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ON THE DIVERGENCE OF THE $(C, 1)$ MEANS OF DOUBLE WALSH-FOURIER SERIES

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Abstract. In 1992, Móricz, Schipp and Wade proved the a.e. convergence of the double $(C, 1)$ means of the Walsh-Fourier series $\sigma_n f \to f$ $(\min(n_1, n_2) \to \infty, n = (n_1, n_2) \in \mathbb{N}^2)$ for functions in $L\log^+ L(I^2)$ (I^2 is the unit square). This paper aims to demonstrate the sharpness of this result. Namely, we prove that for all measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\lim_{t\to\infty} \delta(t) = 0$ we have a function f such as $f \in L\log^+ L\delta(L)$ and $\sigma_n f$ does not converge to f a.e. (in the Pringsheim sense).

Introduction

This paper is devoted to the problem of a.e. divergence of the $(C, 1)$ means of integrable functions with respect to the two-dimensional Walsh-Paley system. The problem of a.e. Cesàro summability is "quite delicate" in any local field setting (Taibleson, [6, p.114]). The dyadic case is no exception (see Fine [1], [2]). For double Walsh-Fourier series, Móricz, Schipp and Wade proved [4] that $\sigma_{(n_1,n_2)} f$ converges to f a.e. in the Pringsheim sense (that is, no restriction on the indices n_1, n_2 other than $\min(n_1, n_2) \to \infty$) for all function f in $L\log^+ L$. Since for the compact set I^2 (the unit square) the space $L\log^+ L$ is a proper subset of $L^1(I^2)$, then it is interesting to ask whether this theorem holds for $L^1(I^2)$ functions also? We give a negative answer. More precisely, we prove that the theorem of Móricz, Schipp and Wade cannot be sharpened. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. We prove the existence of a function $f \in L^1(I^2)$ such as $f \in L\log^+ L\delta(L)$ (i.e., $|f(x)| \log^+(|f(x)|)\delta(|f(x)|) \in L^1(I^2)$) and $\sigma_{(n_1,n_2)} f$ does not converge to f a.e. as $\min(n_2, n_2) \to \infty$.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $I := [0, 1)$. For any set E let E^2 be the cartesian product $E \times E$. Thus $\mathbb{N}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j -dimensional Lebesgue measure (μ) of any set $E \subset I^j$ by $\mu(E)$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which

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terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in I$, the i th coordinate of e_i is 1 , the rest are zeros ($i \in \mathbb{N}$). Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} (x_j + y_j \bmod 2) 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := I$ are the dyadic intervals of I. Denote $\mathcal{I} := \{I_n(x) : x \in I, n \in \mathbb{N}\}$; the elements of $\mathcal{I}$ are called the dyadic intervals on I. Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in I)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})\,(f \in L^1)$. For $t = (t^1, t^2) \in I^2$, $b = (b_1, b_2) \in \mathbb{N}^2$ set the two-dimensional dyadic interval

$$I_b^2(t) := I_{b_1}(t^1) \times I_{b_2}(t^2).$$

If $b \in \mathbb{N}$, then $I_b^2(t) := I_b(t^1) \times I_b(t^2)$. For $n = (n_1, n_2) \in \mathbb{N}^2$ denote by $E_n = E_{(n_1,n_2)}$ the two-dimensional expectation operator with respect to $\mathcal{A}_n = \mathcal{A}_{(n_1,n_2)} = \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$. For $n \in \mathbb{N}$ denote $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in I,\, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in I,\, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. Let us consider the Dirichlet and Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k,\ K_n := \frac{1}{n} \sum_{k=1}^{n} D_k,\ D_0, K_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C,1)$ mean of $f \in L^1(I)$:

$$\hat{f}(n) := \int_I f(x)\omega_n(x) d\mu(x)\ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_I f(x+y) D_n(x) d\mu(x)\ (n \in \mathbb{P},\, S_0 f = 0),$$

$$\sigma_n(y) := \frac{1}{n} \sum_{k=1}^{n} S_k(y) = \int_I f(x+y) K_n(x) d\mu(x)\, (n \in \mathbb{P}, \sigma_0 f = 0, y \in I).$$

Define the two-dimensional Dirichlet and Fejér kernel functions as

$$D_n(x) := \sum_{0 \leq j_1 < n_1, 0 \leq j_2 < n_2} \omega_{j_1}(x^1)\omega_{j_2}(x^2) = D_{n_1}(x^1) D_{n_2}(x^2),$$

$$K_n(x) := \frac{1}{n_1 n_2} \sum_{1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2} D_n(x) = K_{n_1}(x^1) K_{n_2}(x^2),$$

where $x = (x^1, x^2) \in I^2$, $n = (n_1, n_2) \in \mathbb{P}^2$ (for $n_1 n_2 = 0$ set $D_n = K_n = 0$). Also define the Fourier coefficients, the $n \in \mathbb{N}^2$-th partial sum of the Fourier series, the $n \in \mathbb{N}^2$-th $(C,1)$ mean of $f \in L^1(I^2)$:

$$\hat{f}((n_1, n_2)) := \int_{I^2} f(x^1, x^2)\omega_{n_1}(x^1)\omega_{n_2}(x^2) d\mu(x^1, x^2) \ (n \in \mathbb{N}^2),$$

$$\begin{aligned} S_{(n_1,n_2)}f(y) &:= \sum_{k_1 < n_1, k_2 < n_2} \hat{f}((k_1, k_2))\omega_{k_1}(y^1)\omega_{k_2}(y^2) \\ &= \int_{I^2} f(x+y) D_n(x) d\mu(x) \end{aligned}$$

($n \in \mathbb{P}^2$, $S_n f = 0$ for $n_1 n_2 = 0$),

$$\sigma_{(n_1,n_2)}(y) := \frac{1}{n_1 n_2} \sum_{1 \le k_1 \le n_1, 1 \le k_2 \le n_2} S_k(y) = \int_{I^2} f(x+y) K_n(x) d\mu(x)$$

($n \in \mathbb{P}^2, \sigma_n f = 0$ for $n_1 n_2 = 0$, $y \in I^2$).

The construction

Define a subset of the set of the two-dimensional intervalls $\mathcal{I} \times \mathcal{I}$:

$$\mathcal{I}_{n,a}(x) := \{I_{n+j}(x^1) \times I_{n+a-j}(x^2) : j = 0, 1, ..., a\} \quad (x \in I^2, a, n \in \mathbb{N}).$$

It is easy to have

$$\bigcap \mathcal{I}_{n,a}(x) = I_{n+a}(x^1) \times I_{n+a}(x^2), \quad \mu(\bigcap \mathcal{I}_{n,a}(x)) = 2^{-2n-2a};$$

$F \in \mathcal{I}_{n,a}(x)$ implies $\mu(F) = 2^{-2n-a}$. Next we prove

Lemma 1. $\mu(\bigcup \mathcal{I}_{n,a}(x)) = (1 + a/2)2^{-2n-a}$.

Proof. Denote (for the sake of this proof, only)

$$\mu_k := \mu(\bigcup_{j=0}^{k} (I_{n+j}(x^1) \times I_{n+a-j}(x^2)))$$

for $k = 0, 1, ..., a$. Then, $\mu_0 = 2^{-2n-a}$ and for $k > 0$ we have

$$\begin{aligned} \mu_k &= \mu_{k-1} + \mu(I_{n+k}(x^1) \times I_{n+a-k}(x^2)) \\ &\quad - \mu(\bigcup_{j=0}^{k-1} (I_{n+j}(x^1) \times I_{n+a-j}(x^2)) \cap (I_{n+k}(x^1) \times I_{n+a-k}(x^2))) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu(\bigcup_{j=0}^{k-1} (I_{n+k}(x^1) \times I_{n+a-j}(x^2))) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu(I_{n+k}(x^1) \times I_{n+a-k+1}(x^2)) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \frac{1}{2^{2n+a+1}} = \mu_{k-1} + \frac{1}{2^{2n+a+1}}. \end{aligned}$$

This gives

$$\mu(\bigcup \mathcal{I}_{n,a}(x)) = \mu(\bigcup_{j=0}^{a} \big(I_{n+j}(x^1) \times I_{n+a-j}(x^2)\big)) = \mu_a$$

$$= \mu_0 + a\frac{1}{2^{2n+a+1}} = \frac{1}{2^{2n+a}} + a\frac{1}{2^{2n+a+1}} = \frac{1+a/2}{2^{2n+a}}.$$

This completes the proof of Lemma 1. □

For $t \in I^2$, $a, b, k \in \mathbb{N}$ define the sets $J^k_{a,b}(t)$, $\Omega^k_{a,b}(t)$ recursively:

$$J^0_{a,b}(t) := \{t\}, \quad \Omega^0_{a,b}(t) := \bigcup \mathcal{I}_{b,a}(t).$$

Suppose that the sets $J^j_{a,b}(t)$, $\Omega^j_{a,b}(t)$ are defined for $j < k$. Then decompose

$$I^2_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega^j_{a,b}(t)$$

as the disjoint union of dyadic squares of the form $I^2_{b+ka}(x)$. Take from each dyadic rectangle an element to represent. The set of x's corresponding to these squares is $J^k_{a,b}(t)$. That is,

$$I^2_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega^j_{a,b}(t) = \bigcup_{x \in J^k_{a,b}(t)} I^2_{b+ka}(x).$$

Then, set

$$\Omega^k_{a,b}(t) := \bigcup_{x \in J^k_{a,b}(t)} \bigcup \mathcal{I}_{b+ka,a}(x).$$

This gives the a.e. equality $I^2_b(t) = \bigcup_{j=0}^{\infty} \Omega^j_{a,b}(t)$. By this we get

$$I^2 = \bigcup_{\substack{t^1_i, t^2_i \in \{0,1\} \\ i=0,1,\ldots,b-1}} \bigcup_{j=0}^{\infty} \Omega^j_{a,b}(t),$$

where $t = (t^1, t^2) = (t^1_0 e_0 + \ldots + t^1_{b-1} e_{b-1}, t^2_0 e_0 + \ldots + t^2_{b-1} e_{b-1}) \in I^2$. Set for $a, b, d \in \mathbb{N}$ $(b \geq 2)$ the functions ($b^\circ := [b/2] - 1$ ($[x]$ denotes the integer part of x)) $f^d_{a,b} : I^2 \to \mathbb{R}$ as follows:

$$f^d_{a,b}(x) := \begin{cases} (-1)^{x^1_{b^\circ} + x^2_{b^\circ}} 2^a, & \text{if there exists } t \in I^2, k \leq d, y \in J^k_{a,b}(t) \\ & \text{for which } x \in \bigcap \mathcal{I}_{b+ka,a}(y), \\ 0, & \text{otherwise .} \end{cases}$$

Denoting by 1_B the characteristic set of any set $B \subset I^2$ we have

$$f^d_{a,b}(x) = (-1)^{x^1_{b^\circ} + x^2_{b^\circ}} 2^a \sum_{\substack{t^1_i, t^2_i \in \{0,1\} \\ i=0,1,\ldots,b-1}} \sum_{k=0}^{d} \sum_{y \in J^k_{a,b}(t)} 1_{I^2_{b+k(a+1)}(y)}(x).$$

Lemma 2. *For all $a, b, d \in \mathbb{N}$ we have* $\int_{I^2} |f^d_{a,b}| \log^+ |f^d_{a,b}| \leq 2$.

Proof.

$$\int_{I^2} |f^d_{a,b}(x)| \log^+(|f^d_{a,b}(x)|) d\mu(x)$$

$$= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^{d} \sum_{y \in J^k_{a,b}(t)} \mu(1_{I^2_{b+k(a+1)}(y)}(x) = 1)$$

$$= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^{d} \sum_{y \in J^k_{a,b}(t)} \mu(\bigcap \mathcal{I}_{b+ka,a}(y))$$

$$= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^{d} \sum_{y \in J^k_{a,b}(t)} \frac{\mu(\bigcup \mathcal{I}_{b+ka,a}(y))}{2^a(1+a/2)}$$

$$\leq \frac{\log(2^a)}{1+a/2} \mu(I^2) \leq 2.$$

□

By the definition of the functions $f^d_{a,b}$ we have that

$$f^d_{a,b}(.+e_{b^\circ}, .) = f^d_{a,b}(., .+e_{b^\circ}) = -f^d_{a,b}(.,.).$$

It is known [5, p. 46] that for all $n \in \mathbb{P}$, $f \in L^1(I)$

$$\sigma_{2^n} f(x) = (2^n + 1/2) \int_{I_n(x)} f + \sum_{j=0}^{n-1} 2^{j-1} \int_{I_n(x)} f(.+e_j),$$

by which we have for $n = (n_1, n_2) \in \mathbb{P}^2$, $x = (x^1, x^2) \in I^2$, $(I^2_n(x) = I_{n_1}(x^1) \times I_{n_2}(x^2))$, $f \in L^1(I^2)$

$$\sigma_{(2^{n_1}, 2^{n_2})} f(x) = (2^{n_1} + 1/2)(2^{n_2} + 1/2) \int_{I^2_n(x)} f(.)$$

$$+ (2^{n_1} + 1/2) \sum_{j_2=0}^{n_2-1} 2^{j_2-1} \int_{I^2_n(x)} f(., .+e_{j_2})$$

$$+ (2^{n_2} + 1/2) \sum_{j_1=0}^{n_1-1} 2^{j_1-1} \int_{I^2_n(x)} f(.+e_{j_1}, .)$$

$$+ \sum_{j_2=0}^{n_2-1} \sum_{j_1=0}^{n_1-1} 2^{j_1+j_2-2} \int_{I^2_n(x)} f(.+e_{j_1}, .+e_{j_2}). \tag{1}$$

Lemma 3. *Let $a, b, d \in \mathbb{P}$, $a - b^\circ < 0$ $(b^\circ = [b/2] - 1)$, $b \geq 4$,*

$$x \in \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \bigcup_{k=0}^{d} \Omega^k_{a,b}(t);$$

then there exists a unique $k \le d$ *for which*

$$x \in I_{b+ka+j}(y^1) \times I_{b+(k+1)a-j}(y^2), \tag{2}$$

$y \in J^k_{a,b}(t)$. *Setting* $n := (b + ka + j, b + (k + 1)a - j) \in \mathbb{P}^2$ *we have*

$$|\sigma_{(2^{n_1},2^{n_2})} f^d_{a,b}(x)| \ge \frac{1}{4}. \tag{3}$$

Proof. (2) is a straightforward consequence of the definition of $\Omega^k_{a,b}(t)$. Next, verify (3). Define $\tilde f^d_{a,b}$ by $f^d_{a,b} = (-1)^{x^1_{b^\circ}+x^2_{b^\circ}} 2^a \tilde f^d_{a,b}$. Then $\tilde f^d_{a,b}$ is either 0 or 1. Recall that

$$f^d_{a,b}(x^1, x^2 + e_{j_2}) = (-1)^{x^1_{b^\circ}+x^2_{b^\circ}} 2^a \tilde f^d_{a,b}(x^1, x^2 + e_{j_2}),$$

$$f^d_{a,b}(x^1 + e_{j_1}, x^2) = (-1)^{x^1_{b^\circ}+x^2_{b^\circ}} 2^a \tilde f^d_{a,b}(x^1 + e_{j_1}, x^2)$$

for $j_1, j_2 \ne e_{b^\circ}$ and

$$f^d_{a,b}(. + e_{b^\circ}, .) = f^d_{a,b}(., . + e_{b^\circ}) = -f^d_{a,b}(., .).$$

By (1) we have

$$\begin{aligned}
\sigma_{(2^{n_1},2^{n_2})} f^d_{a,b}(x) &= 2^a(-1)^{x^1_{b^\circ}+x^2_{b^\circ}} \Big((2^{n_1} + 1/2)(2^{n_2} + 1/2) \int_{I^2_n(x)} \tilde f^d_{a,b}(.) \\
&+ (2^{n_1} + 1/2) \sum_{j_2=0, j_2\ne b^\circ}^{n_2-1} 2^{j_2-1} \int_{I^2_n(x)} \tilde f^d_{a,b}(., . + e_{j_2}) \\
&+ (2^{n_2} + 1/2) \sum_{j_1=0, j_1\ne b^\circ}^{n_1-1} 2^{j_1-1} \int_{I^2_n(x)} \tilde f^d_{a,b}(. + e_{j_1},.) \\
&+ \sum_{j_2=0, j_2\ne b^\circ}^{n_2-1} \sum_{j_1=0, j_1\ne b^\circ}^{n_1-1} 2^{j_1+j_2-2} \int_{I^2_n(x)} \tilde f^d_{a,b}(. + e_{j_1}, . + e_{j_2}) \Big) \\
&- 2^a(-1)^{x^1_{b^\circ}+x^2_{b^\circ}} \Big((2^{n_1} + 1/2)(2^{b^\circ-1}) \int_{I^2_n(x)} \tilde f^d_{a,b}(.) \\
&+ (2^{n_2} + 1/2)(2^{b^\circ-1}) \int_{I^2_n(x)} \tilde f^d_{a,b}(.) \Big) \\
&- 2^a(-1)^{x^1_{b^\circ}+x^2_{b^\circ}} \Big(\sum_{j_2=0, j_2\ne b^\circ}^{n_2-1} 2^{j_2+b^\circ-2} \int_{I^2_n(x)} \tilde f^d_{a,b}(., . + e_{j_2}) \\
&+ \sum_{j_1=0, j_1\ne b^\circ}^{n_1-1} 2^{j_1+b^\circ-2} \int_{I^2_n(x)} \tilde f^d_{a,b}(. + e_{j_1}, .) \Big) \\
&+ 2^a(-1)^{x^1_{b^\circ}+x^2_{b^\circ}} \Big(2^{b^\circ+b^\circ-2} \int_{I^2_n(x)} \tilde f^d_{a,b}(., .) \Big).
\end{aligned}$$

From the above it follows that

$$\begin{aligned}
&|\sigma_{(2^{n_1},2^{n_2})} f^d_{a,b}(x)| \\
&\geq 2^{a+n_1+n_2}\int_{I_n^2(x)} \tilde{f}^d_{a,b}(.) - 2^a(2^{n_1}+1/2)(2^{b^\circ-1})\int_{I_n^2(x)} \tilde{f}^d_{a,b}(.) \\
&- 2^a(2^{n_2}+1/2)(2^{b^\circ-1})\int_{I_n^2(x)} \tilde{f}^d_{a,b}(.) - 2^a\sum_{j_2=0, j_2\neq b^\circ}^{n_2-1} 2^{j_2+b^\circ-2}\int_{I_n^2(x)} \tilde{f}^d_{a,b}(.,.+e_{j_2}) \\
&- 2^a\sum_{j_1=0, j_1\neq b^\circ}^{n_1-1} 2^{j_1+b^\circ-2}\int_{I_n^2(x)} \tilde{f}^d_{a,b}(.+e_{j_1},.) \\
&\geq 2^a\Big(2^{n_1+n_2}\mu(I_{b+(k+1)a}(y^1)\times I_{b+(k+1)a}(y^2)) \\
&- ((2^{n_1}+1/2)(2^{b^\circ-1}) + (2^{n_2}+1/2)(2^{b^\circ-1}))\mu(I_{b+(k+1)a}(y^1)\times I_{b+(k+1)a}(y^2)) \\
&- (\sum_{j_2=0, j_2\neq b^\circ}^{n_2-1} 2^{j_2+b^\circ-2} + \sum_{j_1=0, j_1\neq b^\circ}^{n_1-1} 2^{j_1+b^\circ-2})\mu(I_n^2(x))\Big) \\
&\geq 2^a\Big(2^{b+ka+j+b+(k+1)a-j}2^{-2b-2(k+1)a} - 2^{b+(k+1)a+b^\circ+1}2^{-2b-2(k+1)a} \\
&- 2^{b^\circ-2}2^{-2b-ka-(k+1)a}(\sum_{j_1=0}^{n_1-1} 2^{j_1} + \sum_{j_2=0}^{n_2-1} 2^{j_2})\Big) \geq \frac{1}{2}.
\end{aligned}$$

Recall that $a < b^\circ = [b/2]-1$. This completes the proof of Lemma 3. □

The construction of the function

Define the sequences $(\beta_n), (\delta_n), (a_n), (b_n), (d_n)$ recursively in the following way: $(\beta_0) = (\delta_0) = (a_0) = (b_0) = (d_0) := 4$. For $n \in \mathbb{P}$

$$\beta_n := 72\max(n, \sum_{k=0}^{n-1}\beta_k 2^{a_k}),$$

$\delta_n := [\sup\{t\in\mathbb{R} : \delta(t) > \frac{1}{2^n\beta_n}\}] + 1$ (if $\{t : \delta(t) > 1/(2^n\beta_n)\} = \emptyset$, then let $\delta_n = 4$),

$a_n := \max(2^{n+1}, \delta_n, \log^+(2\beta_n))$,

$b_n := \max(b_{n-1} + (d_{n-1}+1)a_{n-1} + 2, 2a_n)$.

By construction it is easy to see that the union of the disjoint sets $\Omega^k_{a,b}(t)$ $(k\in\mathbb{N})$ is equal (neglecting a set of measure zero) to $I_b^2(t)$; hence there exists a d_n for which

$$\frac{1}{2^{2b}}\left(1-\frac{1}{2^n}\right) \leq \mu(\bigcup_{k\leq d_n}\Omega^k_{a_n,b_n}(t)) \leq \frac{1}{2^{2b}}. \tag{4}$$

Set

$$G_{a_n,b_n,d_n} := \bigcup_{\substack{t_i^1,t_i^2\in\{0,1\}\\ i=0,1,\dots,b_n-1}}\bigcup_{j=0}^{d_n}\Omega^j_{a_n,b_n}(t)$$

for all $n\in\mathbb{N}$ and $G := \liminf G_{a_n,b_n,d_n} = \bigcup_{k\in\mathbb{N}}\bigcap_{n\geq k} G_{a_n,b_n,d_n}$. That is, $x\in I^2$ is an element of G if and only if x is an element of all but a finite number of

G_{a_n,b_n,d_n}'s. By (4) we have $1-\frac{1}{2^n} \le \mu(G_{a_n,b_n,d_n}) \le 1$, which easily gives that $\mu(I^2 \setminus G) = 0$.

Define the function f as $f = \sum_{n=0}^{\infty} \beta_n f_n := \sum_{n=0}^{\infty} \beta_n f_{a_n,b_n}^{d_n}$. At first we prove

Lemma 4. *$\int_{I^2} |f(x)| \log^+(|f(x)|)\delta(|f(x)|)d\mu(x) < \infty$.*

Proof. Set

$$H_n := \{x \in I^2 : f_n(x) \neq 0, f_{n+j}(x) = 0\,(j \in \mathbb{P})\} \quad (n \in \mathbb{N})$$

and $H_{-1} := \{x \in I^2 : f_j(x) = 0\,(j \in \mathbb{P})\}$. The definiton of $f_{a,b}^d$, (a_n) gives

$$\begin{aligned}
&\mu(H_n)\\
&\ge 1-\mu(\bigcup_{k>n}\{x \in I^2 : f_k(x) \neq 0\}) \ge 1 - \sum_{k>n} \mu(\{x \in I^2 : f_k(x) \neq 0\})\\
&\ge 1 - \sum_{k>n} \frac{1}{2^{a_k}(a_k/2+1)} \ge 1 - \sum_{k>n} \frac{1}{2^{2^k}}.
\end{aligned}$$

It follows that $\bigcup_{n=-1}^{\infty} H_n = I^2$ (neglecting a set of measure zero). Corresponding to this argument if $x \in H_n\,(n \in \mathbb{N})$, then

$$\begin{aligned}
|f(x)| &\le \sum_{k=0}^{n-1} \beta_k 2^{a_k} + \beta_n 2^{a_n}\\
&\le \beta_n + \beta_n 2^{a_n} \le 2\beta_n 2^{a_n} = |\beta_n 2 f_{a_n,b_n}^{d_n}(x)|,
\end{aligned}$$

$$\begin{aligned}
|f(x)| &\ge \beta_n 2^{a_n} - \sum_{k=0}^{n-1} \beta_k 2^{a_k}\\
&\ge \beta_n 2^{a_n} - \frac{1}{2}\beta_n \ge \frac{1}{2}\beta_n 2^{a_n}\\
&= \frac{1}{2}|\beta_n f_{a_n,b_n}^{d_n}(x)|.
\end{aligned}$$

Moreover, for $x \in H_n$ we have $|f(x)| \ge \frac{1}{2}\beta_n 2^{a_n} \ge 2^{a_n} > \delta_n$, which gives $\delta(|f(x)|) \le \frac{1}{2^n \beta_n}$. Consequently, by Lemma 2

$$\begin{aligned}
&\int_{H_n} |f(x)| \log^+(|f(x)|)\delta(|f(x)|)d\mu(x)\\
&\le \int_{H_n} 2|\beta_n f_{a_n,b_n}^{d_n}(x)| \log^+(2|f_{a_n,b_n}^{d_n}(x)|)\frac{1}{2^n\beta_n}\\
&\le \int_{H_n} 2|\beta_n f_{a_n,b_n}^{d_n}(x)| \log^+(|f_{a_n,b_n}^{d_n}(x)|^2)\frac{1}{2^n\beta_n}\\
&\le \frac{4}{2^n}\int_{I^2} |f_{a_n,b_n}^{d_n}(x)| \log^+(|f_{a_n,b_n}^{d_n}(x)|)d\mu(x) \le \frac{8}{2^n}.
\end{aligned}$$

Since for $x \in H_{-1}$ we have $f(x) = 0$, then we get

$$\begin{aligned}
&\int_{I^2} |f(x)| \log^+(|f(x)|)\delta(|f(x)|)d\mu(x)\\
&\le \sum_{n\in\mathbb{N}} \int_{H_n} |f(x)| \log^+(|f(x)|)\delta(|f(x)|)d\mu(x) \le 16.
\end{aligned}$$

□

Now, we are ready to prove

Theorem. *For all measurable function* $\delta : [0,+\infty) \to [0,+\infty)$, $\lim_{t\to\infty} \delta(t) = 0$ *we have a function* f *such that* $f \in L\log^+ L\delta(L)$ *and* $\limsup \sigma_{(2^{n_1},2^{n_2})} f(x) = +\infty$ *(*$\min(n_2,n_2) \to \infty$*), i.e. the two-dimensional* $(C,1)$ *means does not converge to* f *a.e. (in the Pringsheim sense).*

It is not possible to prove more (e.g. to prove that $\lim \sigma_{(2^{n_1},2^{n_2})} f(x) = +\infty$ ($\min(n_2,n_2) \to \infty$)), whereas Móricz, Schipp and Wade proved [4] for functions in $L^1(I^2)$ the a.e. convergence $\sigma_{(2^{n_1},2^{n_2})} f \to f$ as the indices $n_1, n_2 \to \infty$ ($|n_1 - n_2|$ is bounded). The author in a different way proved [3] even more. Namely, for functions in $L^1(I^2)$ the a.e. convergence $\sigma_{(n_1,n_2)} f \to f$ holds as the indices $n_1, n_2 \to \infty$ restricted as $\beta^{-1} \le n_1/n_2 \le \beta$ for some $\beta > 1$ (compare with [7], [8], [9]).

Proof. We apply Lemmas 2,3 and 4. The rest is to prove the "divergence". Namely, we verify that for a.e. $x \in I^2$ $\limsup \sigma_{(2^{n_1},2^{n_2})} f(x) = +\infty$ ($\min(n_2,n_2) \to \infty$). This is the same as $\sup_{n_1,n_2\in\mathbb{N}} \sigma_{(2^{n_1},2^{n_2})} f(x) = +\infty$. Let $x \in G$ (recall that $\mu(I^2 \setminus G) = 0$). Then there is an infinite number $n \in \mathbb{N}$ for which (even for all, but a finite number) $x \in G_{a_n,b_n,d_n}$. Then Lemma 3 gives that there exist a $t \in I^2$, $k \le d_n$ for which $x \in \Omega^k_{a_n,b_n}(t)$, whereby, there are a $y \in J^k_{a_n,b_n}(t)$, a unique $j \in \{0,1,...,a_n\}$ for which $x \in I_{b_n+ka_n+j}(y^1) \times I_{b_n+(k+1)a_n-j}(y^2)$. Set

$$N := (2^{N_1}, 2^{N_2}) := (b_n + ka_n + j, b_n + (k+1)a_n - j) \in \mathbb{P}^2.$$

By Lemma 3 we have

$$|\sigma_{(2^{N_1},2^{N_2})} f^{d_n}_{a_n,b_n}(x)| \ge \frac{1}{2}.$$

In [5, p. 46] one can find that $\|K_n\|_1 \le 3$ for $n \in \mathbb{N}$ which in the standard way (see e.g. [5]) gives $\|\sigma_n f\|_\infty \le 9\|f\|_\infty$ ($f \in L^1(I^2)$, $n \in \mathbb{P}^2$). It follows for $i \in \mathbb{P}$ that

$$\|\sigma_{(2^{N_1},2^{N_2})} f^{d_{n-i}}_{a_{n-i},b_{n-i}}\|_\infty \le 9\|f^{d_{n-i}}_{a_{n-i},b_{n-i}}\|_\infty \le 9 \cdot 2^{a_{n-i}}.$$

The definition of function $f^d_{a,b}$ gives that $E_{(b^\circ,b^\circ)} f^d_{a,b} = 0$ and also that for $j_1, j_2 < b^\circ$ natural numbers $E_{(b^\circ,b^\circ)} f^d_{a,b}(. + e_{j_1}, .) = E_{(b^\circ,b^\circ)} f^d_{a,b}(., . + e_{j_1}) = 0$. Hence, for $k \in \mathbb{P}$ it follows that

$$\begin{aligned} &2^{N_1N_2} \int_{I^2_N(x)} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(. + e_{j_1}, .) \\ &= E_{(N_1,N_2)} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(. + e_{j_1}, .)(x) \\ &= E_{(N_1,N_2)} \left(E_{(b^\circ_{n+k},b^\circ_{n+k})} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(. + e_{j_1}, .) \right)(x) = 0 \end{aligned}$$

and

$$\begin{aligned} &2^{N_1N_2} \int_{I^2_N(x)} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(., . + e_{j_2}) \\ &= E_{(N_1,N_2)} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(., . + e_{j_2})(x) \\ &= E_{(N_1,N_2)} \left(E_{(b^\circ_{n+k},b^\circ_{n+k})} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(., . + e_{j_2}) \right)(x) = 0. \end{aligned}$$

This gives that for all $k \in \mathbb{P}$

$$\sigma_{(2^{N_1},2^{N_2})} f^{d_{n+k}}_{a_{n+k},b_{n+k}}(x) = 0.$$

Consequently,

$$\begin{aligned}
&|\sigma_{(2^{N_1},2^{N_2})} f(x)| \\
&\geq |\sigma_{(2^{N_1},2^{N_2})} \beta_n f^{d_n}_{a_n,b_n}(x)| - \sum_{i=1}^{n-1} |\sigma_{(2^{N_1},2^{N_2})} \beta_i f^{d_i}_{a_i,b_i}(x)| \\
&\geq \frac{1}{2}\beta_n - 9\sum_{i=0}^{n-1} \beta_i \|f^{d_i}_{a_i,b_i}\|_\infty \\
&\geq \frac{1}{2}\beta_n - 9\sum_{i=0}^{n-1} \beta_i 2^{a_i} \geq \frac{1}{4}\beta_n \geq 18n.
\end{aligned}$$

This completes the proof of the theorem. □

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