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UNIQUENESS OF THE LEAST-ENERGY SOLUTION FOR A SEMILINEAR NEUMANN PROBLEM

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Abstract. We prove that the least-energy solution of the problem

$$\begin{cases} -d\Delta u + u = u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \end{cases}$$

where B is a ball, $d > 0$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$, is unique (up to rotation) if d is small enough.

Introduction

In this paper we consider the following problem:

$$\begin{cases} -d\Delta u + u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, ν is the unit outer normal to $\partial\Omega$, $d > 0$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$.

To our knowledge the first existence result to (0.1) is due to Lin, Ni and Takagi (see [LNT]). In this paper the authors consider the functional

$$J_d(u) = \frac{1}{2}\int_\Omega (d|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega (u^+)^{p+1} \tag{0.2}$$

and

$$c_d = \inf_{h\in\Gamma} \max_{t\in[0,1]} J_d(h(t)) \tag{0.3}$$

where $\Gamma = \{f \in C\big([0,1], H^1(\Omega)\big) | f(0) = 0,\ f(1) = e\}$ and $e \neq 0$ is a non-negative function of $H^1(\Omega)$ such that $J_d(e) = 0$. From the mountain-pass theorem (see [AR]) we have that c_d is a positive critical value of J_d. Moreover, it turns out that c_d is the least positive critical value for J_d (see [LNT] or [NT]).

Therefore, we define a critical point u_d of J_d satisfying $J_d(u_d) = c_d$ a *least-energy solution* of (0.1).

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Clearly (0.1) admits the trivial solutions $u \equiv 1$ and $u \equiv 0$. In [LNT] it was proved that $c_d = O(d^{N/2})$ as $d \to 0$ and from this $0 < J_d(u_d) < J_d(1)$, i.e. u_d is a nonconstant solution provided d is small enough.

Another important result concerns the shape of the least-energy solution u_d. In [NT] it was shown that u_d has only one local maximum in $\overline{\Omega}$, and it is achieved at exactly one point which lies on the boundary of Ω, provided d is small enough.

In this paper we will prove the uniqueness of the least-energy solution if Ω is a ball. Of course, in this case, if $u(x)$ is a solution of (0.1), then also $u(Tx)$ solves (0.1) for any T belonging to the orthogonal group $O(N)$; therefore by uniqueness we mean that any two solutions can be obtained from each other using the action of $O(N)$.

Theorem 1. *Let us consider the problem* (0.1) *when* Ω *is a ball. Then there exists a* $d_0 > 0$ *such that for any* $d < d_0$ *the least-energy solution of* (0.1) *is unique.*

We want to point out that uniqueness results for (0.1) cannot be obtained if we do not restrict the class of the solutions of (0.1). Indeed, in [DY] Dancer and Yan prove the existence of k-peak solutions to (0.1) (solutions with more than one local maximum point). Clearly the energy of these solutions is strictly greater than c_d. We mention here the papers of [BDS], [Gu], [W1], and [W2] for the existence of multipeak solutions in other domains.

The paper is organized as follows: in Section 1 we recall some important results which we use in the proof of Theorem 1. In Section 2 we prove Theorem 1.

1. Known results

In this section we recall some known facts which will be useful in the proof of Theorem 1. Here Ω is an arbitrary bounded smooth domain of $\mathbb{R}^N$ and p is a real number such that $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $p > 1$ if $N = 2$. We start by recalling some results on the existence and the shape of the least-energy solution u_d.

Theorem 1.1. *Let* $p > 1$ *if* $N = 1$ *or* 2 *and let* $1 < p < \frac{N+2}{N-2}$ *if* $N \geq 3$. *Then there exists a* $d_0 > 0$ *such that, for any* $d < d_0$, (0.1) *has a positive nonconstant solution* u_d *satisfying*

$$J_d(u_d) = c_d, \tag{1.1}$$

$$\int_\Omega u_d^q \leq C_1(q) d^{N/2} \quad \text{for any } q \geq 1, \tag{1.2}$$

and

$$\sup_\Omega u_d(x) \leq C_2 \tag{1.3}$$

with $C_1(q), C_2$ *independent of* d.

Proof. See [LNT]. □

Theorem 1.2. *Let* u_d *be a least-energy solution to* (1.3). *Then* u_d *has at most one local maximum in* Ω *and it is achieved exactly at one point which must lie on the boundary, provided* d *is sufficiently small.*

Proof. See [NT], Theorem 2.1. □

Theorem 1.3. *There is exactly one solution (up to translation) of the problem*

$$(1.4) \qquad \begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Proof. See [K]. □

Next let us introduce a diffeomorphism which straightens a boundary portion of Ω as follows. We denote by B_R the ball of $\mathbb{R}^N$ centered at the origin and radius R. Through translation and rotation of the coordinate system we may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. Then there exists a smooth function $\psi(x'), x' = (x_1, \ldots, x_{N-1})$, defined for $|x| < \delta_0$ such that $\psi(0) = 0$, $\nabla\psi(0) = 0$ and

$$\partial\Omega \cap \mathcal{N} = \{x = (x', x_N) \mid x_N = \psi(x')\}$$

and

$$\Omega \cap \mathcal{N} = \{x = (x', x_N) \mid x_N > \psi(x')\}.$$

For $|y| < \delta'$ let us define $x = \Phi(y)$, with $\Phi(y) = (\Phi_1(y), \ldots, \Phi_N(y))$, by

$$(1.5) \qquad \Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial x_j}(y') & \text{for } j = 1, \ldots, N-1, \\ y_N + \psi(y') & \text{if } j = N. \end{cases}$$

Since $\nabla\psi(0) = 0$ the differential map $D\Phi$ of Φ satisfies $D\Phi(0) = I$, the identity map. Thus there exists the inverse map $y = \Phi^{-1}(x)$ between $C \subset \mathbb{R}^N$ and $B(0, \delta)$ with $\delta < \delta'$.

Theorem 1.4. *Let u_d be a least-energy solution to* (0.1) *and let us denote by P_d the point where the maximum of u_d is achieved. Then for any $\epsilon > 0$ there exist a positive constant d_0 and a subdomain $\Omega_d^{(i)} \subset \Omega$ such that for $d \in]0, d_0[$ the following statements hold:*

$$(1.6) \qquad P_d \in \partial\Omega_d \quad \text{and} \quad \operatorname{diam}(\Omega_d^{(i)}) \le C\sqrt{d},$$

$$(1.7) \qquad u_d(\Phi(\sqrt{d}y)) \to U(y) \text{ in } C^2_{\text{loc}}(\mathbb{R}^N_+) \text{ as } d \to 0,$$

$$(1.8) \qquad |u_d(x)| \le C_1 \epsilon e^{-\frac{\mu_1 \delta(x)}{\sqrt{d}}} \quad \text{for} \quad x \in \Omega \setminus \Omega_d^{(i)},$$

where $\delta(x) = \min\{\operatorname{dist}(x, \partial\Omega_d), \eta_0\}$, C, C_1, μ_1, η_0 are positive constants depending only on Ω and $U(x)$ is the unique solution of (1.4) *with maximum achieved at $P_0 = \lim_{d\to 0} P_d$.*

Proof. See [NT], (i) and (iii) of Theorem 2.3 and (4.29). □

Theorem 1.5. *Let us consider the problem*

$$(1.9) \qquad \begin{cases} -\Delta z + z = pU(x)^{p-1} z & \text{in } \mathbb{R}^N, \\ ||z||_{L^\infty(\mathbb{R}^N)} \le C, \end{cases}$$

where $U(x)$ is the unique solution of (1.4) *symmetric with respect to the origin. Then the only nontrivial solutions of* (1.9) *are given by*

$$(1.10) \qquad z = \sum_{i=1}^{N} \alpha_i \frac{\partial U}{\partial x_i}, \qquad \alpha_i \in \mathbb{R}.$$

Proof. This result is contained in the proof of Proposition 1 of [D]. □

2. The uniqueness result

In this section we denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ a generic point of $\mathbb{R}^N$. Let us assume that $\Omega = B = \{x \in \mathbb{R}^N \mid |x'|^2 + (x_N - 1)^2 < 1\}$. From Theorem 1.2 the maximum of a least-energy solution u of (0.1) is achieved on the boundary of B if d is small enough. So, since the problem (0.1) is invariant with respect to the orthogonal group of $\mathbb{R}^N$, by a suitable rotation we can assume that u attains its maximum at the origin. Hence $\nabla u(0) = 0$ for any least-energy solution of (0.1).

We suppose by contradiction that there exist a sequence $d_n \searrow 0$ and two distinct least-energy solutions $u_{1,n}$ and $u_{2,n}$ which solve (0.1) with $d = d_n$. So we can suppose that $u_{1,n}$ and $u_{2,n}$ achieve their maximum at the origin for any $n \in \mathbb{N}$. Since $u_{1,n} - u_{2,n} \not\equiv 0$ we can consider

$$z_n(x) = \frac{u_{1,n}(x) - u_{2,n}(x)}{||u_{1,n} - u_{2,n}||_{L^\infty}}. \tag{2.1}$$

By assumption on $u_{1,n}$ and $u_{2,n}$ we have

$$\nabla z_n(0) = 0. \tag{2.2}$$

Note that z_n satisfies

$$\begin{cases} -d_n \Delta z_n + z_n = c_n(x) z_n & \text{in } B, \\ \frac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial B, \\ ||z_n||_{L^\infty(B)} = 1, \end{cases} \tag{2.3}$$

where

$$c_n(x) = p \int_0^1 (t u_{1,n}(x) + (1-t) u_{2,n}(x))^{p-1} dt. \tag{2.4}$$

Let us set

$$\widetilde{z}_n(x) = z_n(\sqrt{d_n} x), \quad \widetilde{z}_n : B/\sqrt{d_n} \mapsto \mathbb{R}. \tag{2.5}$$

We remark that for any compact set $K \subset \mathbb{R}^N_+$, where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x_N > 0\}$, we have that $K \subset B/\sqrt{d_n}$ for n large enough. In the following proposition we study the asymptotic behavior of $\widetilde{z}_n$.

Proposition 2.1. *We have that*

$$\widetilde{z}_n \to z \quad \textit{in } C^1_{\mathrm{loc}}(\mathbb{R}^N_+), \tag{2.6}$$

where z is a bounded solution of

$$-\Delta z + z = p U^{p-1} z \quad \textit{in } \mathbb{R}^N$$

and U is the unique solution of (1.4) *symmetric with respect to the origin.*

Proof. Let us consider the diffeomorphism Φ between $B(0, \delta)$ and $C \subset \mathbb{R}^N$ which straightens the boundary of B (see (1.5)). Now, as in [NT], let us set

$$v_n(y) = z_n(\Phi(y)), \qquad y \in \overline{B^+_{2k}}, \tag{2.7}$$

where $B^+_{2k} = \{y \in B_{2k} \mid y_N > 0\}$.

Step 1. In this step we prove that

$$v_n(\sqrt{d_n} y) \to z(y) \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N_+). \tag{2.8}$$

First of all we extend v_n to $B(0, 2k)$ by reflection,

$$\widetilde{v}_n(y) = \begin{cases} v_n(y) & \text{if } y \in \overline{B^+_{2k}}, \\ v_n(y', -y_N) & \text{if } y \in \overline{B^-_{2k}}, \end{cases}$$

with $\overline{B^-_{2k}} = \{y \in B_{2k} \mid y_N < 0\}$. Moreover we define a scaled function $w_n(z)$ by

$$w_n(y) = \widetilde{v}_n(\sqrt{d_n} y) \quad \text{for } y \in \overline{B}_{\frac{k}{\sqrt{d_n}}}. \tag{2.9}$$

It is easily seen that

$$w_n \in C^2\big(\overline{B}_{\frac{k}{\sqrt{d_n}}} \setminus \{y_N = 0\}\big) \cap C^1(\overline{B}_{\frac{k}{\sqrt{d_n}}})$$

since $\frac{\partial v_n}{\partial y_N}$ on $y_N = 0$ and that w_n satisfies the elliptic equation

$$-L_n w_n + w_n = \widetilde{c}_n(y) w_n \tag{2.10}$$

with

$$L_n w_n = \sum_{i,j=1}^N a^n_{ij}(y) \frac{\partial^2 w_n}{\partial z_i \partial z_j} + \sqrt{d_n} \sum_{j=1}^N b^n_j(y) \frac{\partial w_n}{\partial z_j} \tag{2.11}$$

in $B_{\frac{k}{\sqrt{d_n}}} \setminus \{y_N = 0\}$, where a^n_{ij}, b^n_j and $\widetilde{c}_n(y)$ are defined as follows. First, put

$$\Psi = \Phi^{-1},$$

$$a_{ij}(y) := \sum_{l=1}^N \frac{\partial \Psi_i}{\partial x_l}(\Phi(y)) \frac{\partial \Phi_j}{\partial x_l}(\Phi(y)), \qquad 1 \le i, j \le N,$$

$$b_j(y) := (\Delta \Psi_j)(\Phi(y)), \qquad j = 1, \dots, N,$$

$$\hat{c}_n(y) = c_n(\Phi(y)).$$

Then set

$$a^n_{ij}(y) := \begin{cases} a_{ij}(\sqrt{d_n} y) & \text{if } y_N \ge 0, \\ (-1)^{\delta^i_N + \delta^j_N} a_{ij}(\sqrt{d_n} y', -\sqrt{d_n} y_N) & \text{if } y_N < 0, \end{cases}$$

$$b^n_j(y) := \begin{cases} b_j(\sqrt{d_n} y) & \text{if } y_N \ge 0, \\ (-1)^{\delta^j_N} b_j(\sqrt{d_n} y', -\sqrt{d_n} y_N) & \text{if } y_N < 0, \end{cases}$$

$$\widetilde{c}_n(y) := \begin{cases} \hat{c}_n(\sqrt{d_n} y) & \text{if } y_N \ge 0, \\ (-1)^{\delta^j_N} \hat{c}_n(\sqrt{d_n} y', -\sqrt{d_n} y_N) & \text{if } y_N < 0, \end{cases}$$

where δ^i_j is the Kroneker symbol.

Note that by (1.3) the functions w_n and c_n are uniformly bounded with respect to n. Then, by using the standard L^r-estimate and the interior Schauder estimate as in [NT], pp. 834-836, we obtain that

$$w_n \to z \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N) \tag{2.12}$$

and

$$-L_n w_n \to -\Delta z. \tag{2.13}$$

Hence, since by (1.7) and (2.4) we get $\widetilde{c}_n \to p U^{p-1}$, z is a solution of

$$-\Delta z + z = p U^{p-1} z \quad \text{in } \mathbb{R}^N. \tag{2.14}$$

By the definition of w_n and $\widetilde{v}_n$ (2.8) follows.

Step 2. In this step we prove the claim of the proposition.

The function $\widetilde{z}_n$ satisfies the equation

$$\begin{cases} -\Delta\widetilde{z}_n + \widetilde{z}_n = c_n(\sqrt{d_n})\widetilde{z}_n & \text{in } B/\sqrt{d_n}, \\ \widetilde{z}_n > 0 & \text{in } B/\sqrt{d_n}, \\ \frac{\partial\widetilde{z}_n}{\partial\nu} = 0 & \text{on } \partial B/\sqrt{d_n}. \end{cases} \tag{2.15}$$

From (1.3) we get $||c_n\widetilde{z}_n||_\infty \le C$ and then by standard L^p-estimates we deduce, for a subsequence, $||\widetilde{z}_n||_{C^1(K)} \le C$ for any compact set $K \subset \mathbb{R}^N$.

By Step 1 we know that $||z_{d_n}(\Phi(\sqrt{d_n}y)) - U(y)||_{C^0_{\text{loc}}(\mathbb{R}^N_+)} \to 0$. So if we show that

$$||z_{d_n}(\Phi(\sqrt{d_n}y)) - \widetilde{u}_n(y)||_{C^0_{\text{loc}}(\mathbb{R}^N_+)} \to 0, \tag{2.16}$$

the claim follows.

Let us fix a compact set $K \subset \mathbb{R}^N$. By the definition of Φ (see (1.5)) we get

$$|\Phi(\sqrt{d_n}y) - \sqrt{d_n}y|^2 = |d_n y_N^2|\nabla_{y'}\psi(\sqrt{d_n}y)|^2 + \psi^2(\sqrt{d_n}y)| \to \psi^2(0) = 0 \quad \text{uniformly in } K. \tag{2.17}$$

Then, since $|\nabla u_{d_n}| \le C$ in K, we obtain that

$$||z_{d_n}(\Phi(\sqrt{d_n}y)) - \widetilde{z}_n(y)||_{C^0(K)} \le C||\Phi(\sqrt{d_n}y) - \sqrt{d_n}y||_{C^0(K)} \to 0, \tag{2.18}$$

and this finishes the proof.

□

Proof of Theorem 1. Let us again consider the functions z_n and $\widetilde{z}_n$ defined by (2.1) and (2.5). First of all we remark that since $u_{1,n}$ and $u_{2,n}$ solve (0.1) we have

$$\int_\Omega u_{1,n}u_{2,n}(u_{1,n}^{p-1} - u_{2,n}^{p-1}) = 0$$

and then $\widetilde{z}_n$ does change sign.

Since we have that $||\widetilde{z}_n||_{L^\infty(B/\sqrt{d_n})} = 1$ we can assume that there exists a sequence of points $\widetilde{x}_n \in B/\sqrt{d_n}$ such that $\widetilde{z}_n(\widetilde{x}_n) = \max_{x \in B/\sqrt{d_n}} \widetilde{z}_n(x) = 1$. Moreover we have that

$$\nabla\widetilde{z}_n(\widetilde{x}_n) = 0 \quad \text{and} \quad \Delta\widetilde{z}_n(\widetilde{x}_n) \le 0 \tag{2.19}$$

(this is obvious if $\widetilde{x}_n \in B/\sqrt{d_n}$ and it follows by $\frac{\partial\widetilde{z}_n}{\partial\nu}(\widetilde{x}_n) = 0$ if $\widetilde{x}_n \in \frac{\partial B}{\sqrt{d_n}}$).

From Proposition 2.1 the following alternative occurs: either

$$\widetilde{z}_n \to 0 \tag{2.20}$$

or

$$\widetilde{z}_n \to \sum_{i=1}^N \alpha_i \frac{\partial U}{\partial x_i} \qquad \text{for some } (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \setminus \{0\}. \tag{2.21}$$

We will prove that in any case we reach a contradiction.

Case 1. $|\widetilde{z}_n| \to z \equiv 0$.

Here we have the following alternative:

i) $|\widetilde{x}_n| \le C$,

ii) $|\widetilde{x}_n| \to \infty$.

Case i). $|\widetilde{x}_n| \le C$.

Let $x_0 = \lim_n \widetilde{x}_n$. If $x_0 \notin \partial\mathbb{R}^N_+$, the claim follows easily. If $x_0 \in \partial\mathbb{R}^N_+$, we again consider the diffeomorphism Φ between $B(0,\delta')$ and $B(0,\delta)$ which straightens the boundary. Since $\widetilde{x}_n \in B/\sqrt{d_n}$ we have that $\widetilde{x}_n = \frac{x_n}{\sqrt{d_n}}$ with $x_n \in B$ and $x_n \to 0$. Then there exists $P_n \in B^+_\delta$ such that

$$\Phi(P_n) = x_n. \tag{2.22}$$

Set $y_n = \frac{P_n}{\sqrt{d_n}}$. Then, since $\Phi(0) = 0$ and Φ is a diffeomorphism in B^+_δ,

$$|\sqrt{d_n} y_n| = |P_n| = |\Phi^{-1}(x_n)| = |\Phi^{-1}(\sqrt{d_n}\widetilde{x}_n)| \le C|\sqrt{d_n}\widetilde{x}_n|, \tag{2.23}$$

so $|y_n| \le C$. Finally, by (2.7)-(2.9)

$$1 = \widetilde{z}_n(\widetilde{x}_n) = z_n(x_n) = z_n(\Phi(P_n)) = z_n(\Phi(\sqrt{d_n} y_n)) \to 0 \tag{2.24}$$

and this gives a contradiction.

Case ii). $|\widetilde{x}_n| \to \infty$.

Let us prove that $c_n(\sqrt{d_n}\widetilde{x}_n) \to 0$. We have

$$c_n(\sqrt{d_n}\widetilde{x}_n) = p\int_0^1 (tu_{1,n}(\sqrt{d_n}\widetilde{x}_n) + (1-t)u_{2,n}(\sqrt{d_n}\widetilde{x}_n))^{p-1} dt.$$

Again since $\widetilde{x}_n \in B/\sqrt{d_n}$ there exists $x_n \in B$ such that $x_n = \sqrt{d_n}\widetilde{x}_n$ and

$$\frac{|x_n|}{\sqrt{d_n}} \to \infty. \tag{2.25}$$

Then (1.6) and (1.7) of Theorem 1.4 applies: since by (2.25) $x_n \notin B^{(i)}_{d_n}$ we obtain, for any $\epsilon > 0$,

$$u_{1,n}(x_n) < \epsilon \text{ and } u_{2,n}(x_n) < \epsilon \text{ for } n \text{ large}. \tag{2.26}$$

Hence $c_n(\sqrt{d_n}\widetilde{x}_n) = c_n(x_n) \to 0$. Finally from (2.19) we obtain

$$0 \le -\Delta\widetilde{z_n}(\widetilde{x}_n) = c_n(x_n) - 1 \to -1, \tag{2.27}$$

and this gives a contradiction.

Case 2. $|\widetilde{z}_n| \to z = \sum_{i=1}^N \alpha_i \frac{\partial U}{\partial x_i}$ for some $(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \setminus \{0\}$.

First of all let us prove that

$$\frac{\partial^2 U}{\partial x_j^2}(0) < 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_i \partial x_j}(0) = 0 \text{ if } i \ne j,\ i,j \in \{1,\dots,N\}. \tag{2.28}$$

This is a well-known fact (see [NT] for example), but we repeat the proof for the reader's convenience. Since the function $U(\rho)$ is radially decreasing for $\rho > 0$ (see [GNN]), we have that $\frac{\partial U}{\partial x_j} = U'(\rho)\frac{x_j}{\rho} > 0$ in $T_j = \{x \in \mathbb{R}^N \mid x_j < 0\}$. Hence $\frac{\partial U}{\partial x_j}\big|_{\partial T_j} \equiv 0$ and then $\frac{\partial^2 U}{\partial x_i \partial x_j}(0) = 0$ for any $i \ne j$. Moreover $f = \frac{\partial U}{\partial x_j}$ solves

$$\begin{cases} -\Delta f + f = pU(x)^{p-1} f & \text{in } T_j, \\ f > 0 & \text{in } T_j, \\ f = \frac{\partial U}{\partial x_j} = 0 & \text{on } \partial T_j. \end{cases}$$

So, by Hopf's Lemma, $\frac{\partial f}{\partial x_j} = \frac{\partial^2 U}{\partial x_j^2}(0) < 0$ on ∂T_j, and then (2.28) is proved.

Now let us recall that $\nabla \widetilde{z}_n(0) = 0$. Since $\Phi(0) = 0$ we have

$$\nabla(z_n \circ \Phi)(0) = 0. \tag{2.29}$$

Then since $(\alpha_1, \ldots, \alpha_N) \neq (0, \ldots, 0)$ there exists $\alpha_j \neq 0$. Finally, from (2.8), (2.28) and (2.29) we get

$$0 = \frac{\partial(z_n \circ \Phi)}{\partial x_j}(0) \to \sum_{i=1}^{N} \alpha_i \frac{\partial^2 U}{\partial x_i \partial x_j}(0) = \alpha_j \frac{\partial^2 U}{\partial x_j^2} \neq 0, \tag{2.30}$$

a contradiction. The proof of Theorem 1 is finished.

References

[AR] A. Ambrosetti and P. H. Rabinowitz, "*Dual variational methods in critical point theory and applications*", J Funct. Anal. **14**, 349-381, (1973). MR **51:**6412

[BDS] P. Bates, E.N. Dancer and J. Shi, "*Multi-spike stationary solutions on the Cahn-Hilliard equation in higher dimension and instability*", preprint.

[D] E.N. Dancer, "*On the uniqueness of the positive solution of a singularly perturbed problem*", Rocky Mountain Journal of Mathematics **25** (1995), 957–975. MR **96j:**35021

[DY] E.N. Dancer and S. Yan, "*Multipeak solutions for a singularly perturbed Neumann problem*" (to appear).

[GNN] B. Gidas, W.N. Ni and L. Nirenberg, "*Symmetry of positive solutions of nonlinear elliptic equation in* $\mathbb{R}^N$", Advances in Math. Studies **7 A**, 369-402, (1981). MR **84a:**35083

[GT] D. Gilbarg and N. Trudinger, "*Elliptic partial differential equations of second order*", Springer Verlag (1983). MR **86c:**35035

[Gu] C.Gui, "*Multipeak solutions for a semilinear Neumann problem*", Duke Math. J. **84**, 739-769, (1996). MR **97i:**35052

[K] M.K. Kwong, "*Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$", Arch. Rat. Mech. Anal., **105**, 243-266, (1989). MR **90d:**35015

[LN] C.S. Lin and W.M. Ni, "*On the diffusion coefficient of a semilinear Neumann problem, Calculus of Variations and Partial Differential Equations*", S. Hildebrandt, D. Kinderlehrer and M. Miranda, eds., Lecture Notes in Math. 1340, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 160-174, (1988). MR **90d:**35101

[LNT] C.S. Lin, W.M. Ni and I. Takagi, "*Large amplitude stationary solutions to a chemotaxis system*", J. Diff. Eqns. **72**, 1-27, (1988). MR **89e:**35075

[NT] W.M. Ni and I. Takagi, "*On the shape of Least Energy Solutions to a Semilinear Neumann Problem*", Comm. Pure Math. Appl., **Vol XLIV**, 819-851, (1991). MR **92i:**35052

[W1] Z.Q. Wang, "*On the existence of multiple single-peaked solution for a semilinear Neumann problem*", Arch. Rat. Mech. Anal., **120**, 375-399, (1992). MR **93k:**35109

[W2] Z.Q. Wang, "*Nonradial solutions of nonlinear Neumann problems in radially symmetric domains*", Topology in Nonlinear analysis (Warsaw, 1994), 85-96, Banach Center Publications, **35**, Polish Acad. Sci., Warsaw, (1996). MR **98e:**35075

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