

EXISTENCE OF SOLUTIONS FOR FIRST ORDER SINGULAR PROBLEMS

M. CHERPION AND C. DE COSTER

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ABSTRACT. We develop the lower and upper solutions method for first order initial value problems as well as for first order periodic problems in case the nonlinearity presents singularities. More precisely we prove that if we have a lower solution α and an upper solution β of these problems, which are not necessarily continuous nor ordered, we have a solution wedged between $\min\{\alpha, \beta\}$ and $\max\{\alpha, \beta\}$.

1. INTRODUCTION

Let us consider the problem

$$(1) \quad u' = f(t, u) \text{ on }]0, T], \quad u(0) = 0,$$

where $f :]0, T] \times]0, +\infty[\rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, singular in $t = 0$ and in $u = 0$. Due to the singularity, we are not even sure of the local existence of a solution of (1). In this work we develop the lower and upper solutions method for (1).

To our knowledge, the first works devoted to lower and upper solutions for (1) are the papers of G. Peano [20] in 1885 and O. Perron [21] in 1915. They consider the problem (1) with f continuous, assume the existence of continuous functions α and β with $\alpha(0) = 0 = \beta(0)$, $\alpha \leq \beta$ and

$$D_{\pm}\alpha(t) \leq f(t, \alpha(t)), \quad D_{\pm}\beta(t) \geq f(t, \beta(t)),$$

and prove the existence of a minimal and a maximal solution of (1) between α and β . The formulation of G. Peano is a little different but the idea is the same. In 1931, G. Scorza Dragoni [23] considers systems and allows $\alpha(0) \leq 0 \leq \beta(0)$. The extension to the Carathéodory case of Scorza Dragoni's result concerning one equation is due to F. Cafiero [3] in 1947. Recently, M. Frigon and D. O'Regan [6] have considered noncontinuous α and β and C. Marcelli and P. Rubbioni [15] have studied the situation where α and β can cross.

For the periodic problem

$$(2) \quad u' = f(t, u) \text{ on } [0, T], \quad u(0) = u(T),$$

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we have already found the lower and upper solutions approach in the book of F. Tricomi [24] in 1953, where we also find an application of this approach to a singular problem. In 1958, S. Moretto [18] extends Tricomi's result to a situation where f is a Carathéodory function which satisfies a L^1 -Lipschitz condition and α , β are supposed to be well ordered T -periodic lower and upper solutions. This is extended to systems with f continuous and locally Lipschitz in u by H.W. Knobloch [12] and for f continuous by J. Mawhin [16, 17]. The Carathéodory case without Lipschitz condition was considered by A. Adje [1], A. Cabada [2] and M. Nkashama [19]. In [4], C. De Coster considered the Carathéodory case with α and β not necessarily continuous.

Except for the book of Tricomi, we have already found the lower and upper solutions method for singular problems in the paper of G. Prodi [22] in 1953 devoted to parabolic equations. More recently, an extensive literature appears on second order O.D.E. with different kind of boundary conditions; see for example C. De Coster, M. Grossinho and P. Habets [5], P. Habets and L. Sanchez [7], P. Habets and F. Zanolin [8, 9], I.T. Kiguradze and B.L. Shekhter [11], T.C. Lee and D. Willett [13], A.G. Lomtadze [14] and the references therein, but much less is known for problem (1).

In this work we develop the lower and upper solutions method for (1) and for (2), i.e. we prove that if we have a lower solution α and an upper solution β of (1) (resp. (2)) which are not necessarily continuous nor ordered, we have a solution of (1) (resp. (2)) between $\min\{\alpha, \beta\}$ and $\max\{\alpha, \beta\}$. The results are illustrated by several examples.

2. THE LOWER AND UPPER SOLUTIONS METHOD

In this section we consider the problem

$$(3) \quad u' = f(t, u) \text{ on }]0, T], \quad u(0) = 0.$$

Basic assumptions on f are described in the following definitions.

Definition 2.1. A function $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ is a *Carathéodory function* on $D \subset]0, T] \times]0, \infty[$ if

(i) for a.e. $t \in]0, T]$, the function $f(t, \cdot)$ with domain $\{u \in]0, \infty[: (t, u) \in D\}$ is continuous;

(ii) for all $u \in]0, \infty[$, the function $f(\cdot, u)$ with domain $\{t \in]0, T] : (t, u) \in D\}$ is measurable.

Definition 2.2. A function $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ is *L^1 -Carathéodory* on $D \subset]0, T] \times]0, \infty[$ if f is a Carathéodory function on D and

(iii) for all $R > 0$, there exists $h_R \in L^1(0, T)$ such that for a.e. $t \in]0, T]$, all $u \in [\frac{1}{R}, R]$ with $(t, u) \in D$ we have

$$|f(t, u)| \leq h_R(t).$$

In case $D =]0, T] \times]0, \infty[$, we just say that f is L^1 -Carathéodory.

We denote by $W^{1,1}(a, b)$ the set of continuous functions on $[a, b]$ with weak derivative in $L^1(a, b)$ and by $W_{loc}^{1,1}(a, b)$ the set of functions $u \in W^{1,1}(c, d)$ for some $a < c < d < b$.

Let us introduce next the concepts of lower and upper solutions.

Definition 2.3. A bounded function $\alpha :]0, T] \rightarrow]0, \infty[$ is a *lower solution* of (3) if

- (i) $\lim_{t \rightarrow 0} \alpha(t) = 0$;
- (ii) there exist $0 = t_1 < t_2 < \dots < t_n = T$ such that
 - (a) for all $i = 1, \dots, n - 1$, $\alpha \in W^{1,1}(t_i, t_{i+1})$ and for a.e. $t \in]t_i, t_{i+1}[$,

$$\alpha'(t) \leq f(t, \alpha(t));$$
 - (b) for all $i = 2, \dots, n - 1$,

$$\alpha(t_i) = \lim_{t \rightarrow t_i^-} \alpha(t) > \lim_{t \rightarrow t_i^+} \alpha(t).$$

Definition 2.4. A bounded function $\beta :]0, T] \rightarrow]0, \infty[$ is an *upper solution* of (3) if there exist $0 = s_1 < s_2 < \dots < s_m = T$ such that

- (a) for all $i = 1, \dots, m - 1$, $\beta \in W^{1,1}(s_i, s_{i+1})$ and for a.e. $t \in]s_i, s_{i+1}[$,

$$\beta'(t) \geq f(t, \beta(t));$$
- (b) for all $i = 2, \dots, m - 1$,

$$\beta(s_i) = \lim_{t \rightarrow s_i^-} \beta(t) < \lim_{t \rightarrow s_i^+} \beta(t).$$

Remark 2.1. (i) Observe that, as $\beta(t) > 0$ on $]0, T]$, $\lim_{t \rightarrow 0} \beta(t) \geq 0$.

- (ii) In particular, functions $\alpha, \beta \in W^{1,1}(]0, T],]0, +\infty[)$ such that $\alpha(0) = 0$ and

$$\alpha'(t) \leq f(t, \alpha(t)), \quad \beta'(t) \geq f(t, \beta(t)), \quad \text{a.e. on }]0, T],$$

are respectively lower and upper solutions of (3).

Our main result is the following one.

Theorem 2.1. *Let $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ and α and β be lower and upper solutions of (3). Assume f is L^1 -Carathéodory on*

$$D = \{(t, u) \in]0, T] \times]0, \infty[: \min(\alpha(t), \beta(t)) \leq u \leq \max(\alpha(t), \beta(t))\}.$$

Then the problem (3) has at least one solution $u \in \mathcal{C}([0, T]) \cap W_{loc}^{1,1}(0, T)$ such that for all $t \in [0, T]$,

$$\min(\alpha(t), \beta(t)) \leq u(t) \leq \max(\alpha(t), \beta(t)).$$

Before proving Theorem 2.1, let us make some comments.

Remark 2.2. Observe that we do not make any assumptions on the number of crossings of α and β .

Remark 2.3. If f is not L^1 -Carathéodory on D , there is no hope to have a solution as shown by the problem

$$(4) \quad u' = -\frac{1}{t}(u - 1) \text{ on }]0, T], \quad u(0) = 0.$$

This problem has no solution although $\alpha(t) = \frac{t}{2T}$ and $\beta(t) = 2$ are lower and upper solutions of (4).

Remark 2.4. Without additional assumptions, we cannot expect more regularity on the solutions (such as $u \in W^{1,1}(0, T)$). We just have to consider the problem

$$(5) \quad u' = f(t, u) \text{ on }]0, 1], \quad u(0) = 0,$$

with

$$\begin{aligned}
 f(t, u) &= 2t(2 + \sin \frac{1}{t^2}) - \left(\frac{2(2u - t^2)}{t^3(3 + 2 \sin \frac{1}{t^2})} \right) \cos \frac{1}{t^2}, & \text{if } u < t^2(2 + \sin \frac{1}{t^2}), \\
 &= 2t(2 + \sin \frac{1}{t^2}) - \left(\frac{2(4t^2 - u)}{t^3(2 - \sin \frac{1}{t^2})} \right) \cos \frac{1}{t^2}, & \text{if } u \geq t^2(2 + \sin \frac{1}{t^2}).
 \end{aligned}$$

It is easy to observe that $\alpha(t) = \frac{t^2}{2}$ is a lower solution, $\beta(t) = 4t^2$ is an upper solution and $u(t) = t^2(2 + \sin \frac{1}{t^2})$ if $t \neq 0$, $u(0) = 0$ is a solution of (5) with $u \in \mathcal{C}([0, 1]) \cap W_{loc}^{1,1}(0, 1)$ but $u \notin W^{1,1}(0, 1)$.

Remark 2.5. In case α and β cross each other, the situation is quite different according to f being continuous and $\alpha, \beta \in \mathcal{C}^1(]0, T])$ or f being just L^1 -Carathéodory and $\alpha, \beta \in W^{1,1}(0, T)$. In the continuous case, if α and β cross at t_0 with α lower than β on the left of t_0 and β lower than α on the right of t_0 , we deduce from the definition of lower and upper solutions that α and β are tangent at t_0 . This is different in the Carathéodory case as shown by the following example. The problem

$$u' = \frac{(\cos t - 2)\sqrt[3]{u - t}}{\sqrt[3]{\sin t}} + 2 \text{ on }]0, T], \quad u(0) = 0,$$

has $\alpha(t) = t$ as a lower solution and $\beta(t) = \sin t + t$ as an upper solution. Observe that α and β cross at $t = \pi$ as above but without being tangent. Moreover, if $T > 2\pi$, α and β have points where they cross one way and points where they cross the other way.

The next example shows the interest of allowing jumps in the lower and upper solutions.

Example 2.1. Consider the problem

$$(6) \quad \varepsilon u' + u^3 = f(t) \text{ on }]0, 2], \quad u(0) = 0,$$

where $\varepsilon > 0$ is a small parameter and

$$\begin{aligned}
 f(t) &= t^3, & \text{if } 0 < t \leq 1, \\
 &= 0, & \text{if } t > 1.
 \end{aligned}$$

We are interested in the existence of solution u_ε and the asymptotic behaviour of u_ε when ε tends to zero. It is easy to observe that

$$\begin{aligned}
 \alpha(t) &= \max\{0, t - \varepsilon^{1/3}\}, & \text{if } 0 < t \leq 1, \\
 &= 0, & \text{if } t > 1,
 \end{aligned}$$

is a lower solution of (6) and

$$\begin{aligned}
 \beta(t) &= t + k, & \text{if } 0 < t \leq 1, \\
 &= (1 + k) \left(\frac{\varepsilon}{2t - 2 + \varepsilon} \right)^{1/2}, & \text{if } t > 1,
 \end{aligned}$$

with $k = \frac{\sqrt{\varepsilon}}{2}(\sqrt{2 + \varepsilon} + \sqrt{\varepsilon})$ is an upper solution. Hence there is a solution $u_\varepsilon \in [\alpha, \beta]$ from which we deduce the pointwise limit

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} u_\varepsilon(t) &= t, & \text{if } 0 < t \leq 1, \\
 &= 0, & \text{if } t > 1.
 \end{aligned}$$

Moreover, as the nonlinearity is decreasing in u , it can be proved that the solution u_ε is unique. This is the goal of the next result.

Theorem 2.2. Let $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ be L^1 -Carathéodory and assume there exists $L \in L^1(0, T)$ such that, for a.e. $t \in]0, T]$ and all $u, v \in]0, +\infty[$,

$$(u - v)(f(t, u) - f(t, v)) \leq L(t)(u - v)^2.$$

Then the problem (3) has at most one solution.

Proof. Let u and v be two solutions of (3). Define $w(t) = \frac{1}{2}(u(t) - v(t))^2$. We have that

$$\begin{aligned} \frac{d}{dt}(w(t) \exp(-2 \int_0^t L(s) ds)) \\ = \exp(-2 \int_0^t L(s) ds)[(u(t) - v(t))(u'(t) - v'(t)) - 2L(t)w(t)] \leq 0. \end{aligned}$$

As $u(0) - v(0) = 0$, we deduce $u = v$ on $[0, T]$. □

Using Theorem 2.1 we can consider repulsive or attractive singular forces.

Example 2.2. The problems

$$(7) \quad u' = \frac{1}{u} - \frac{2}{\sqrt{t}} - \frac{1}{4} \text{ on }]0, 1], \quad u(0) = 0,$$

as well as

$$(8) \quad u' = -\frac{1}{u} + \frac{2}{\sqrt{t}} + \frac{1}{4} \text{ on }]0, 1], \quad u(0) = 0,$$

have at least one solution in $W^{1,1}(0, 1)$ as $\alpha(t) = \frac{\sqrt{t}}{4}$ and $\beta(t) = 4$ are lower and upper solutions of (7) and $\alpha(t) = 2\sqrt{t}$ and $\beta(t) = 4\sqrt{t}$ are lower and upper solutions of (8). We can prove the existence of a second solution for (8) as $\beta(t) = \frac{\sqrt{t}}{4}$ is an upper solution and $\alpha(t) = \sqrt{t}$ is a lower solution. But we deduce from Theorem 2.2 the uniqueness of the solution of problem (7).

Remark 2.6. In the previous example we have a nonlinearity which is singular in u , but not really singular in t as for every $u, f(\cdot, u) \in L^1(0, T)$. We can also consider stronger singularities in t if α and β are chosen adequately as shown in the next example.

Example 2.3. Let us consider the problem

$$(9) \quad u' = \frac{1}{u^2} - \frac{1}{t^2} \text{ on }]0, 1], \quad u(0) = 0.$$

This problem has a solution, as $\alpha(t) = \frac{t}{2}$ and $\beta(t) = t$ are lower and upper solutions of (9) and f is L^1 -Carathéodory on $D = \{(t, u) \in]0, 1] \times]0, \infty[: \frac{t}{2} \leq u \leq t\}$ even though $f(\cdot, u) \notin L^1(0, 1)$.

Now let us prove Theorem 2.1. To this aim we need the following lemmata.

Lemma 2.3. Let the assumptions of Theorem 2.1 be satisfied and t_1, t_2 be such that $0 \leq t_1 < t_2 \leq T$ and $\alpha \leq \beta$ a.e. on $[t_1, t_2]$. For every $x_1 > 0$ with $\alpha(t_1) \leq x_1 \leq \beta(t_1)$, the problem

$$(10) \quad u' = f(t, u), \quad u(t_1) = x_1,$$

has at least one solution $u \in W^{1,1}(t_1, t_2)$ such that for all $t \in [t_1, t_2]$,

$$\alpha(t) \leq u(t) \leq \beta(t).$$

Proof. Consider the modified problem

$$(11) \quad u' = f(t, \gamma(t, u)), \quad u(t_1) = x_1,$$

where $\gamma(t, u) = \alpha(t) + (u - \alpha(t))^+ - (u - \beta(t))^+$, $u^+ := \max\{0, u\}$. Extend f to $]0, 2T[\times]0, +\infty[$.

By [10, Theorem I-5.1], there exist $\bar{t}_1 < t_1 < \bar{t}_2 \leq 2T$ such that the problem (11) has a maximal solution $u :]\bar{t}_1, \bar{t}_2[\rightarrow \mathbb{R}$, $u \in W_{loc}^{1,1}(\bar{t}_1, \bar{t}_2)$. If we prove that on $[t_1, t_2] \cap [t_1, \bar{t}_2[$, $\alpha \leq u \leq \beta$, we deduce from [10, Theorem I-5.2] that $\bar{t}_2 > t_2$. So $u \in W^{1,1}(t_1, t_2)$ and the result follows as u is also a solution of (10).

Let us prove that $u \geq \alpha$ on $[t_1, t_2] \cap [t_1, \bar{t}_2[$. If it were not true and as $\alpha(t_1) \leq u(t_1)$ and the jumps of α are downward, then there exist $t_1 \leq t_3 < t_4 < \bar{t}_2$ such that $\alpha(t_3) = u(t_3)$, $\alpha \in W^{1,1}(t_3, t_4)$ and for all $t \in]t_3, t_4[$, $\alpha(t) > u(t)$. This gives us the contradiction

$$0 > u(t_4) - \alpha(t_4) = \int_{t_3}^{t_4} [u'(s) - \alpha'(s)] ds \geq \int_{t_3}^{t_4} [f(s, \gamma(s, u(s))) - f(s, \alpha(s))] ds = 0.$$

In a similar way, we prove that $u \leq \beta$ on $[t_1, t_2] \cap [t_1, \bar{t}_2[$. □

Lemma 2.4. *Let the assumptions of Theorem 2.1 be satisfied and t_1, t_2 be such that $0 \leq t_1 < t_2 \leq T$ and $\beta \leq \alpha$ a.e. on $[t_1, t_2]$. For every $x_2 > 0$ with $\beta(t_2) \leq x_2 \leq \alpha(t_2)$, the problem*

$$(12) \quad u' = f(t, u), \quad u(t_2) = x_2,$$

has at least one solution $u \in W^{1,1}(t_1, t_2)$ such that for all $t \in [t_1, t_2]$,

$$\beta(t) \leq u(t) \leq \alpha(t).$$

Proof. By the change of variables $s = t_1 + t_2 - t$, we are reduced to Lemma 2.3. □

Lemma 2.5. *Under the assumptions of Theorem 2.1, for every $t_1 \in [0, T]$, every $x_1 > 0$ with $\alpha(t_1) \leq x_1 \leq \beta(t_1)$, the problem*

$$(13) \quad u' = f(t, u), \quad u(t_1) = x_1,$$

has at least one solution $u \in W^{1,1}(t_1, T)$ such that for all $t \in [t_1, T]$,

$$\min(\alpha(t), \beta(t)) \leq u(t) \leq \max(\alpha(t), \beta(t)).$$

Sketch of the proof. Let $A = \{t \in]t_1, T[\mid \alpha(t) < \beta(t)\}$, $B = \{t \in]t_1, T[\mid \beta(t) < \alpha(t)\}$ and $C =]t_1, T[\setminus (A \cup B)$. It is well known that $\text{int}(A)$ and $\text{int}(B)$ are countable unions of disjoint open intervals, say

$$\text{int}(A) = \bigcup_{i=1}^{+\infty}]a_i, b_i[, \quad \text{int}(B) = \bigcup_{j=1}^{+\infty}]c_j, d_j[.$$

For $t_0 = a_i$ or d_j , we consider the nonempty interval

$$\left[\lim_{t \rightarrow t_0^+} \alpha(t), \lim_{t \rightarrow t_0^+} \beta(t) \right] \cap \left[\lim_{t \rightarrow t_0^-} \beta(t), \lim_{t \rightarrow t_0^-} \alpha(t) \right]$$

and we define $m(t_0)$ to be the minimum of this interval if $t_0 \neq t_1$ and $m(t_1) = x_1$. On each interval $[a_i, b_i]$, we define, from Lemma 2.3, a function u_i which is a solution of

$$u' = f(t, u) \text{ on } [a_i, b_i], \quad u(a_i) = m(a_i),$$

and such that $\alpha \leq u_i \leq \beta$. On each interval $[c_j, d_j]$, we define, from Lemma 2.4, a function v_j which is a solution of

$$u' = f(t, u) \text{ on } [c_j, d_j], \quad u(d_j) = m(d_j),$$

and such that $\beta \leq v_j \leq \alpha$.

It can be proved, along the lines of [15], that the function

$$\begin{aligned} u(t) &= u_i(t), && \text{if } t \in [a_i, b_i] \text{ for some } i, \\ &= v_j(t), && \text{if } t \in [c_j, d_j] \text{ for some } j, \\ &= \alpha(t) = \beta(t), && \text{otherwise,} \end{aligned}$$

is the required solution. □

Proof of Theorem 2.1. We consider separately the cases $\lim_{t \rightarrow 0} \beta(t) > 0$ or $\lim_{t \rightarrow 0} \beta(t) = 0$. Set $A = \{t \in]0, T] \mid \alpha(t) < \beta(t)\}$, $B = \{t \in]0, T] \mid \beta(t) < \alpha(t)\}$ and $C =]0, T] \setminus (A \cup B)$.

Case 1: $\lim_{t \rightarrow 0} \beta(t) = \bar{\varepsilon} > 0$. Let $(\varepsilon_n)_n$ be a decreasing sequence of positive numbers such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and $\varepsilon_1 \leq \bar{\varepsilon}$. Consider the approximations problem

$$(P_n) \quad u' = f(t, u) \text{ on }]0, T], \quad u(0) = \varepsilon_n.$$

As a solution of (P_{n-1}) is an upper solution of (3), it is not difficult to prove, by induction using Lemma 2.5, that for every n , the problem (P_n) has a solution u_n such that

$$\begin{aligned} \min(\alpha, \beta) &\leq \dots \leq u_n \leq u_{n-1} \leq \dots \leq u_1 \leq \max(\alpha, \beta) && \text{on } A, \\ \min(\alpha, \beta) &\leq u_1 \leq \dots \leq u_{n-1} \leq u_n \leq \dots \leq \max(\alpha, \beta) && \text{on } B. \end{aligned}$$

As in [9], we conclude that the pointwise limit of $(u_n)_n$ is a function $u \in \mathcal{C}([0, T]) \cap W_{loc}^{1,1}(0, T)$, a solution of (3) such that, for all $t \in [0, T]$,

$$\min(\alpha(t), \beta(t)) \leq u(t) \leq \max(\alpha(t), \beta(t)).$$

Case 2: $\lim_{t \rightarrow 0} \beta(t) = 0$ and there exists a decreasing sequence $(t_n)_n$ of positive numbers such that $\lim_{n \rightarrow \infty} t_n = 0$ and $\beta(t_n) \geq \alpha(t_n)$. Consider the approximations problem

$$(Q_n) \quad u' = f(t, u) \text{ on }]t_n, T], \quad u(t_n) = \beta(t_n).$$

By Lemma 2.5, the problem (Q_1) has a solution $u_1 \in W^{1,1}(t_1, T)$ such that for all $t \in [t_1, T]$, $\min(\alpha(t), \beta(t)) \leq u_1(t) \leq \max(\alpha(t), \beta(t))$. Let us define β_1 , with $\min(\alpha, \beta) \leq \beta_1 \leq \max(\alpha, \beta)$, an upper solution of (3) in the following way:

$$\begin{aligned} \beta_1(t) &= \beta(t), && \text{if } t \in [0, t_1], \\ &= u_1(t), && \text{if } t \in]t_1, T]. \end{aligned}$$

Arguing by induction and using Lemma 2.5, we have a sequence $(u_n)_n$ of solutions of (Q_n) and a sequence $(\beta_n)_n$ of upper solutions of (3) defined by

$$\begin{aligned} \beta_n(t) &= \beta(t), && \text{if } t \in [0, t_n], \\ &= u_n(t), && \text{if } t \in]t_n, T], \end{aligned}$$

such that

$$\begin{aligned} \min(\alpha, \beta) &\leq \dots \leq \beta_n \leq \beta_{n-1} \leq \dots \leq \beta_1 \leq \max(\alpha, \beta) && \text{on } A, \\ \min(\alpha, \beta) &\leq \beta_1 \leq \dots \leq \beta_{n-1} \leq \beta_n \leq \dots \leq \max(\alpha, \beta) && \text{on } B. \end{aligned}$$

We conclude again by applying the argument of [9].

Case 3: $\lim_{t \rightarrow 0} \beta(t) = 0$ and there exists $\varepsilon > 0$ such that for all $t \in]0, \varepsilon]$, $\beta(t) < \alpha(t)$.

In that case, if $\beta < \alpha$ on $]0, T]$, by Lemma 2.4, there exists a solution u of

$$u' = f(t, u) \text{ on }]0, T], \quad u(T) = \beta(T),$$

with $\beta \leq u \leq \alpha$. As $\beta(0) = \alpha(0) = 0$, u is a solution of (3).

If β and α cross, let $\bar{t} = \sup\{t \in [0, T] \mid \beta(s) < \alpha(s) \text{ for all } s \in [0, t]\}$. By Lemmata 2.4 and 2.5, there exist u_1 and u_2 such that

$$\begin{aligned} u'_1 &= f(t, u_1) \text{ on }]0, \bar{t}], & u_1(\bar{t}) &= \alpha(\bar{t}), \\ u'_2 &= f(t, u_2) \text{ on }]\bar{t}, T], & u_2(\bar{t}) &= \alpha(\bar{t}). \end{aligned}$$

As $\alpha(0) = \beta(0)$, the function

$$\begin{aligned} u(t) &= u_1(t), & \text{if } t \in]0, \bar{t}], \\ &= u_2(t), & \text{if } t \in]\bar{t}, T], \end{aligned}$$

is the required solution. □

3. CONSTRUCTION OF LOWER AND UPPER SOLUTIONS

We now give conditions to ensure the existence of the required lower and upper solutions.

Proposition 3.1. *Let $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ be L^1 -Carathéodory and assume (H_1) there exist $a \in W^{1,1}(0, T)$, $L \in L^1(0, T)$ such that $a(0) \geq 0$, $a' > 0$, $L > 0$ and for a.e. $t \in]0, T]$, all $u \in]0, a(t)]$,*

$$f(t, u) \geq L(t).$$

Then there exists a lower solution α of (3).

Remark 3.1. Condition (H_1) is equivalent to (H'_1) for every $K \subset]0, T]$ compact, there exist $L_K \in L^1(K)$, $L_K > 0$ on K and $\varepsilon_K > 0$ such that for a.e. $t \in K$ and all $u \in]0, \varepsilon_K]$,

$$f(t, u) \geq L_K(t).$$

Proof. By replacing L by $\min\{a', L\}$ if necessary we can assume that for all $t \in]0, T]$, $\int_0^t L(s) ds \leq a(t)$. It is then easy to observe that

$$\alpha(t) = \int_0^t L(s) ds$$

is a lower solution of (3). □

Example 3.1. The condition (H_1) is of course satisfied if there exists $\delta > 0$ such that $f(t, u) \geq L(t) > 0$ on $]0, T] \times]0, \delta]$. But it is also satisfied for functions of the type

$$f(t, u) = \frac{1}{u^s} - \frac{1}{t^r},$$

with $r > 0$ and $s > 0$.

Remark 3.2. We do not really need $L > 0$ a.e. on $]0, T]$ but only that

$$\int_0^t L(s) ds > 0 \quad \text{on }]0, T].$$

Now let us give some conditions which ensure the existence of upper solution.

Proposition 3.2. *Let $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ be L^1 -Carathéodory and assume there exists $R > 0$ such that for a.e. $t \in]0, T]$,*

$$f(t, R) \leq 0.$$

Then there exists an upper solution β of (3) such that $\beta(t) > 0$ on $[0, T]$.

Proof. We choose $\beta(t) = R$.

When $f \geq 0$ on $]0, T] \times]0, +\infty[$, we have an upper solution if f does not tend to infinity too quickly when u tends to infinity. \square

Proposition 3.3. *Let $f :]0, T] \times]0, \infty[\rightarrow \mathbb{R}$ be L^1 -Carathéodory and assume (H_2) there exist $R > 0$, $h \in \mathcal{C}([R, \infty[)$ and $q \in L^1(0, T)$ such that $h > 0$, $q \geq 0$ and for a.e. $t \in]0, T]$, all $u \in [R, \infty[$,*

$$(14) \quad f(t, u) \leq q(t)h(u),$$

and

$$(15) \quad \int_0^T q(s) ds < \int_R^\infty \frac{1}{h(s)} ds.$$

Then there exists an upper solution β of (3) such that $\beta(t) > 0$ on $[0, T]$.

Proof. We define β to be the solution of

$$u' = q(t)h(u), \quad u(0) = R.$$

From (15), we deduce that β is defined on $[0, T]$. As $\beta \geq R$ on this interval, using (14), we prove it is an upper solution of (3). \square

Example 3.2. The prototypes for condition (H_2) are nonlinearities $f(t, u)$ of the form

$$f(t, u) = \frac{u^s}{t^r}, \quad \text{for } u \gg 1,$$

with $0 < r < 1$ and $s \leq 1$. This restricts the behaviour of $f(t, \cdot)$ for u large but not for u near 0.

4. SIMILAR RESULTS FOR THE PERIODIC PROBLEM

Using the ideas of Section 2, we can consider other problems such as the periodic one. In this section, we indicate how to deal with the problem

$$(16) \quad u' = f(t, u), \quad u(0) = u(T).$$

Definition 4.1. A bounded function $\alpha : [0, T] \rightarrow \mathbb{R}$ is a *lower solution* of (16) if condition (ii) of Definition 2.3 is satisfied and

$$\alpha(0) \leq \alpha(T).$$

In the same way, a bounded function $\beta : [0, T] \rightarrow \mathbb{R}$ is an *upper solution* of (16) if the conditions of Definition 2.4 are satisfied and

$$\beta(0) \geq \beta(T).$$

Theorem 4.1. *Let $f : D \subset]0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and α and β be lower and upper solutions of (16). Assume $E = \{(t, u) \in]0, T] \times \mathbb{R} \mid \min(\alpha(t), \beta(t)) \leq u \leq \max(\alpha(t), \beta(t))\} \subset D$, f is Carathéodory on E and there exists $h \in L^1(0, T)$ with*

$$|f(t, u)| \leq h(t) \text{ on } E.$$

Then the problem (16) has at least one solution $u \in \mathcal{C}([0, T]) \cap W^{1,1}(0, T)$ such that for all $t \in [0, T]$,

$$\min(\alpha(t), \beta(t)) \leq u(t) \leq \max(\alpha(t), \beta(t)).$$

Proof. *Case 1:* $\alpha(0) < \beta(0)$. If $\alpha \leq \beta$ on $[0, T]$, we consider the modified problem

$$(17) \quad u' + u = f(t, \gamma(t, u)) + \gamma(t, u), \quad u(0) = u(T),$$

where $\gamma(t, u) = \alpha(t) + (u - \alpha(t))^+ - (u - \beta(t))^+$. It is easy to prove by Schauder's theorem that (17) has at least one solution. The proof that the solution u of (17) satisfies $\alpha \leq u \leq \beta$ on $[0, T]$ uses the same ideas as Lemma 2.3.

If $\beta(T) \leq \alpha(T)$, let $\bar{t} = \sup\{t \in [0, T] \mid \alpha(s) < \beta(s) \text{ for all } s \in [0, t]\}$. Observe that, by the boundary restrictions imposed on α and β , there exists $\bar{x} \in [\alpha(0), \beta(0)] \cap [\beta(T), \alpha(T)]$. By Lemma 2.3, there exists a solution u_1 of

$$u' = f(t, u) \text{ on }]0, \bar{t}], \quad u(0) = \bar{x},$$

and, by the change of variables $s = \bar{t} + T - t$, we deduce from Lemma 2.5 that there exists a solution u_2 of

$$u' = f(t, u) \text{ on }]\bar{t}, T], \quad u(T) = \bar{x}.$$

Hence, the result is proved with

$$\begin{aligned} u(t) &= u_1(t), & \text{if } t \in [0, \bar{t}], \\ &= u_2(t), & \text{if } t \in]\bar{t}, T]. \end{aligned}$$

If $\beta(T) > \alpha(T)$ and $\alpha \not\leq \beta$, let \bar{t} be defined as above. We have $[\alpha(T), \beta(T)] \subset [\alpha(0), \beta(0)]$ and using Lemmata 2.5 and 2.3, we prove the existence of functions u_2 and u_1 such that

$$u'_2 = f(t, u_2) \text{ on }]\bar{t}, T], \quad u_2(\bar{t}) = \alpha(\bar{t}) = \beta(\bar{t}),$$

and

$$u'_1 = f(t, u_1) \text{ on }]0, \bar{t}], \quad u_1(0) = u_2(T).$$

We conclude as in the previous situation.

Case 2: $\alpha(0) = \beta(0)$. As $\beta(T) \leq \beta(0) = \alpha(0) \leq \alpha(T)$, the solution u of

$$u' = f(t, u) \text{ on }]0, T], \quad u(T) = \alpha(0) = \beta(0),$$

given by Lemma 2.5, is periodic.

Case 3: $\alpha(0) > \beta(0)$. The boundary conditions imply $\alpha(T) > \beta(T)$ and we are reduced to Case 1 by the change of variables $s = T - t$. □

Example 4.1. Consider the equivalent of problems (7) and (8), i.e.

$$(18) \quad u' = \frac{1}{u} - \frac{2}{\sqrt{t}} - \frac{1}{4}, \quad u(0) = u(1),$$

as well as

$$(19) \quad u' = -\frac{1}{u} + \frac{2}{\sqrt{t}} + \frac{1}{4}, \quad u(0) = u(1).$$

We can prove that both (18) and (19) have at most one solution. Such a proof relies on an argument which recalls the proof of Theorem 2.2. Moreover $\alpha(t) = \frac{\sqrt{t}}{4}$ and $\beta(t) = 4$ are lower and upper solutions of (18), $\alpha(t) = 4$ and $\beta(t) = \frac{\sqrt{t}}{4}$ if $t \leq \frac{1}{2}$, $\beta(t) = \frac{\sqrt{1-t}}{4}$ if $t > \frac{1}{2}$ are lower and upper solutions of (19) and we apply Theorem 4.1 to prove that (18) and (19) have a unique solution. Observe that this situation is quite different from the one observed for the Cauchy problem.

Remark 4.1. Observe that in Theorem 4.1 we allow less singularities than in Theorem 2.1. This is due to the fact that, in applications, lower and upper solutions for the periodic case can usually be chosen “far enough” from the singularity as in the next result. Nevertheless, more singularity as in Theorem 2.1 can be considered in the periodic case also in the following way. First observe that in Theorem 2.1 we allow more singularity mainly in case $\alpha(0) = \beta(0) = 0$. In the periodic case, this implies $\beta(T) = 0$. Hence two cases have to be considered. Either $\alpha(T) = 0$ and the result follows by arguments similar to Theorem 2.1 as the solution of (3) with $\min(\alpha, \beta) \leq u \leq \max(\alpha, \beta)$ has to satisfy $u(T) = 0 = u(0)$, or in the case $\alpha(T) > 0$, we define $\bar{t} = \inf\{t \mid \beta(s) < \alpha(s) \text{ for all } s \in]t, T]\}$ and again arguing as in Section 2, we prove the existence of a solution u_1 of

$$u' = f(t, u) \text{ on }]0, \bar{t}], \quad u(0) = 0,$$

with $\min(\alpha, \beta) \leq u_1 \leq \max(\alpha, \beta)$ and of a solution u_2 of

$$u' = f(t, u) \text{ on }]\bar{t}, T[, \quad u(T) = 0,$$

with $\beta \leq u_2 \leq \alpha$ on $[\bar{t}, T]$. Hence,

$$\begin{aligned} u(t) &= u_1(t), & \text{if } t \in [0, \bar{t}], \\ &= u_2(t), & \text{if } t \in]\bar{t}, T], \end{aligned}$$

is the required solution.

To conclude, consider the following result for a singular periodic problem.

Theorem 4.2. *Let $f : [0, T] \times]0, \infty[\rightarrow \mathbb{R}$ be L^1 -Carathéodory. Assume*

- (i) *for some $\varepsilon > 0$, $\liminf_{u \rightarrow 0^+} f(t, u) > \varepsilon$ uniformly in $t \in [0, T]$;*
- (ii) *there exist $R > 0$ and $g_0 \in L^1(0, T)$ such that for a.e. $t \in [0, T]$, all $u \geq R$,*

$$f(t, u) \leq g_0(t),$$

and

$$\int_0^T g_0(t) dt \leq 0.$$

Then the problem (16) has at least one solution.

Proof. Notice that if $\alpha < R$ is a sufficiently small positive constant, it is a lower solution of (16).

Let w be the solution of

$$u' = g_0(t) - \frac{1}{T} \int_0^T g_0(t) dt, \quad u(0) = u(T).$$

It is easy to see that for $K > 0$ large enough, $\beta(t) = w(t) + K$ is an upper solution of (16) such that $\beta > R > \alpha$ on $[0, T]$. We conclude by Theorem 4.1. □

Using lower and upper solutions with the reversed order according to Theorem 4.2, we can also consider the dual result.

Example 4.2. Model examples for Theorems 4.2 and its dual version are the problems

$$u' = \frac{1}{u} - g(t), \quad u(0) = u(T),$$

and

$$u' = -\frac{1}{u} + g(t), \quad u(0) = u(T),$$

where $g \in L^\infty(0, T)$ with $\int_0^T g(t) dt > 0$.

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UNIVERSITÉ CATHOLIQUE DE LOUVAIN, INSTITUT DE MATHÉMATIQUE PURE ET APPLIQUÉE,
CHEMIN DU CYCLOTRON 2, 1348 LOUVAIN-LA-NEUVE, BELGIQUE
E-mail address: `cherpion@amm.ucl.ac.be`

UNIVERSITÉ DU LITTORAL - CÔTE D'OPALE, CENTRE UNIVERSITAIRE DE LA MI-VOIX, 50 RUE
F. BUISSON, B.P. 699, 62228 CALAIS CÉDEX, FRANCE
E-mail address: `decoster@lma.univ-littoral.fr`