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# ON INTEGERS OF THE FORM $2^k \pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$

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Abstract. In this paper we prove that the set of positive odd integers which have no representation of the form $2^n \pm p^\alpha q^\beta$, where $p$, $q$ are distinct odd primes and $n, \alpha, \beta$ are nonnegative integers, has positive lower asymptotic density in the set of all positive odd integers.

## 1. Introduction

The problem of expressing an odd integer $m > 1$ in the form $2^n + p$, where $p$ is a prime and $n$ is a nonnegative integer, is an old one. Romanoff [8] showed that the set of positive odd numbers which can be expressed in the form $2^n + p$ has positive asymptotic density in the set of all positive odd numbers. P. Erdős [6] exhibited a residue class of odd integers each of which has no representation of the form $2^n + p$. Cohen and Selfridge [5] proved that there exist (infinitely many ) odd numbers which are neither the sum nor the difference of a power of two and a prime power.

For a positive integer $n$ and an integer $a$ let

$$a(n) = \{a + nk : k \in \mathbb{Z}\}.$$

We call $\{a_i(n_i)\}_{i=1}^t$ a covering system if every integer $y$ satisfies $y \equiv a_i \pmod{n_i}$ for at least one value of $i$. For the construction of covering systems one may refer to S. L. G. Choi [4]. For further related information one may see Guy [7], A19, B21 and F13.

For convenience we give the following definitions.

**Definition 1.** A positive integer $d$ is called a $(a, b)$-primitive divisor of order $n$ if $d | a^n - b^n$ and $d \nmid a^m - b^m$ for all $1 \le m < n$.

**Definition 2.** $\{a_i(n_i)\}_{i=1}^t$ is called an $m$-covering system if every integer belongs to at least $m$ of $a_1(n_1)$, $a_2(n_2)$, $\cdots$, $a_t(n_t)$.

**Definition 3.** $\{a_i(n_i)\}_{i=1}^t$ is called a $(2, 1)$-primitive $m$-covering system if $\{a_i(n_i)\}_{i=1}^t$ is an $m$-covering system and there exist distinct primes $p_1, p_2, \cdots, p_t$ such that, for each $i$, $p_i$ is a $(2, 1)$-primitive divisor of order $n_i$ $(1 \le i \le t)$.

For example $0(2), 3(4), 5(8), 9(16), 17(32), 33(64), 1(64)$ is a $(2, 1)$-primitive 1-covering system (corresponding primes are 3, 5, 17, 257, 65537, 641, 6700417 respectively).

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From the form $\pm 2^n \pm p^\alpha$ to $\pm 2^n \pm p^\alpha q^\beta$ there are some essential difficulties. The first is to construct a (2, 1)-primitive 2-covering system. We employ Choi's method to complete such a construction. From this we need only consider $p$,$q$ in a finite set for our purpose. The second difficulty is to give additional conditions as in the proof of Theorem 2 of [2] or to do a similar thing. To avoid this we employ a result of Baker to show that for $p, q$ in a finite set, the set of integers of the form $\pm 2^n \pm p^\alpha q^\beta$ is very thin in the set of all positive odd integers. For later use we give the argument in a general form.

In this paper the following results are proved.

**Theorem.** *Suppose that there exists a (2, 1)-primitive $r$-covering system. Then the set of odd positive integers which have neither the form $2^n + q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ nor the form $2^n - q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ , where $q_1, q_2, \cdots, q_r$ are distinct positive odd primes and $n, \alpha_1, \cdots, \alpha_r$ are nonnegative integers, has positive lower asymptotic density.*

**Corollary.** *The set of odd positive integers which have neither the form $2^n + p^\alpha q^\beta$ nor the form $2^n - p^\alpha q^\beta$ , where $p, q$ are distinct positive odd primes and $n, \alpha, \beta$ are nonnegative integers, has positive lower asymptotic density.*

*Remark 1.* For given integers $a, b$ with $ab$ prime to corresponding primes $p_1, \cdots, p_r$ in Definition 3, the same conclusion is true when one replaces odd integers by the integers prime to $2ab$ and one replaces $2^n \pm q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ by $a2^n \pm b q_1^{\alpha_1} \cdots q_r^{\alpha_r}$.

*Remark 2.* I believe that for every $r \geq 1$ there exists a (2, 1)-primitive $r$-covering system.

## 2. Proofs of the Theorem and the Corollary

For the case with minus we need a result of A. Baker.

Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers. Let $K$ be their splitting field over $\mathbf{Q}$, and let $D = [K : \mathbf{Q}]$. We denote $A_1, \cdots, A_n$ upper bounds for the heights of $\alpha_1, \cdots, \alpha_n$ respectively, and assume that $A_j \geq 2$ for $1 \leq j \leq n$. Let

$$\omega' = \prod_{j=1}^{n-1} \log A_j, \quad \omega = \omega' \log A_n.$$

**Lemma 1** (Baker [1])**.** *There exist effectively computable absolute constants $c_1 > 0$ and $c_2 > 0$ such that the inequalities*

$$0 < |\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| < \exp\big( - (c_1 n D)^{c_2 n} \omega \log \omega' \log B\big)$$

*have no solutions in rational integers $b_1, \cdots, b_n$ with absolute values at most $B \geq 2$.*

As a result of Lemma 1 we have

**Lemma 2.** *Let $p_1, \cdots, p_r$ be given positive primes. Then the Diophantine inequality*

$$|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| \leq x$$

*has at most $(2 \log x)^{r+1}$ nonnegative integral solutions $n, \alpha_1, \cdots, \alpha_r$ for $x \geq c$, where $c$ is a constant depending only on $p_1, \cdots, p_r$.*

*Proof.* Suppose that $n, \alpha_1, \cdots, \alpha_r$ are not all zero. Let

$$\delta = (c_1(r+1))^{c_2(r+1)} \prod_{i=1}^{r} \log p_i \cdot \log 2 \cdot \log \prod_{i=1}^{r} \log p_i.$$

By Lemma 1 we have

$$\begin{aligned}|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| &= 2^n |1 - 2^{-n} p_1^{\alpha_1} \cdots p_r^{\alpha_r}| \\ &\geq \frac{2^n}{(2 \max\{n, \alpha_1, \cdots, \alpha_r\})^\delta}\end{aligned}$$

and

$$\begin{aligned}|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| &= p_1^{\alpha_1} \cdots p_r^{\alpha_r} |2^n p_1^{-\alpha_1} \cdots p_r^{-\alpha_r} - 1| \\ &\geq \frac{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}{(2 \max\{n, \alpha_1, \cdots, \alpha_r\})^\delta}.\end{aligned}$$

Thus

$$\begin{aligned}|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| &= 2^n |1 - 2^{-n} p_1^{\alpha_1} \cdots p_r^{\alpha_r}| \\ &\geq \frac{2^{\max\{n, \alpha_1, \cdots, \alpha_r\}}}{(2 \max\{n, \alpha_1, \cdots, \alpha_r\})^\delta}.\end{aligned}$$

Then Lemma 2 follows immediately. □

*Proof of the Theorem.* Suppose that $\{a_i(n_i)\}_{i=1}^t$ is a $(2,1)$-primitive $r$-covering system and $p_1, \cdots, p_t$ are corresponding primes in Definition 3. Take an integer $M$ satisfying

$$M \equiv 2^{a_i} \pmod{p_i}, \quad i = 1, \cdots, t. \tag{1}$$

For any positive integer $n$ there exist $i_1, \cdots, i_r$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq t$ and

$$n \in a_{i_j}(n_{i_j}), \quad j = 1, 2, \cdots, r.$$

Then by (1) and

$$2^{n_{i_j}} \equiv 1 \pmod{p_{i_j}}, \quad j = 1, \cdots, r,$$

we have

$$M \equiv 2^n \pmod{p_{i_j}}, \quad j = 1, \cdots, r.$$

Thus

$$M = 2^n + p_{i_1}^{\alpha_{i_1}} \cdots p_{i_r}^{\alpha_{i_r}} b, \quad \alpha_{i_j} > 0 \ (j = 1, 2, \cdots r), b \in \mathbb{Z}.$$

Hence , if $M$ has the form

$$M = 2^n \pm q_1^{\beta_1} \cdots q_r^{\beta_r},$$

where $q_1, \cdots, q_r$ are primes, then $q_i \in \{p_1, \cdots, p_t\}$ $(i = 1, 2, \cdots r)$. It is clear that the number of integers $M$ with

$$M = 2^n + q_1^{\beta_1} \cdots q_r^{\beta_r} \leq x, \quad q_i \in \{p_1, \cdots, p_t\}, i = 1, 2, \cdots, r,$$

is less than $c_t^r (2 \log x)^{r+1}$ for $x \geq X_1$. By Lemma 2 the number of positive integers $M$ with

$$M = 2^n - q_1^{\beta_1} \cdots q_r^{\beta_r} \leq x, \quad q_i \in \{p_1, \cdots, p_t\}, i = 1, 2, \cdots, r,$$

is less than $c_t^r (2 \log x)^{r+1}$ for $x \geq X_2$. It is well known that the number of positive odd integers $M$ with (1) and $M \leq x$ is more than

$$\frac{x}{2 p_1 \cdots p_t} - 1, \quad x \geq X_3.$$

Therefore , for $x \geq \max\{X_1, X_2, X_3\}$ , there exist at least $\frac{x}{2p_1\cdots p_t} - 1 - 2c_t^r(2\log x)^{r+1}$ positive odd numbers $M \leq x$ which have neither the form $2^n + q_1^{\beta_1}\cdots q_r^{\beta_r}$ nor the form $2^n - q_1^{\beta_1}\cdots q_r^{\beta_r}$, where $q_1, \cdots, q_r$ are primes. This completes the proof of the Theorem. □

*Proof of the Corollary.* Since

$$A = \{0(2), 3(4), 5(8), 9(16), 17(32), 33(64), 1(64)\}$$

and

$$\begin{aligned} B =\{&0(3), 4(9), 2(12), 10(18), 8(24), 16(36), 34(36), 20(48), 44(48),\\ &1(5), 5(10), 3(15), 7(20), 4(25), 14(25), 13(30), 17(40), 9(50),\\ &19(50), 23(60), 53(60), 37(80), 49(100), 77(120), 99(150), 157(160),\\ &199(200), 199(300), 237(480), 157(240), 299(600)\}\end{aligned}$$

are both 1-covering systems, $A \cup B$ is a 2-covering system. By Birkhoff and Vandiver [3] (or Bang [2], Zsigmondy [9]), for $n \geq 2, n \neq 6$, there exist (2, 1)-primitive prime divisors of order $n$. Again, 641, 6700417 are (2, 1)-primitive prime divisors of order 64; write this as $64 \leftrightarrow 641, 6700417$. Similarly, $36 \leftrightarrow 37, 109$; $48 \leftrightarrow 97, 673$; $25 \leftrightarrow 601, 1801$; $50 \leftrightarrow 251, 4051$; $60 \leftrightarrow 61, 1321$. Thus $A \cup B$ is a (2, 1)-primitive 2-covering system. Now the corollary follows from the theorem. □

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