} L_\alpha^* L_\alpha E + O_2(E) \) with \( O_2 \) being an operator of weight \( \leq 2 \) and \( \alpha = 4 + i \sqrt{3} \). Since \( L_\alpha \) is subelliptic, the above expression was used in [CL1] to show the short-time solution of a certain regularized evolution equation. Using Lemma 2.1 repeatedly, we can write the highest-weight term of (3.1) as follows:

\[
(3.2)
\]

\[
\frac{1}{3} E_{11,11111} - E_{11,00} - \frac{2i}{3} E_{11,0111} = \frac{1}{3} E_{11,11111} - E_{11,00} - i E_{11,0111}
\]

\[
+ \frac{i}{3} (E_{11,11} A_{11,11} + \frac{2i}{3} E_{11,11} A_{11,11} - \frac{1}{3} (RE_{11,11})_{11}.
\]

On the other hand, we compute

\[
(3.3)
\]

\[
\int_M R[E_{11,11}]^2 d\theta = - \int_M (RE_{11,11} E_{11} + R_{11} E_{11,1} E_{11}) d\theta
\]

(by integration by parts)

\[
= - \int_M [R(E_{11,11} + i E_{11,0} + 2RE_{11}) E_{11,1} + R_{11} E_{11,1} E_{11}] d\theta \quad (by \ (2.4)).
\]

To see how we treat (3.1) in general, we first deal with the torsion=0 case. By the local slice theorem ([CL2]), there exists a smooth family of contact diffeomorphisms \( \psi_t \) with \( \psi_0 = \text{identity} \) so that \( J(t) = \psi_t^* J(t) \) lies in the local slice passing through \( J \). Since the infinitesimal slice at \( J \) is parametrized by the kernel of \( D_j^* \), we have \( D_j^* E = 0 \) for \( 2E = \frac{d}{dt}|_{t=0} J(t) \).
Now applying (3.1) to such a deformation tensor $E$: $D_\gamma^* E = 0$, substituting (3.2) in (3.1), and taking the $L^2$-inner product with $E$, we obtain

\begin{equation}
0 = \langle -DQ \beta(2E), E \rangle \tag{3.4}
\end{equation}

\begin{align*}
&= \int_M 2 \Re \left\{ \frac{1}{3} |E_{11,11}|^2 + |E_{11,0}|^2 - i E_{11,0} E_{11,11} + \frac{1}{6} R |E_{11,11}|^2 \\
&\quad + \frac{2i}{3} R E_{11,0} E_{11,11} - \frac{1}{6} R E_{11,11} E_{11,11} - \frac{1}{6} R_i E_{11,11} E_{11,11} \\
&\quad + \frac{1}{6} R_{11,11} - 2 R_{11,11} + \frac{1}{3} R^2 |E_{11,11}|^2 \right\} d\theta \\
&\text{by putting } A_{11} = 0, \text{ using (3.3) and integration by parts repeatedly.}
\end{align*}

Note that we reduce (3.4) to the formula (6.3) in [CL1] for $R$ being a constant $\tilde{R}$. Furthermore, if $\tilde{R} > 0$, the right-hand side of (3.4) is a positive definite quadratic hermitian form in $E_{11,11}$, $iE_{11,0}$, and $\tilde{R} E_{11,11}$.

It follows that $E = 0$ and $0 = \frac{d}{d\theta} |J_{(t)}(t) = \frac{d}{d\theta} (\phi_{\theta} \phi_{\psi_{\theta}}) \frac{d}{d\theta} J(t) = L_X J + \frac{d}{d\theta} |J_{(t)}(t)$ where the vector field $X = \frac{d}{d\theta} (\phi_{\theta} \phi_{\psi_{\theta}})$. So $J$ is rigid. For general $R$, we require that the integrand in (3.4) is a (pointwise) positive definite quadratic hermitian form in $E_{11,11}$, $iE_{11,0}$, $E_{11,11}$, $E_{11,11}$. Now the conditions in Corollary C can be deduced from basic linear algebra.

When the torsion does not vanish, the formula for $\langle -DQ \beta(2E), E \rangle$ with $D_\gamma^* E = 0$ reads

\begin{equation}
0 = \langle -DQ \beta(2E), E \rangle = \int_M \left\{ \frac{2}{3} |E_{11,11}|^2 + 2 |E_{11,0}|^2 \\
+ \frac{1}{3} R |E_{11,11}|^2 + \frac{2}{3} R^2 + \frac{1}{6} \Delta_b R + 6 |A_{11}|^2 + \frac{8i}{3} (A_{11,11} - A_{11,11}) |E_{11}|^2 \\
+ 2 \Re \left[ -i E_{11,0} E_{11,11} + \frac{2i}{3} R E_{11,0} E_{11,11} - \frac{1}{6} R E_{11,11} E_{11,11} \\
+ \frac{1}{6} R_{11,11} - 2i A_{11,11} E_{11,11} - \frac{2i}{3} A_{11,11} E_{11,11} E_{11,11} - \frac{5i}{3} A_{11,11} E_{11,11} |E_{11,11}| \right] \right\} d\theta.
\end{equation}

There are non-cross terms like $A_{11,11} E_{11,11}$ and $A_{11,11} E_{11,11} E_{11,11}$ in (3.5). We need an inequality to deal with $A_{11,11} E_{11,11} E_{11,11}$.

**Lemma 3.1.** Let $\lambda$, $\rho$ be real numbers. Then

\begin{equation}
2 \lambda \rho \int_M R |E_{11,11}|^2 d\theta \leq \lambda^2 \int_M |E_{11,11}|^2 d\theta + \rho^2 \int_M R^2 |E_{11,11}|^2 d\theta \\
- \lambda \rho \int_M (R_1 E_{11,11} E_{11,11} + R_1 E_{11,11} E_{11,11}) d\theta.
\end{equation}

For the term $A_{11,11} E_{11,11} E_{11,11}$, we use the following estimate:

\begin{equation}
2 \Re \left[ -\frac{2i}{3} A_{11,11} E_{11,11} E_{11,11} \right] \geq - \frac{2}{3} |A_{11,11}|^2 |E_{11,11}|^2 - \frac{2}{3} |A_{11,11}|^2 |E_{11,11}|^2.
\end{equation}

(To deduce (3.7), replace $a, b$ by $\omega_1, \omega_2$ in the basic inequality $2 \Re(ab) \geq -|a|^2 - |b|^2$ with $a = -i E_{11,11}$, $b = E_{11,11}$, and $\omega^3 = A_{11,11}$.)

The reason for taking fractional exponents in (3.7) is to make our conditions invariant under the scale change of contact form by a positive constant multiple as we’ll see later. Take a small amount of $\int_M |E_{11,11}|^2 d\theta$ and $\int_M R^2 |E_{11,11}|^2 d\theta$ to gain the term $\int_M R |E_{11,11}|^2 d\theta$ by (3.6) in the right-hand side of (3.5) while keeping
the quadratic hermitian form in $E_{11,11}$, $iE_{11,0}$, $E_{11,1}$, $E_{11,1}$, $E_{11,1}$ positive definite at least for $R = constant > 0$, $A_{11} = 0$. For instance, we can take $\lambda = \rho = \frac{1}{2}$ in (3.6) and then use it and (3.7) in estimating the right-hand side of (3.5). The final result reads

$$\begin{align*}
0 \geq \int_M \left( \frac{29}{48} |E_{11,11}|^2 + 2|E_{11,0}|^2 + \frac{1}{3} R |E_{11,1}|^2 \\
+ \left( \frac{1}{8} R - \frac{2}{3} |A_{11,1,1}|^2 \right) |E_{11,1}|^2 + \frac{29}{48} R^2 + \frac{11}{48} \Delta_b R \\
+ 6 |A_{11}|^2 + \frac{8i}{3} (A_{11,1} - A_{11,1,1}) - \frac{2}{3} |A_{11,1}|^2 |E_{11}|^2 \\
+ 2 Re [-iE_{11,0} E_{11,11} + \frac{2}{3} i RE_{11,0} E_{11} - \frac{1}{6} RE_{11,1} E_{11} \\
+ \left( \frac{5}{48} R - \frac{1}{2} i A_{11,1} \right) E_{11,1} E_{11} - \frac{5i}{3} A_{11,1} E_{11,1} E_{11,1}] dv_9. \right)
\end{align*}$$

The integrand in (3.8) is a quadratic hermitian form in $E_{11,11}$, $iE_{11,0}$, $E_{11,1}$, $E_{11,1}$, and $E_{11,1}$, $E_{11,1}$. By basic linear algebra it is positive definite if and only if $R > 0$,

$$\begin{align*}
\begin{vmatrix}
\frac{29}{48} & -1 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
0 & 0 & \frac{1}{8} R & \frac{5}{3} A_{11,1} \\
0 & 0 & -\frac{5i}{3} A_{11} & \frac{1}{8} R - \frac{1}{2} |A_{11,1}|^2
\end{vmatrix} > 0, \text{ and}
\end{align*}$$

$$\begin{align*}
\begin{vmatrix}
\frac{29}{48} & -1 & 0 & 0 & -\frac{1}{6} R \\
-1 & 2 & 0 & 0 & \frac{2}{3} R \\
0 & 0 & \frac{1}{8} R & \frac{5}{3} A_{11,1} & \frac{5}{48} R - 2i A_{11,1} \\
0 & 0 & -\frac{5i}{3} A_{11} & \frac{1}{8} R - \frac{1}{2} |A_{11,1}|^2 & 0 \\
-\frac{1}{6} R & \frac{2}{3} R & \frac{5}{3} A_{11,1} & 2i A_{11,1} & M
\end{vmatrix} > 0.
\end{align*}$$

is larger than 0. Here $M = \frac{29}{48} R^2 + \frac{11}{48} \Delta_b R + 6 |A_{11}|^2 - \frac{2}{3} |A_{11,1}|^2 + \frac{8i}{3} (A_{11,1} - A_{11,1,1})$. A straightforward computation shows that (3.9) is equivalent to

$$3 \frac{8}{8} R^2 - 2 R |A_{11,1}|^2 - 25 |A_{11}|^2 > 0$$

while (3.10) is equivalent to

$$\begin{align*}
\left( \frac{3}{8} R^2 - 2 R |A_{11,1}|^2 - 25 |A_{11}|^2 \right) \left\{ \frac{83}{3456} R^2 \\
+ \frac{55}{1152} \Delta_b R + \frac{5}{4} |A_{11}|^2 - \frac{5}{36} |A_{11,1}|^2 + \frac{5i}{9} (A_{11,1} - A_{11,1,1}) \\
- \frac{15}{8} \left( \frac{1}{8} R - \frac{2}{3} |A_{11,1}|^2 \right) \left[ \frac{5}{48} R - 2i A_{11,1} \right]^2 > 0. \right.
\end{align*}$$

Observe that if $\theta$ changes by a positive constant multiple $k$, $R$ and $A_{11}$ change by multiplying $k^{-1}$ while $A_{11,1}, A_{11,1,1}$, and $R_{11}$ change by multiplying $k^{-\frac{3}{2}}$. Similarly $A_{11,1,1}, A_{11,1,1,1}$, and $\Delta_b R$ change by multiplying $k^{-2}$. So the conditions (3.11), (3.12) are invariant under the change of contact form by a positive constant multiple. Now Theorem B follows from (3.8) under the conditions (3.11), (3.12).
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