

ENDOMORPHISMS OF THE PLANE SENDING LINEAR COORDINATES TO COORDINATES

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ABSTRACT. Let k be a field of characteristic zero. We show that an endomorphism of $k[X_1, X_2]$ which sends each linear coordinate to a coordinate is an automorphism of $k[X_1, X_2]$.

In [3] the authors raised the following question (referred to as Problem 1): let k be a field of characteristic zero and $A := k[X_1, \dots, X_n]$ the polynomial ring over k . Is every k -endomorphism of A which sends each coordinate of A to a coordinate of A an automorphism of A ? (A polynomial f of A is called a *coordinate* of A if there exist F_2, \dots, F_n in A such that $A = k[f, F_2, \dots, F_n]$.) Problem 1 was answered affirmatively for $n = 2$ in [3]. The case $n \geq 3$ remains open for arbitrary k . However, as was observed by Derksen, a negative answer to Problem 1 for algebraically closed fields would give a counterexample to the Jacobian Conjecture. More explicitly, he shows that ([3, Lemma 2.4]) if a k -endomorphism φ of A sends linear coordinates to coordinates, then $\det J\varphi(x) \neq 0$ for all $x \in k^n$, so $\det J\varphi \in k^*$ if k is algebraically closed! (A polynomial in A is called a *linear coordinate* if it is of the form $c_1X_1 + \dots + c_nX_n$ for some c_i in k , not all zero.) Based on Derksen's Lemma, Jelonek[7] gives a positive answer in any dimension to Problem 1 for algebraically closed fields! (He shows that the coordinate preservation property implies that φ is proper which together with $\det J\varphi \in k^*$ shows that φ is an automorphism of A .)

In [10], Mikhalev, Yu and Zolotykh, motivated by Derksen's Lemma, consider the following stronger version of Problem 1, referred to as Problem 2: is every endomorphism of A which sends every linear coordinate to a coordinate an automorphism? Of course (as observed above) in case that k is an algebraically closed field the hypothesis implies that $\det J\phi \in k^*$. So if the Jacobian Conjecture is true, the answer to Problem 2 is yes. Consequently, a negative solution would give a counter-example to the Jacobian Conjecture. For $n = 2$ the algebraically closed case was solved affirmatively in [3]. The case $n \geq 3$ remained open (if $k = \bar{k}$).

On the other hand, in [10] the authors show that the answer to Problem 2 is negative in case $n \geq 3$ and k is any non-algebraically closed field. However, the case $n = 2$ remained open for k a non-algebraically closed field.

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Now the remarkable point is that (in spite of the negative result for $n \geq 3$) the answer is yes for all fields k ($\text{char } k = 0$ in case $n = 2$). This is the main result of this paper (Theorem 1.1). The importance of this seemingly marginal result (compared with [3], [7] and [10]) is that it shows again the enormous difference between two-dimensional space and n -dimensional space, where $n \geq 3$.

Another example is the Markus-Yamabe Conjecture (Jacobian Conjecture for differential equations) which turned out to be true for all C^1 -vector fields on \mathbb{R}^2 (see [4], [5], [6]) and false for all $n \geq 3$ even for polynomial vectorfields on \mathbb{R}^n . Also the existence of exotic \mathbb{C}^n 's for $n \geq 3$ (i.e., smooth affine algebraic varieties of dimension n homeomorphic to \mathbb{C}^n but not algebraic isomorphic to \mathbb{C}^n), while they do not exist for $n = 2$ (see [2] and [8]), shows the difference between $n = 2$ and $n \geq 3$.

Summarizing: In light of the above considerations, the positive solution of Problem 2 in case $n = 2$ should inspire the reader to look for a (possible negative) solution of Problem 2 for $n \geq 3$ and k algebraically closed (which in turn has important implications for the Jacobian Conjecture!).

The proof of this result is based on a modified version of a result due to Rabier[11], where a useful characterization of a coordinate of $k[X, Y]$ is given. An easy proof of this modified version is given in Section 3.

1. THE MAIN RESULT

Throughout this section, k denotes a field of characteristic zero and $R := k[X, Y]$ the polynomial ring in two variables over k . The main result of this paper is the following.

Theorem 1.1. *Let φ be a k -endomorphism of R which sends each linear coordinate to a coordinate. Then φ is an automorphism of R .*

The proof of this theorem is based on the following proposition, a modified version of Theorem 2.1 in Rabier[11], whose proof will be given in Section 3 (see Proposition 3.1).

Proposition 1.2. *Let P be a coordinate of R with $n := \deg_Y P > 0$ and let $D := P_Y \partial_X - P_X \partial_Y$. Then $D^n(X) \in k$.*

Proof of Theorem 1.1. i) Let $P := \varphi(X)$. So P is a coordinate of R . Hence there exists an automorphism ψ of R with $\psi(X) = P$. Then $\psi^{-1}\varphi$ is an endomorphism of R sending X to X . Furthermore, if f is a linear coordinate of R , then $\varphi(f)$ is a coordinate, whence $\psi^{-1}\varphi(f)$ is a coordinate too. Consequently, replacing φ by $\psi^{-1}\varphi$ we may assume that $\varphi(X) = X$.

ii) So let $\varphi(X) = X$ and $g = \varphi(Y)$. The hypothesis implies that $g - cX = \varphi(Y - cX)$ is a coordinate of R for all $c \in k$. It follows easily that $n := \deg_Y g > 0$. Write $g = g_n Y^n + g_{n-1} Y^{n-1} + \cdots + g_1 Y + g_0$, with $g_i \in k[X]$ for all i . Since g is a coordinate, it follows from Corollary 14 of [9] that its Newton polygon is a triangle and, consequently, $g_n \in k^*$. Replacing Y by $Y - \frac{g_{n-1}}{n g_n}$, we may assume that $g_{n-1} = 0$. Replacing g by $g_n^{-1}g$, we may also assume that $g_n = 1$. In the remainder of this proof we will show that $n = 1$ (which concludes the proof, since if $n = 1$, then $\varphi = (X, g_1 Y + g_0)$ with $g_1 \in k^*$, so φ is an automorphism).

iii) So from now on we assume that $n \geq 2$, $g_n = 1$ and $g_{n-1} = 0$, and we will derive a contradiction. Since $g - cX$ is a coordinate of R for all $c \in k$, it follows

from Proposition 1.2 that

$$(1.1) \quad (D + c\partial_Y)^n(X) \in k$$

for all $c \in k$ where $D := g_Y\partial_X - g_X\partial_Y$. Let T be a new variable and consider the polynomial

$$H(T) := (D + T\partial_Y)^n(X) \in k[X, Y][T].$$

It follows from (1.1) and Lemma 1.3 below that $H(T) \in k[T]$. In particular, $A :=$ the T^{n-2} -coefficient of $H(T)$ belongs to k . From Proposition 2.1 of Section 2, we deduce that $g_X \in k$. Consequently, $g = \lambda X + a(Y)$, for some $\lambda \in k$ and $a(Y) \in k[Y]$. However, $g - cX$ is a coordinate of R for all $c \in k$. So, in particular, $a(Y) = g - \lambda X$ is a coordinate of R , a contradiction since $\deg a(Y) = n \geq 2$. \square

Lemma 1.3. *Let k be an infinite field and $f_0, \dots, f_n \in k[X, Y]$. If $f(T) := \sum f_p T^p$ (where T is a new variable) is such that $f(c) \in k$ for all $c \in k$, then $f_p \in k$ for all k , i.e., $f(T) \in k[T]$.*

Proof. Let $i, j \in \mathbf{N}$ with $i + j > 0$, and let $f_{p,i,j}$ be the coefficient of the monomial $X^i Y^j$ appearing in f_p . Then the hypothesis implies that $\sum f_{p,i,j} c^p = 0$ for all $c \in k$. Since k is infinite, this implies that $f_{p,i,j} = 0$ for all p and all $i, j \geq 0$ with $i + j > 0$, i.e., $f_p \in k$ for all p . \square

2. A TECHNICAL PROPOSITION

Throughout this section we have the following notation: $g := Y^n + g_{n-2}Y^{n-2} + \dots + g_1Y + g_0 \in k[X, Y]$ with $n \geq 2$ and $g_i \in k[X]$ for all i . Put $D := g_Y\partial_X - g_X\partial_Y$, $D(T) := D + T\partial_Y$, where T is a new variable and $\partial := \partial_Y$. Furthermore, define $D_i := \partial^i(g_Y)\partial_X - \partial^i(g_X)\partial_Y$, for all $i \geq 0$. By Leibniz' rule (and induction on k) one obtains

$$(2.1) \quad \partial^k D = \sum_{i=0}^k \binom{k}{i} D_i \partial^{k-i}$$

for all $k \geq 0$. Let $A :=$ the T^{n-2} -coefficient of $D(T)^n(X)$. Since $D(T)^n(X) = X(T)^{n-1}(G_Y)$, we obtain

$$(2.2) \quad A = \sum_{k=0}^{n-2} \partial^k D \partial^{n-k-2}(g_Y) = \sum_{k=0}^{n-2} \partial^k D \partial^{n-k-1}(g).$$

So by (2.1) and (2.2) we get

$$(2.3) \quad A = \sum_{k=0}^{n-2} \sum_{i=0}^k \binom{k}{i} D_i \partial^{n-i-1}(g).$$

Proposition 2.1. *If $A \in k$, then $g_X \in k$.*

Proof. i) We first show that $g'_j = 0$ for all $j \geq 1$. Suppose the contrary and let $j \geq 1$ be maximal such that $g'_j \neq 0$. We will show that the Y^j -coefficient of A is nonzero, contradicting that $A \in k$. Since $g_X = g'_j Y^j +$ lower order Y -terms, one easily verifies that $\deg_Y D_i \partial^{n-i-1}(g) \leq j$, for all $0 \leq i \leq n - 2$. By (2.3),

$\deg_Y A \leq j$. So to show that the Y^j -coefficient of A is nonzero it suffices to show that $\partial^j A \neq 0$. From (2.2) and (2.1) we deduce

$$\begin{aligned} \partial^j A &= \sum_{k=0}^{n-2} \partial^{j+k} D \partial^{n-k-1}(g) \\ &= \sum_{k=0}^{n-2} \left(\sum_{i=0}^{j+k} \binom{j+k}{i} D_i \partial^{j+k-i} \partial^{n-k-1}(g) \right) \\ &= \sum_{k=0}^{n-2} \sum_{i=0}^{j+k} \binom{j+k}{i} D_i \partial^{j+n-i-1}(g). \end{aligned}$$

Furthermore, it is left to the reader to verify (using again that $g_X = g'_j Y^j +$ lower order Y -terms) that

$$(2.4) \quad D_i(\partial^{j+n-i-1}(g)) = 0$$

for $i \neq j$, and

$$(2.5) \quad D_j(\partial^{n-1}(g)) = -j!n!g'_j.$$

Consequently, $\partial^j A = \sum_{k=0}^{n-2} \binom{j+k}{j} (-j!)n!g'_j \neq 0$, as desired.

ii) By i) we get $g = a(Y) + g_0(X)$, where $a(Y) = Y^n +$ lower order Y -terms. Consequently, $D = a'(Y)\partial_X - g'_0(X)\partial_Y$ and by (2.2)

$$\begin{aligned} A &= \sum_{k=0}^{n-2} \partial^k D a^{(n-k-1)}(Y) \\ &= \sum_{k=0}^{n-2} \partial^k (-g'_0(X) a^{(n-k)}(Y)) \\ &= -g'_0(X)(n-1)a^{(n)}(Y) \\ &= -(n-1)(n!)g'_0(X). \end{aligned}$$

Since $n \geq 2$ and $A \in k$, this implies that $g'_0 \in k$, whence $g_X \in k$. □

3. A MODIFICATION OF A RESULT OF RABIER

In [11], Rabier proved the following result: let $P \in R := k[X_1, X_2]$ with $\deg P = n \geq 1$ and put $D_P := P_{X_2}\partial_{X_1} - P_{X_1}\partial_{X_2}$. If P is a coordinate of R , then $(D_P)^n(X_1)$ and $(D_P)^n(X_2)$ belong to k and not both are zero. Furthermore, it is shown in [11] that the converse is true as well. In the proof of Theorem 1.1 we need the following modified version of the result mentioned above.

Proposition 3.1. *Let P be a coordinate of R with $\deg_{X_2} P = n \geq 1$. Then $(D_P)^n(X_1) \in k^*$.*

Proof. Put $F_1 := P$ and let (F_1, F_2) be an automorphism of R with $\det J(F_1, F_2) = 1$. Then $D_P = -\frac{\partial}{\partial F_2}$ and if (G_1, G_2) denotes the inverse of (F_1, F_2) , then $X_1 = G_1(F_1, F_2)$. So

$$(3.1) \quad (D_P)^n(X_1) = (-1)^n \left(\frac{\partial}{\partial F_2} \right)^n G_1(F_1, F_2).$$

By (1.6) and (1.8) of [1] we have

$$n = \deg_{X_2} F_1 = \deg_{Y_2} G_1(Y_1, Y_2) = \deg_{F_2} G_1(F_1, F_2)$$

or

$$(3.2) \quad n = \deg_{F_2} G_1(F_1, F_2).$$

From (3.1) we deduce that $(D_P)^n(X_1) = (-1)^n n! c$, where c is the coefficient of F_2^n in $G_1(F_1, F_2)$. By Corollary 1.4 [1], $c \in k^*$. So the desired result follows from (3.2). \square

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