NONVANISHING OF SYMMETRIC SQUARE $L$-FUNCTIONS OF CUSP FORMS INSIDE THE CRITICAL STRIP

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(Communicated by Dennis A. Hejhal)

Abstract. We shall give a certain nonvanishing result for the symmetric square $L$-function of an elliptic cuspidal Hecke eigenform w.r.t. the full modular group inside the critical strip.

1. Introduction

Let $f$ be a normalized cuspidal Hecke eigenform of integral weight $k$ on the full modular group $SL_2(\mathbb{Z})$ and denote by $D_f^s(s \in \mathbb{C})$ the symmetric square $L$-function of $f$ completed with its archimedean $\Gamma$-factors. As is well-known [7], [8], $D_f^s(s)$ has a holomorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k - 1 - s$. Note that, by [3], $D_f^s(s)$ (up to a variable shift) also is the standard zeta function of a cuspidal automorphic representation of $GL(3)$, and so by [4] zeros of $D_f^s(s)$ can occur only inside the critical strip $k - 1 < \text{Re}(s) < k$. According to the generalized Riemann hypothesis, the zeros of $D_f^s(s)$ should all lie on the critical line $\text{Re}(s) = k - \frac{1}{2}$.

The last statement of course is far from being settled. On the other hand, it turns out to be comparatively easy to prove nonvanishing results for $D_f^s(s)$ on the average. For example, in [6] Xian-Jin Li used an approximate functional equation for an average sum of the $D_f^s(s)$ to show that for any given $s$ with $k - 1 < \text{Re}(s) < k$, $s \neq k - \frac{1}{2}$, $\zeta(s - k + 1) \neq 0$, there are infinitely many different $f$ such that $D_f^s(s)$ is not zero.

In the present note, using a different approach we will prove that given any $s$ with $k - 1 < \text{Re}(s) < k$, $\text{Re}(s) \neq k - \frac{1}{2}$, then for all $k$ large enough there exists a Hecke eigenform $f$ of weight $k$ such that $D_f^s(s) \neq 0$. For the proof we use a “kernel function” for $D_f^s(s)$ as given by Zagier in [8] and then proceed in a similar way as in [3], where a corresponding result for Hecke $L$-functions was proved.

2. Notation

For $s \in \mathbb{C}$ we usually write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$.
3. Statement of result

Let $k$ be an even integer $\geq 12$ and let $S_k$ be the space of cusp forms of weight $k$ w.r.t. the full modular group $\Gamma_1 = SL_2(\mathbb{Z})$, equipped with the usual Petersson scalar product $(\cdot, \cdot)$. For $f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$ ($z \in \mathcal{H}$ = upper half plane) a normalized Hecke eigenform in $S_k$ (recall that normalized means $a(1) = 1$), we denote by

$$D_f(s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}$$

($\sigma > k$)

the symmetric square $L$-function of $f$, where the product is taken over all rational primes $p$ and $\alpha_p, \beta_p$ are defined by

$$\alpha_p + \beta_p = a(p), \quad \alpha_p \beta_p = p^{k-1}. $$

By [7], [8], $D_f(s)$ has a holomorphic continuation to $\mathbb{C}$, and the function

$$D_f^*(s) = 2^{-s} \pi^{-3s/2} \Gamma(s) \Gamma(s - k + 2) D_f(s)$$

satisfies the functional equation

$$D_f^*(2k - 1 - s) = D_f^*(s).$$

Let $\{f_{k,1}, \ldots, f_{k,g_k}\} (g_k = \dim S_k)$ be the basis of normalized Hecke eigenforms of $S_k$.

**Theorem.** Let $t_0 \in \mathbb{R}$ and $0 < \epsilon < \frac{1}{2}$. Then there exists a positive constant $C(t_0, \epsilon)$ depending only on $t_0$ and $\epsilon$ such that for $k > C(t_0, \epsilon)$ the function

$$\sum_{\nu=1}^{g_k} \frac{1}{(f_{k,\nu}, f_{k,\nu})} D^*_{f_{k,\nu}}(s)$$

does not vanish at any point $s = \sigma + it_0$, $k - 1 < \sigma < k - \frac{1}{2} - \epsilon$, $k - \frac{1}{2} + \epsilon < \sigma < k$.

**Corollary.** Let $s \in \mathbb{C}$ be fixed with $k - 1 < \sigma < k$, $\sigma \neq k - \frac{1}{2}$. Then for all $k$ large enough there exists a normalized Hecke eigenform $f$ in $S_k$ such that $D_f^*(s) \neq 0$.

4. Proof

The proof proceeds along similar lines as in [5]. We consider the cusp forms dual w.r.t. the Petersson scalar product to the values $D_f^*(s)$ ($2 - k < \sigma < k - 1$) where $f$ is any normalized Hecke eigenform in $S_k$. These have been constructed by Zagier [8] and will be denoted $\Phi_s$ as in [8] in what follows. To state the relevant properties of $\Phi_s$, we need to introduce several notations.

Let $\Delta$ be a discriminant, i.e. $\Delta \in \mathbb{Z}$ and $\Delta \equiv 0, 1 \pmod{4}$. Put

$$L(s, \Delta) = \begin{cases} \zeta(2s - 1), & \text{if } \Delta = 0, \\ L_D(s) \sum_{d|f, d > 0} \mu(d) \left( \frac{D}{d} \right) d^{-s} \sigma_{1-2s}(\frac{d}{f}), & \text{if } \Delta \neq 0, \end{cases}$$

where if $\Delta \neq 0$ we have written $\Delta = Df^2$ with $f \in \mathbb{N}$ and $D$ the discriminant of $\mathbb{Q}(\sqrt{\Delta})$, $(\mathbb{Z})$ is the Kronecker symbol, $L_D(s)$ the associated $L$-function defined by analytic continuation of the series $\sum_{n \geq 1} \left( \frac{D}{n} \right) n^{-s}$ ($\sigma > 1$), $\mu$ is the Möbius function and $\sigma_{1-2s}(d) = \sum_{d|m, d > 0} d^r \sigma_{1-2s}(m)$ ($m \in \mathbb{N}, r \in \mathbb{C}$).
Furthermore, for \( t \) an integer with \( \Delta < t^2 \) and \( s \in \mathbb{C} \) with \( \frac{1}{2} < \sigma < k \) we define

\[
I_k(\Delta, t; s) = \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}}{(x^2 + y^2 + ity - \frac{1}{4}\Delta)^k} \, dx \, dy
\]

(2)

\[
= \frac{\Gamma(k - \frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2 + ity - \frac{1}{4}\Delta)^{k-\frac{1}{2}}} \, dy
\]

where the second integral converges absolutely for \( 1 - k < \sigma < k \) if \( \Delta \neq 0 \) [8, Proposition 4]. We are now in a position to state Zagier’s theorem.

**Theorem (8).** Let \( k \geq 4 \) be an even integer. For \( m \in \mathbb{N}, s \in \mathbb{C} \) set

\[
c_m(s) = m^{k-1} \sum_{t \in \mathbb{Z}} (I_k(t^2 - 4m, t; s) + I_k(t^2 - 4m, -t; s))L(s, t^2 - 4m)
\]

\[
+ \begin{cases} 
(-1)^{\frac{k}{2}} \frac{\pi \zeta(2s)}{2^{s-k+1}} m^{k-s-1} & \text{if } m = u^2, u > 0, \\
0 & \text{if } m \text{ is not a perfect square}.
\end{cases}
\]

Then the following assertions hold:

i) The series converges absolutely and uniformly for \( 2 - k < \sigma < k - 1 \).

ii) The function

\[
\Phi_s(z) = \sum_{m \geq 1} c_m(s) e^{2\pi i m z} \quad (z \in \mathcal{H}, 2 - k < \sigma < k - 1)
\]

is in \( S_k \).

iii) Let \( f \) be a normalized Hecke eigenform in \( S_k \). Then the Petersson scalar product of \( \Phi_s \) and \( f \) is given by

\[
(\Phi_s, f) = c_k \frac{\pi \zeta(s+k-1)}{2^{s+k-1} \Gamma(\frac{s+k-1}{2})} D^*_f(s+k-1)
\]

where

\[
c_k = \frac{(-1)^{\frac{k}{2}} \pi}{2^{k-3}(k-1)}
\]

From the theorem, taking \( m = 1 \) we deduce

\[
c_1(s) = c_k \frac{\pi \zeta(s+k-1)}{2^{s+k-1} \Gamma(\frac{s+k-1}{2})} \sum_{\nu=1}^{\nu_k} \frac{1}{|f_{k,\nu}|} D^*_{f_{k,\nu}}(s+k-1)
\]

(3)

\[
(2 - k < \sigma < k - 1).
\]

In view of the functional equation (1), it is sufficient to prove the theorem in the range \( k - \frac{1}{2} + \epsilon < \sigma < k \). Suppose that the right-hand side of (3) vanishes at \( s = \frac{1}{2} + \delta + it_0 \) where \( \epsilon < \delta < \frac{1}{2} \). Then from the definition of \( c_1(s) \) we obtain

\[
\sum_{t \in \mathbb{Z}} (I_k(t^2 - 4, -\frac{1}{2} + \delta + it_0) + I_k(t^2 - 4, -\frac{1}{2} + \delta - it_0))L(\frac{1}{2} + \delta + it_0, t^2 - 4) + (-1)^{\frac{k}{2}} \frac{\Gamma(k - \frac{1}{2} + \delta + it_0) \zeta(1 + 2\delta + 2it_0)}{2^{k-2+2\delta+2it_0} \pi^{-\frac{1}{2}+\delta+it_0} \Gamma(k)} = 0,
\]
or

\begin{equation}
(4) \quad \frac{(-1)^k}{\Gamma(k - \frac{1}{2} + \delta + it_0)} \sum_{l \in \mathbb{Z}} (I_k(t^2 - 4; 1; \frac{1}{2} + \delta + it_0) + I_k(t^2 - 4; -1; \frac{1}{2} + \delta + it_0)) \\
\cdot L(\frac{1}{2} + \delta + it_0, t^2 - 4) = \frac{\zeta(1 + 2\delta + 2it_0)}{2^{2\delta-2+2it_0} \pi^{-\frac{1}{2} + \delta + it_0}}.
\end{equation}

Clearly the right-hand side of (4) does not depend on \(k\) and is never zero for \(\epsilon \le \delta \le \frac{1}{2}\). Therefore in absolute value it is bounded from below by a positive absolute constant depending only on \(\epsilon\).

We will show that the left-hand side of (4) goes to zero uniformly for \(\epsilon < \delta < \frac{1}{2}\) as \(k \to \infty\), thereby arriving at a contradiction.

We first look at the terms \(I_k(t^2 - 4; 1; \frac{1}{2} + \delta + it_0) + I_k(t^2 - 4; -1; \frac{1}{2} + \delta + it_0)\) in (4).

If \(t = 0\), we obtain from (2)

\[2I_k(-4, 0; \frac{1}{2} + \delta + it_0) = 2 \frac{\Gamma(k - \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k-\frac{1}{2}+\delta+it_0}}{(y^2 + 1)^{k-\frac{1}{2}}}\, dy
\]

where \(B(z, w)\) is the Beta function and in the last line we have used [1] 6.2.1. Since \(B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}\) we deduce

\begin{equation}
(5) \quad 2I_k(-4, 0; \frac{1}{2} + \delta + it_0) = \sqrt{\pi} \frac{\Gamma(\frac{k-\frac{1}{2}+\delta+it_0}{2}) \Gamma(\frac{k-\frac{1}{2}-\delta-it_0}{2})}{\Gamma(k)}.
\end{equation}

Next suppose that \(t \neq 0, \pm 2\). We will then use the fact that \(I_k(t^2 - 4, t; s)\) can be expressed in terms of standard Legendre functions \(P^n_s(z)\) [8] Proof of Proposition 4. More precisely, one has

\[I_k(t^2 - 4, t; s) = \left(\frac{|t^2 - 4|}{4}\right)^{\frac{1}{2}} \frac{\Gamma(k - \frac{1}{2})\sqrt{\pi}}{\Gamma(k)} \cdot \begin{cases} I_{k,s}(\frac{|t|}{\sqrt{|t^2 - 4|}}), & \text{if } t^2 < 4, \\ e^{\frac{s}{2}(z-k)\sigma_0(t)} I_{k,s}(\frac{|t|}{\sqrt{|t^2 - 4|}}), & \text{if } t^2 > 4, \end{cases}
\]

where

\[I_{k,s}(z) = \frac{2^{1-k}\sqrt{\pi}}{\Gamma(k - \frac{1}{2})} \Gamma(k - 1 + s) \Gamma(k - s) (z^2 - 1)^{-\frac{k-1}{2}} P_{-s}^{1-k}(z)
\]

\((1 - k < \sigma < k, z \in \mathbb{C} \setminus (-\infty, 1])\). Also for \(|z - 1| < 2\) the identity

\[P_{-s}^{1-k}(z) = \frac{1}{\Gamma(k)} \left(\frac{z + 1}{z - 1}\right)^{1-k} \Gamma(1 - s, k; \frac{1-z}{2}) F(s, 1 - s, k; \frac{1-z}{2})
\]

holds, where \(F(a, b; c; z) \ (|z| < 1)\) denotes the Gauss hypergeometric series [11] 8.1.2.

Therefore for \(t = \pm 1\) we easily find that

\begin{equation}
(6) \quad 2\left(I_k(-3, 1; \frac{1}{2} + \delta + it_0) + I_k(-3, -1; \frac{1}{2} + \delta + it_0)\right)
\leq \frac{\left|\Gamma(k - \frac{1}{2} + \delta + it_0) \Gamma(k - \frac{1}{2} - \delta - it_0)\right|}{2^{2\delta} \Gamma(k)^2}.
\end{equation}
and for $\pm t \geq 3$ we get
\[ 2(I_k(t^2 - 4, t; \frac{1}{2} + \delta + it_0) + I_k(t^2 - 4, -t; \frac{1}{2} + \delta + it_0)) \ll t_0 (t^2 - 4)^{-\frac{3}{4} + \frac{\varepsilon}{4}} \]

(7)
\[
\frac{(|t| - \sqrt{t^2 - 4})^{\frac{1}{2} + \delta + it_0}}{|t| + \sqrt{t^2 - 4}} \frac{|\Gamma(k - \frac{1}{2} + \delta + it_0)\Gamma(k - \frac{1}{2} - \delta - it_0)|}{2^k \Gamma(k)^2}
\]

where the constants implied in $\ll t_0$ depend only on $t_0$.

Finally, for $t = \pm 2$ we have by [8 Proposition 4]
\[
2(I_k(0, 2; \frac{1}{2} + \delta + it_0) + I_k(0, -2; \frac{1}{2} + \delta + it_0))
\]
\[
= 2(e^{\frac{k}{2}(k - k + \delta + it_0)} + e^{-\frac{k}{2}(k - k + \delta + it_0)}) \Gamma(\frac{1}{2} + \delta + it_0)
\]
\[
= \Gamma(\delta + it_0) \Gamma(k - \frac{1}{2} - \delta - it_0) 2^{-k + \frac{1}{2} + \delta + it_0} \]
\[
\ll t_0, \varepsilon \frac{|\Gamma(k - \frac{1}{2} - \delta - it_0)|}{2^k \Gamma(k)}
\]

We now look at the quantities $L(\frac{1}{2} + \delta + it_0, t^2 - 4)$ on the left-hand side of (4).

For $|t| = 2$, we have by definition
\[ L(\frac{1}{2} + \delta + it_0, 0) = \zeta(2\delta + 2it_0) \]

which is a continuous function in the range $\varepsilon \leq \delta \leq \frac{1}{2}$, provided $t_0 \neq 0$. If $t_0 = 0$, the same applies to the function
\[ 2t^k \cos\left(\frac{\pi}{2} \left(\frac{1}{2} + \delta + it_0\right)\right) \cdot \zeta(2\delta + 2it_0) \]

where
\[ 2t^k \cos\left(\frac{\pi}{2} \left(\frac{1}{2} + \delta + it_0\right)\right) = e^{\frac{\pi}{4}(k - k + \delta + it_0)} + e^{-\frac{\pi}{4}(k - k + \delta + it_0)} \]

is the second factor on the right-hand side of (8).

On the other hand, if $|t| \neq 2$, then for all $\delta \geq 0$ and all $\varepsilon' > 0$ one has

(9)
\[ L(\frac{1}{2} + \delta + it_0, t^2 - 4) \ll t_0, \varepsilon', |t^2 - 4|^{\frac{1}{2} + \varepsilon'}. \]

In fact, if $t^2 - 4$ is a fundamental discriminant, this follows from [24 Chapter 12, Example 22 (b)], and the general case then easily follows from the definitions.

Denote the left-hand side of (4) by $L_{k, \delta, t_0}$. Then from (5) to (9) (fixing any small $\varepsilon' > 0$ in (9)) and the separate discussion in the case $|t| = 2$, $t_0 = 0$ above we deduce that

(10)
\[ |L_{k, \delta, t_0}| \ll t_0, \varepsilon \left| \frac{2^k \Gamma(k - \frac{1}{2} + \delta + it_0) \Gamma(k - \frac{1}{2} - \delta - it_0)}{\Gamma(k - \frac{1}{2} + \delta + it_0)} \right| + \left| \frac{\Gamma(k - \frac{1}{2} - \delta - it_0)}{\Gamma(k)} \right| \]

\[ + \left| \frac{\Gamma(k - \frac{1}{2} - \delta - it_0)}{\Gamma(k - \frac{1}{2} + \delta + it_0)} \right| + \left| \frac{\Gamma(k - \frac{1}{2} - \delta - it_0)}{\Gamma(k)} \right| \sum_{t \geq 3} (t^2 - 4)^{\frac{1}{2} + \varepsilon'} \left( \frac{t - \sqrt{t^2 - 4}}{t + \sqrt{t^2 - 4}} \right)^{\frac{1}{2} + \varepsilon'}. \]

Elementary considerations show that the sum over $t \geq 3$ converges and is bounded by an absolute constant independent of $k$. 


Note that by Legendre’s duplication formula for the \( \Gamma \)-function, the first term on the right-hand side of (10) can also be written in the form
\[
2^{3-\delta} \sqrt{\pi} \frac{\Gamma\left(\frac{k}{2} - \frac{3}{4} - \frac{\delta}{2} - \frac{d_0}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{1}{4} + \frac{\delta}{2} + \frac{d_0}{2}\right)}.
\]
Using the fact that
\[
\lim_{x \to \infty} x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 \quad (a, b \in \mathbb{C} \setminus \mathbb{R}; x \to \infty)
\] as given in [1, 6.1.46, 6.1.47], we now see indeed that
\[
(11) \quad L_{k, \delta, t_0} \to 0 \quad (k \to \infty).
\]
Moreover, using more precisely the explicit asymptotics of \( x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} \) for \( x \to \infty \) given in [1, 6.1.47], one sees that the convergence in (11) is uniform in \( \delta \), since \( \delta > \epsilon > 0 \) is bounded away from zero.

This proves the theorem.

References


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