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NONSTANDARD SOLVABILITY FOR LINEAR OPERATORS BETWEEN SECTIONS OF VECTOR BUNDLES

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ABSTRACT. Given a certain kind of linear operator A (possibly a differential operator or a properly supported pseudodifferential operator) between sections of Hermitian vector bundles over a Riemannian manifold, a necessary and sufficient condition is obtained for the operator A to be solvable in a class of nonstandard sections in a generalized sense of weak solutions. The existence of a fundamental-solution-like internal section is established in the solvable case.

1. INTRODUCTION

Let E and F be C^∞ Hermitian vector bundles over a Riemannian manifold M, and let $A : \Gamma_0^\infty(E) \to \Gamma^\infty(F)$ be a $\mathbb{C}$-linear operator from the space $\Gamma_0^\infty(E)$ of C^∞ sections of E with compact support to the space $\Gamma^\infty(F)$ of C^∞ sections of F, with formal adjoint $A^\# : \Gamma_0^\infty(F) \to \Gamma^\infty(E)$ such that $A^\#\varphi$ is an L^2-section of E for every $\varphi \in \Gamma_0^\infty(F)$ (§3, (3.1)). (For example, the operator A may be a linear differential operator or a properly supported pseudodifferential operator.) Let T be a generalized section of F (see §2 or [2]).

Consider the equation

$$Au = T \tag{1.1}$$

in the sense that

$$(u, A^\#\varphi)_E = (T, \varphi)_F \quad \text{for all} \quad \varphi \in \Gamma_0^\infty(F). \tag{1.2}$$

Here $(T, \varphi)_F$ is defined by extending the inner product $(,)_F$ in the space of L^2-sections of F (§4, (4.1)). The left-hand side of the equation in (1.2) can be defined if u is an L^2-section of E, or if u is a generalized section of E and if the support of $A^\#\varphi$ is compact for every $\varphi \in \Gamma_0^\infty(F)$. Then for the equation (1.1) to be solvable, it is clearly necessary that the following condition be satisfied:

$$\forall\varphi \in \Gamma_0^\infty(F) \quad [A^\#\varphi = 0 \Longrightarrow (T, \varphi)_F = 0]. \tag{C1}$$

We generalize the notion of solvability in order to be able to deal with nonstandard sections by reformulating (1.1) and (1.2) properly using nonstandard analysis.

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We then show that the condition (C1) is *necessary and sufficient* for (the reformulated equation of) (1.1) to have a solution in the nonstandard extension ${}^*(\Gamma^\infty(E))$ of $\Gamma^\infty(E)$ (§4, Theorem 4.2). Our present work is partly motivated by some results in T. Todorov [6] and [7], where existence results are established for linear partial differential equations (with smooth coefficients) on open sets of $\mathbb{R}^d$ (in other words, he studies in the product complex line bundle case $E = F = \Omega \times \mathbb{C}$ with $M = \Omega$ an open set of $\mathbb{R}^d$).

In this paper, we first show that for the nonstandard extension ${}^*A$ of the operator A and for each internal ${}^*C^\infty$ section f of the nonstandard extension ${}^*F$ of F (that is, $f \in {}^*(\Gamma^\infty(F))$), the equation

$$ {}^*Au = f \tag{1.3} $$

is solvable in a certain generalized sense of weak solutions if and only if f satisfies a condition similar to (C1) (with T replaced by f) (§3, Theorem 3.1). Then we show that this treatment applies to the equation (1.1); moreover, we also obtain a result on the existence of a fundamental-solution-like internal section for (1.1) (§4, Theorem 4.2). Finally we prove the existence of an internal section (depending on A) that plays a fundamental role in constructing a solution of (1.1) for each generalized section T of F satisfying (C1) (Theorem 4.3) .

We obtain our results using the notion of hyperfinite-dimensional internal vector space in the framework of nonstandard analysis. As for the preliminary knowledge of nonstandard analysis, see for example [3], [4]. We work with a sufficiently saturated nonstandard model.

2. Notational preliminaries

Let $\mathbb{N}$ be the set of all (strictly) positive integers, and let $\mathbb{R}$ [resp. $\mathbb{C}$] be the set of all real numbers [resp. complex numbers]. We denote by ${}^*\mathbb{N}_\infty := {}^*\mathbb{N} \setminus \mathbb{N}$ the infinite elements in the nonstandard extension ${}^*\mathbb{N}$ of $\mathbb{N}$.

Given a complex vector bundle $\pi_E : E \to M$, we write $E_x := \pi_E^{-1}(x)$, the fiber of E over $x \in M$. Let $\Gamma^\infty(E)$ denote the space of all C^∞ sections of E and $\Gamma_0^\infty(E)$ the space of C^∞ sections of E with compact support. A *generalized section* of E is defined as a continuous linear functional on the space $\Gamma_0^\infty(E^\star \otimes |\wedge_M|)$, where $E^\star$ is the dual bundle of E and $|\wedge_M|$ the complex line bundle of densities over M (see [2]). The space of generalized sections of E is denoted by $\Gamma^{-\infty}(E)$.

For two vector bundles $E_1 \to M_1$ and $E_2 \to M_2$, we denote by $E_1 \boxtimes E_2$ the vector bundle over $M_1 \times M_2$ such that $(E_1 \boxtimes E_2)_{(x,y)} = (E_1)_x \otimes (E_2)_y$ for every $(x, y) \in M_1 \times M_2$. For simplicity of notation, we write $\otimes$ for ${}^*\otimes$ (the nonstandard extension of $\otimes$). Similarly, we write $\boxtimes$ for ${}^*\boxtimes$.

In what follows, let (M, g) be a (finite-dimensional, σ-compact) C^∞ Riemannian manifold, and let $\pi_E : E \to M$ and $\pi_F : F \to M$ be C^∞ Hermitian vector bundles with Hermitian metrics h_E and h_F, respectively. Let $dv_g \in \Gamma^\infty(|\wedge_M|)$ be the Riemannian volume density associated with g (see [5]). For $s_1, s_2 \in \Gamma^\infty(E)$, we set

$$ (s_1, s_2)_E = \int_M h_E(s_1, s_2)\, dv_g, \qquad \|s_1\|_E = (s_1, s_1)_E^{1/2}, \tag{2.1} $$

if they exist. ($(\,,\,)_E$ is extended to be an L^2-inner product.) Define $(\,,\,)_F$ and $\|\;\|_F$ in a similar way. For simplicity, we denote by the same notation $(\,,\,)_E$ [resp. $(\,,\,)_F$] the nonstandard extension of $(\,,\,)_E$ [resp. $(\,,\,)_F$]; furthermore, ${}^*\|\;\|_E := {}^*(\|\;\|_E)$ stands for the nonstandard extension of $\| \|_E$. The set of standard elements of ${}^*(\Gamma_0^\infty(F))$ is denoted by ${}^\sigma(\Gamma_0^\infty(F))$, that is,

$$ {}^\sigma(\Gamma_0^\infty(F)) = \{{}^*\varphi : \varphi \in \Gamma_0^\infty(F)\}. $$

3. Nonstandard solvability of (1.3)

Let

$$ A : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(F) $$

be a linear operator (a $\mathbb{C}$-linear map) with formal adjoint

$$ A^\# : \Gamma_0^\infty(F) \longrightarrow \Gamma^\infty(E), $$

so that

$$ (As, \varphi)_F = (s, A^\#\varphi)_E \in \mathbb{C} \quad (s \in \Gamma_0^\infty(E),\, \varphi \in \Gamma_0^\infty(F)), $$

such that

$$ \|A^\#\varphi\|_E < \infty \quad (\text{that is,} \quad \|A^\#\varphi\|_E \in \mathbb{R}) \quad \text{for all} \quad \varphi \in \Gamma_0^\infty(F). \tag{3.1} $$

By the transfer principle, we have

$$ ({}^*A\sigma, \zeta)_F = (\sigma, {}^*A^\#\zeta)_E \in {}^*\mathbb{C} \quad (\sigma \in {}^*(\Gamma_0^\infty(E)),\, \zeta \in {}^*(\Gamma_0^\infty(F))\,), $$

where ${}^*A^\# := {}^*(A^\#)$.

For $f \in {}^*(\Gamma^\infty(F))$, we study the existence of an element $u \in {}^*(\Gamma^\infty(E))$ with ${}^*\|u\|_E \in {}^*\mathbb{R}$ that satisfies the equation

$$ {}^*Au = f \tag{3.2} $$

in the sense that

$$ (u, {}^*(A^\#\varphi))_E = (f, {}^*\varphi)_F \quad \text{for all} \quad \varphi \in \Gamma_0^\infty(F). \tag{3.3} $$

(${}^*\varphi :$ the nonstandard extension of φ.) Note that ${}^*(A^\#\varphi) = {}^*A^\#\,{}^*\varphi$ holds for every $\varphi \in \Gamma_0^\infty(F)$. Note also that in (3.3), since ${}^*\|{}^*(A^\#\varphi)\|_E = \|A^\#\varphi\|_E \in \mathbb{R}$ by (3.1), ${}^*\|u\|_E \in {}^*\mathbb{R}$ implies that $(u, {}^*(A^\#\varphi))_E \in {}^*\mathbb{C}$ by transfer of the Schwarz inequality.

The following theorem concerns the solvability of (3.2).

Theorem 3.1. *Let $f \in {}^*(\Gamma^\infty(F))$. Then the equation* (3.2) *in the sense of* (3.3) *has a solution $u \in {}^*(\Gamma^\infty(E))$ with ${}^*\|u\|_E \in {}^*\mathbb{R}$ if and only if f satisfies the following condition* (C2):

$$ \forall \varphi \in \Gamma_0^\infty(F) \quad [A^\#\varphi = 0 \Longrightarrow (f, {}^*\varphi)_F = 0]. \tag{C2} $$

Proof. Suppose an element $u \in {}^*(\Gamma^\infty(E))$ satisfies ${}^*\|u\|_E \in {}^*\mathbb{R}$ and (3.3). Then clearly f satisfies (C2).

Conversely, suppose f satisfies (C2). In the case $A^\# = 0$, we have only to let $u = 0$; so assume $A^\# \neq 0$. Set

$$ W = \{\zeta \in {}^*(\Gamma_0^\infty(F)) : {}^*A^\#\zeta = 0 \Longrightarrow (f, \zeta)_F = 0\}. \tag{3.4} $$

This is an internal vector space over ${}^*\mathbb{C}$. The set ${}^\sigma(\Gamma_0^\infty(F))$ of standard elements of ${}^*(\Gamma_0^\infty(F))$ is an external subset of W; note that ${}^*A^{\#}{}^*\varphi = 0$ holds if and only if $A^\#\varphi = 0$. By the saturation principle, there exists a hyperfinite-dimensional internal vector subspace V of W such that ${}^\sigma(\Gamma_0^\infty(F))$ is a subset of V. Let the

$*$-dimension of V be ${}^*\dim V = \nu$, which is an infinite hypernatural number ($\nu \in {}^*\mathbb{N}_\infty$). We can pick a basis $\{\psi_1, \ldots, \psi_\nu\}$ for V such that

$$(\psi_i, \psi_j)_F = \delta_{ij} \quad (i, j = 1, \ldots, \nu), \tag{3.5}$$

$$\{{}^*A^\#\psi_1, \ldots, {}^*A^\#\psi_{\nu_0}\} \text{ is a basis for } {}^*A^\#(V) = \{{}^*A^\#\zeta : \zeta \in V\}, \tag{3.6}$$

$${}^*A^\#\psi_j = 0 \quad \text{for} \quad j = \nu_0 + 1, \ldots, \nu. \tag{3.7}$$

By applying the transfer principle to (3.1), we have ${}^*\|{}^*A^\#\psi_i\|_E \in {}^*\mathbb{R}$ for $i = 1, \ldots, \nu_0$. Therefore

$$b_{ij} := ({}^*A^\#\psi_i, {}^*A^\#\psi_j)_E \in {}^*\mathbb{C} \quad (i, j = 1, \ldots, \nu_0)$$

by transfer of the Schwarz inequality, and the $\nu_0 \times \nu_0$ internal Hermitian matrix $B = (b_{ij})$ $(i, j = 1, \ldots, \nu_0)$ is nonsingular by (3.6).

Let $C = (c_{ij})$ be the inverse matrix of B. Define $u_i \in {}^*(\Gamma^\infty(E))$ $(i = 1, \ldots, \nu_0)$ by the internal sum

$$u_i = \sum_{k=1}^{\nu_0} c_{ik}{}^*A^\#\psi_k. \tag{3.8}$$

Then for $i, j = 1, \ldots, \nu_0$, it holds that

$$(u_i, {}^*A^\#\psi_j)_E = \delta_{ij} \text{ (Kronecker delta)}, \qquad (u_i, u_j)_E = c_{ij}.$$

Thus

$$u = \sum_{i=1}^{\nu_0} (f, \psi_i)_F u_i \tag{3.9}$$

satisfies ${}^*\|u\|_E \in {}^*\mathbb{R}$ and

$$(u, {}^*A^\#\psi_j)_E = (f, \psi_j)_F \quad (j = 1, \ldots, \nu). \tag{3.10}$$

(Note that both sides of (3.10) are 0 for $j = \nu_0 + 1, \ldots, \nu$.) Since every ${}^*\varphi \in {}^\sigma(\Gamma_0^\infty(F))$ is expressed in the form

$${}^*\varphi = \sum_{j=1}^{\nu} \alpha_j \psi_j \quad (\alpha_j \in {}^*\mathbb{C}), \tag{3.11}$$

it follows from (3.10) that the above u in (3.9) satisfies (3.3). □

Remark 3.1. From the above proof, we see that if $f \in {}^*(\Gamma^\infty(F))$ satisfies (C2) and if the support of $A^\#\varphi$ is compact for every $\varphi \in \Gamma_0^\infty(F)$, then it holds that ${}^*A^\#\psi_j \in {}^*(\Gamma_0^\infty(E))$ $(j = 1, \ldots, \nu_0)$ and thus we can choose u in Theorem 3.1 in such a way that $u \in {}^*(\Gamma_0^\infty(E))$.

Remark 3.2. The operator A admits a $*$-integral representation with ${}^*C^\infty$ kernel ([1, Proposition 2.2]).

4. Nonstandard solvability of (1.1)

For a section $\varphi \in \Gamma_0^\infty(F)$, define ${}^h\varphi \in \Gamma_0^\infty(F^\star)$ by

$${}^h\varphi = h_F(\cdot, \varphi).$$

For a generalized section $T \in \Gamma^{-\infty}(F)$ and a section $\varphi \in \Gamma_0^\infty(F)$, put

$$(T, \varphi)_F = T({}^h\varphi \otimes dv_g) \in \mathbb{C}. \tag{4.1}$$

Note that this is consistent with the notation $(,)_F$ in (2.1) if $T \in \Gamma^{\infty}(F)$ ($\subset \Gamma^{-\infty}(F)$).

Using the nonstandard extension ${}^*h_F$ of h_F, define, for $\zeta \in {}^*(\Gamma_0^{\infty}(F))$,

$$ {}^h\zeta = {}^*h_F(\cdot, \zeta) \in {}^*(\Gamma_0^{\infty}(F^{\star})), $$

so that ${}^h\zeta \otimes {}^*dv_g \in {}^*(\Gamma_0^{\infty}(F^{\star} \otimes |\wedge_M|))$. Then for the nonstandard extension ${}^*T$ of $T \in \Gamma^{-\infty}(F)$, we have

$$ ({}^*T, \zeta)_F = {}^*T({}^h\zeta \otimes {}^*dv_g) \in {}^*\mathbb{C}. $$

The following key lemma enables us to apply Theorem 3.1 to the study of the equation (1.1).

Lemma 4.1 (Key Lemma)**.** *There exists a $\mathbb{C}$-linear injection*

$$ \beta : \Gamma^{-\infty}(F) \longrightarrow {}^*(\Gamma_0^{\infty}(F)) $$

such that for each $T \in \Gamma^{-\infty}(F)$, $\beta(T)$ satisfies

$$ (\beta(T), {}^*\varphi)_F = (T, \varphi)_F \quad (\varphi \in \Gamma_0^{\infty}(F)). \tag{4.2} $$

Proof (cf. [1, Theorem 2.3]). Consider ${}^*(\Gamma_0^{\infty}(F))$ instead of W given in (3.4) and proceed as in the proof of Theorem 3.1 to obtain a hyperfinite-dimensional internal vector subspace V of ${}^*(\Gamma_0^{\infty}(F))$ such that ${}^{\sigma}(\Gamma_0^{\infty}(F))$ is an external subset of V. Pick a basis $\{\psi_1, \dots, \psi_{\nu}\}$ (where ${}^*\dim V = \nu \in {}^*\mathbb{N}_{\infty}$) for V satisfying (3.5), (3.6), and (3.7). (We need (3.6) and (3.7) for later use in the proof of Theorem 4.3, but not here.) Define a $\mathbb{C}$-linear map $\beta : \Gamma^{-\infty}(F) \longrightarrow {}^*(\Gamma_0^{\infty}(F))$ by

$$ \beta(T) = \sum_{i=1}^{\nu} ({}^*T, \psi_i)_F \psi_i \quad (T \in \Gamma^{-\infty}(F)). $$

Then

$$ (\beta(T), \psi_j)_F = ({}^*T, \psi_j)_F \quad \text{for} \quad j = 1, \dots, \nu. $$

Since each ${}^*\varphi \in {}^{\sigma}(\Gamma_0^{\infty}(F))$ is expressed in the form (3.11), we see that

$$ (\beta(T), {}^*\varphi)_F = ({}^*T, {}^*\varphi)_F = {}^*((T, \varphi)_F) = (T, \varphi)_F. $$

The injectivity of β is clear. □

Now we reformulate (1.1) and (1.2). The following theorem concerns not only the nonstandard solvability of (1.1) but also the existence of a related fundamental-solution-like internal section.

Theorem 4.2. *Let $A : \Gamma_0^{\infty}(E) \longrightarrow \Gamma^{\infty}(F)$ be as in §3.*

(a) *Let $T \in \Gamma^{-\infty}(F)$, and consider the equation*

$$ {}^*Au = {}^*T \tag{4.3} $$

in the sense that

$$ (u, {}^*(A^{\#}\varphi))_E = ({}^*T, {}^*\varphi)_F (= (T, \varphi)_F) \quad \textit{for all} \quad \varphi \in \Gamma_0^{\infty}(F). \tag{4.4} $$

Then (4.3) has a solution $u \in {}^*(\Gamma^{\infty}(E))$ with ${}^*\|u\|_E \in {}^*\mathbb{R}$ if and only if T satisfies the condition (C1).

(b) *There exists an internal* ${}^*C^\infty$ *section*

$$G \in {}^*(\Gamma^\infty(E \boxtimes F^\star))$$

such that, for each $x \in {}^*M$, *the internal support of* $G(x,\cdot)$ *is* $*$*-compact and such that, for* $T \in \Gamma^{-\infty}(F)$ *satisfying the condition* (C1), *the* $*$*-integral*

$$u(x) = \int_{{}^*M} G(x,y)\beta(T)(y)\, {}^*dv_g(y) \quad (x \in {}^*M) \tag{4.5}$$

gives a solution of (4.3) *in the sense of* (4.4).

Proof. To prove (a), let $T \in \Gamma^{-\infty}(F)$. By (4.2), $\beta(T)$ satisfies the condition (C2) (with f replaced by $\beta(T)$) if and only if T satisfies the condition (C1). Then (a) follows from Theorem 3.1 and Lemma 4.1.

To prove (b), observe that, for $T \in \Gamma^{-\infty}(F)$ satisfying the condition (C1), the internal set

$$W_T := \{\zeta \in {}^*(\Gamma_0^\infty(F)) : {}^*A^\#\zeta = 0 \Longrightarrow ({}^*T, \zeta)_F = 0\}$$

coincides with ${}^*(\Gamma_0^\infty(F))$ by the transfer principle. Consequently, W_T does not depend on the choice of T satisfying (C1). We proceed as in the proof of Lemma 4.1. Let ψ_i $(i = 1, \ldots, \nu)$ be as in the proof of Lemma 4.1, and using these, define u_i $(i = 1, \ldots, \nu_0)$ as in (3.8). Define $G \in {}^*(\Gamma^\infty(E \boxtimes F^\star))$ by

$$G(x,y)\gamma(y) = \sum_{i=1}^{\nu_0} {}^*h_F(\gamma(y), \psi_i(y))u_i(x) \quad (x, y \in {}^*M)$$

for all $\gamma \in {}^*(\Gamma^\infty(F))$. Then the internal support of $G(x,\cdot)$ is $*$-compact for each $x \in {}^*M$. Furthermore, u defined by the formula (3.9) with f replaced by $\beta(T)$ is a solution of (4.3) in the sense of (4.4) and is represented as (4.5). □

Remark 4.1. Let $u \in {}^*(\Gamma^\infty(E))$ be a solution of (4.3) with ${}^*\|u\|_E \in {}^*\mathbb{R}$.

(1) If u happens to be standard, so that $u = {}^*v$ for some $v \in \Gamma^\infty(E)$ with $\|v\|_E \in \mathbb{R}$ (that is, $\|v\|_E < \infty$), then, by (4.4),

$$(v, A^\#\varphi)_E = (T, \varphi)_F \quad \text{for all} \quad \varphi \in \Gamma_0^\infty(F).$$

In this sense, we have $Av = T$.

(2) Consider the case where the support of $A^\#\varphi$ is compact for every $\varphi \in \Gamma_0^\infty(F)$. If there exists a generalized section $U \in \Gamma^{-\infty}(E)$ such that

$$(u, {}^*s)_E = (U, s)_E \quad \text{for all} \quad s \in \Gamma_0^\infty(E),$$

then, by (4.4),

$$(U, A^\#\varphi)_E = (T, \varphi)_F \quad \text{for all} \quad \varphi \in \Gamma_0^\infty(F).$$

In this sense, we have $AU = T$.

Finally, we have the following.

Theorem 4.3. *Let* $A : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(F)$ *be as in* §3. *Then there exists an internal* ${}^*C^\infty$ *section*

$$K \in {}^*(\Gamma^\infty(E \boxtimes (F^\star \otimes |\wedge_M|)))$$

with the following properties:

(a) *For each* $x \in {}^*M$, *the internal support of* $K(x,\cdot)$ *is* $*$*-compact.*

(b) *For each generalized section* $T \in \Gamma^{-\infty}(F)$ *satisfying the condition* (C1)*, the internal section* $u \in {}^*(\Gamma^\infty(E))$ *defined by*

$$u(x) = {}^*T(K(x,\cdot)) \quad (x \in {}^*M)$$

satisfies ${}^*\|u\|_E \in {}^*\mathbb{R}$ *and is a solution of* (4.3) *in the sense of* (4.4)*. Here* ${}^*T$ *is extended so as to be* ${}^*E_x$*-valued in a natural manner* $({}^*E_x = {}^*(\pi_E)^{-1}(x))$.

Proof. Let ψ_i $(i = 1, \ldots, \nu)$ and u_i $(i = 1, \ldots, \nu_0)$ be as in the proof of Theorem 4.2. We define

$$K \in {}^*(\Gamma^\infty(E \boxtimes (F^\star \otimes |\wedge_M|)))$$

by

$$K(x,y) = \sum_{i=1}^{\nu_0} u_i(x) \otimes ({}^h\psi_i \otimes {}^*dv_g)(y) \quad (x, y \in {}^*M).$$

Then the internal support of $K(x,\cdot)$ is $*$-compact for each $x \in {}^*M$. Put

$$u(x) = {}^*T(K(x,\cdot)) = \sum_{i=1}^{\nu_0} {}^*T({}^h\psi_i \otimes {}^*dv_g) u_i(x) = \sum_{i=1}^{\nu_0} ({}^*T, \psi_i)_F u_i(x)$$

for $x \in {}^*M$. Then $u \in {}^*(\Gamma^\infty(E))$ and ${}^*\|u\|_E \in {}^*\mathbb{R}$. Moreover, we have

$$(u, {}^*A^\# \psi_j)_E = ({}^*T, \psi_j)_F \quad (j = 1, \ldots, \nu). \tag{4.6}$$

(Note that both sides of (4.6) are 0 for $j = \nu_0 + 1, \ldots, \nu$.) Hence u satisfies (4.4) by the expression (3.11) of ${}^*\varphi \in {}^\sigma(\Gamma_0^\infty(F))$. □

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