UNIVERSAL UNIFORM EBERLEIN COMPACT SPACES

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ABSTRACT. A universal space is one that continuously maps onto all others of its own kind and weight. We investigate when a universal Uniform Eberlein compact space exists for weight \( \kappa \). If \( \kappa = 2^{\omega_1} \), then they exist whereas otherwise, in many cases including \( \kappa = \omega_1 \), it is consistent that they do not exist. This answers (for many \( \kappa \) and consistently for all \( \kappa \)) a question of Benyamini, Rudin and Wage of 1977.

1. INTRODUCTION

Let \( \mathcal{C} \) be a class of compact Hausdorff spaces. The weight of a space \( X \) is the least cardinality of a base for \( X \). A universal element of \( \mathcal{C} \) for weight \( \kappa \) is a space \( X \in \mathcal{C} \) with weight \( \kappa \) such that every \( Y \in \mathcal{C} \) of weight \( \leq \kappa \) is a continuous image of \( X \). In topology, the two most famous examples of universal elements are: the Cantor space \( 2^\omega \) is a universal compact space for weight \( \omega \) and, under the Continuum Hypothesis, \( \beta \omega \setminus \omega \) is a universal compact space for weight \( \omega_1 \). Regarding the second example, we mention that Shelah \cite{Sh84, Sh90} has shown that it is consistent that there is no universal boolean algebra (one that contains copies of all others as subalgebras) of size \( \omega_1 \); hence by duality, it follows that it is consistent that there is no universal compact space of weight \( \omega_1 \).

An Eberlein compact space (abbreviated EC) is one that is homeomorphic to a weakly compact subspace of a Banach space. A Uniform Eberlein compact space (abbreviated UEC) is one that is homeomorphic to a weakly compact subspace of a Hilbert space. EC spaces have been extensively studied by analysts and topologists; for a good introduction see the survey article of Negrepontis \cite{Ng84}. Benyamini, Rudin and Wage \cite{BRW77} raised the questions: for which \( \kappa \) do there exist universal EC and universal UEC spaces for weight \( \kappa \)? Argyros and Benyamini \cite{Ab87} have shown that if \( \kappa^{\omega_1} = \kappa \) or \( \kappa = \omega_1 \), then a universal EC for weight \( \kappa \) does not exist, whereas if \( \kappa \) is a strong limit of countable cofinality, then it does exist. Hence under GCH, they exist iff \( \text{cf}(\kappa) = \omega \). In section 3 we prove that if \( \kappa = 2^{\omega_1} \), then a universal UEC space exists for weight \( \kappa \) (hence under GCH, universal UEC spaces exist for all weights \( \kappa \)). In section 4 we note that it follows from results of Shelah \cite{Sh84, Sh90} on universal graphs that it is consistent that universal UEC spaces...
Proof. Assume that \( f \) is a closed subspace of \( X \). We denote the clopen algebra of all clopen subsets of \( X \) by \( CO(X) \). A boolean space is a compact Hausdorff space such that \( CO(X) \) is a basis. If \( B \) is a boolean algebra, then \( st(B) \) is the space of all ultrafilters of \( B \). The weight of \( st(B) \) is the cardinality or size of \( B \). When we say “duality”, we refer to the well-known equivalences of boolean spaces and continuous maps and boolean algebras and homomorphisms.

2. Uniform Eberleins and c-algebras

Let \( \alpha \kappa = \kappa \cup \{\infty\} \) be the one-point compactification of the discrete space \( \kappa \) and let \( \alpha \kappa^\omega \) be the \( \omega \)th power of \( \alpha \kappa \). The following theorem is a key to our investigation.

**Theorem 2.1** (Benyamini, Rudin and Wage [BRW77]). \( X \) is a UEC space of weight \( \leq \kappa \) iff \( X \) is a Hausdorff continuous image of a closed subspace of \( \alpha \kappa^\omega \).

So, if there is a universal UEC space for weight \( \kappa \), then there is one that is a closed subspace of \( \alpha \kappa^\omega \). Moreover, to construct a universal UEC for weight \( \kappa \), it suffices to find a closed subspace of \( \alpha \kappa^\omega \) that continuously maps onto all other closed subspaces. We decide to work in the realm of boolean algebras.

A c-algebra is a boolean algebra \( B \) which has a generating set \( \bigcup_{n<\omega} B_n \) satisfying

a) each \( B_n \) consists of elements pairwise disjoint and \( B_n \cap B_m = \emptyset \) for \( n \neq m \) and

b) the nice property that \( \forall F \neq 1 \) for each finite \( F \subset \bigcup_{n<\omega} B_n \). When we mention a c-algebra \( B \), we always have in mind a fixed generating set which we denote by \( \operatorname{gen}(B) = \bigcup_{n<\omega} B_n \) and which satisfies a) and b) and we let \( \pi : \operatorname{gen}(B) \to \omega \) be defined by \( \pi(x) = n \) where \( x \in B_n \).

The next theorem was, in essence, communicated to the author by Petr Simon. We say that a family \( B \) of subsets of a space \( X \) is \( T_\theta \)-separating if whenever \( x \neq y \) are points of \( X \), then there exists \( B \in B \) such that \( B \) contains exactly one of \( x, y \). By duality, a generating set \( G \) of a boolean algebra \( B \) corresponds to a \( T_\theta \)-separating family of clopen subsets of \( X = st(B) \).

**Theorem 2.2.** \( X \) is homeomorphic to a closed subspace of \( \alpha \kappa^\omega \) iff \( X \) is a boolean space and \( CO(X) \) is a c-algebra of size at most \( \kappa \).

**Proof.** Assume that \( X \) is a closed subspace of \( \alpha \kappa^\omega \). For each \( n \) and for each \( \alpha < \kappa \), put \( B_\alpha^n = \{ f \in X : f(n) = \alpha \} \). The family \( B = \{ B_\alpha^n : \text{all } n \text{ and all } \alpha \} \) generates \( CO(X) \). Fix a point \( x \in X \). For each \( n \) such that \( x(n) = \infty \) put \( B_{3n} = \{ B_\alpha^n : \alpha < \kappa \} \) and for each \( n \) such that \( x(n) \neq \infty \) put \( B_{3n+1} = \{ X \setminus B_\alpha^n : x(n) = \alpha \} \) and \( B_{3n+2} = \{ B_\beta^n : \beta \neq \alpha \} \). Then \( \bigcup_{n<\omega} B_n \) generates \( CO(X) \), has the nice property, each \( B_n \) consists of elements pairwise disjoint, and we can thin each \( B_n \) to get \( \{ B_n : n < \omega \} \) a pairwise disjoint family; so, \( CO(X) \) is a c-algebra of size at most \( \kappa \).

Assume \( X \) is a boolean space and that \( CO(X) \) is a c-algebra of size at most \( \kappa \) and let \( \bigcup_{n<\omega} B_n \) be a generating set of \( CO(X) \) witnessing this. For each \( n \), list \( B_n \) as \( \{ B_\alpha^n : \alpha \in S_n \subset \kappa \} \). Define \( \phi : X \to \alpha \kappa^\omega \) by \( \phi(x)(n) = \infty \) if \( x \notin \bigcup_{\alpha \in S_n} B_\alpha^n \).
and \( \phi(x)(n) = \alpha \) if \( x \in B_n^\alpha \). \( \phi \) is an embedding because \( \bigcup_{n<\omega} B_n \) is a \( T_0 \)-separating family.

We require some c-algebra notions. A c-algebra \( A \) is a c-subalgebra of \( B \), denoted by \( A < B \), if \( A \) is a boolean subalgebra of \( B \) and \( A_n \subset B_n \) for each \( n \). A function \( i : A \to B \) between two c-algebras is a c-embedding if \( i \) is an embedding of boolean algebras and \( i[A_n] \subset B_n \) for each \( n \); if, in addition, \( i[A_n] = B_n \) for each \( n \), then \( i \) is a c-isomorphism. \( U \) is a universal c-algebra of size \( \kappa \) if every c-algebra of size \( \leq \kappa \) can be c-embedded into \( U \).

A nut \( N \) of a c-algebra \( B \) is an element of \( B \) that is given by a description denoted by \( (N^+, N^-) \) so that \( N = \bigwedge N^+ - \bigvee N^- \), where \( N^+ \) and \( N^- \) are disjoint finite subsets of \( \text{gen}(B) \). Since \( \text{gen}(B) \) generates \( B \), knowledge of precisely which nuts are 0 completely determines (up to c-isomorphism) the c-algebra \( B \). A fact that we will use in section 3 is that since a c-algebra \( B \) has the nice property, if \( N \) is a nut of \( B \) and \( N = 0 \), then \( N^+ \neq \emptyset \).

We say that \( A \) is a closed c-subalgebra of a c-algebra \( B \), denoted by \( A <_{c} B \), if \( A < B \) and

1. whenever \( N \) is a nut of \( A \) and \( F \) is a finite subset of \( \text{gen}(B) \) and \( N \leq \bigvee F \), then there exists a finite subset \( G \) of \( \text{gen}(A) \) such that \( N \leq \bigvee G \) and \( \pi[G] = \pi[F] \),

2. whenever \( N \) is a nut of \( A \) and \( F \) is a finite subset of \( \text{gen}(B) \) and \( N \wedge \bigvee F \neq 0 \), then there exists a finite subset \( G \) of \( \text{gen}(A) \) such that \( N \wedge \bigvee G \neq 0 \) and \( \pi[G] = \pi[F] \).

To use closed c-subalgebras (i.e. elementarity) in our proof of Lemma 3.1 was suggested to the author by Alan Dow. Three important properties of \( A <_{c} B \) are:

- \( A < B \) imply that there exists \( C <_{c} B \) such that \( A < C \) and \( |C| = |A| \),
- \( A <_{c} B \) and \( A < C < B \) imply that \( A <_{c} C \),
- if \( A_\alpha < A_\beta \) for \( \alpha < \beta < \kappa \) and for each \( \alpha < \kappa \), \( A_\alpha <_{c} B \), then \( \bigcup_{\alpha<\kappa} A_\alpha <_{c} B \).

To get a universal UEC space for certain weights \( \kappa \), we will construct a universal c-algebra \( U \) of size \( \kappa \), then, by Theorem 2.2 and Theorem 5.1 \( \text{st}(U) \) will be our universal UEC. Whether this is necessary for the existence of universal UEC spaces of weight \( \kappa \) is not known. Recent research of Dzamonja [Dz08] gives sufficient conditions for the non-existence of universal c-algebras of weight \( \kappa \) (in the absence of GCH).

3. Existence of universal uniform Eberleins

It is convenient and economical to decree that \( \emptyset \) is a c-algebra and that for any c-algebra \( B \), \( \emptyset <_{c} B \) and \( \emptyset : \emptyset \to B \) is a c-embedding.

**Lemma 3.1** (Amalgamation Lemma). Let \( B \) and \( C \) be c-algebras. If \( A <_{c} C \) and \( i : A \to B \) is a c-embedding, then there exists a c-algebra \( E \) such that \( B < E \), \( |E| \leq |B| + |C| \), and there exists a c-embedding \( j : C \to E \) such that \( j \upharpoonright A = i \).

**Proof.** Without loss of generality we assume that \( i \) is an inclusion, so \( A < B \) (therefore, \( A_n \subset B_n \) for each \( n \)) and we assume that \( (\text{gen}(C) \setminus \text{gen}(A)) \cap B = \emptyset \). Define, for each \( n \), \( E_n = (C_n \setminus A_n) \cup B_n \). Our goal is to define a c-algebra structure \( E \) with \( \text{gen}(E) = \bigcup_{n<\omega} E_n \) which simultaneously extends the c-algebra structure of
be the ideal in $B \ast D$ generated by $I_1 \cup I_2$ where $I_1 = \{(x, y) : x \in B_n \setminus A_n$ and $y \in C_n \setminus A_n$, for some $n\}$ and $I_2 = \{(N, M) : N$ is a nut of $A$ and $M$ is a nut of $D$ and $N \cap M = 0 \in C\}$. Put $E = B \ast D/I$ and let $\phi : B \ast D \to B \ast D/I$ be the quotient homomorphism. $E$ is generated by $\bigcup_n E_n$ where $E_n = \{\phi(x, 1) : x \in B_n \} \cup \{\phi(1, y) : y \in C_n \setminus A_n\}$. $I_1$ ensures that, for each $n$, $E_n$ consists of elements pairwise disjoint.

Claim 1. $E$ is a c-algebra.

Proof of Claim. We must check that $E$ has the nice property. Striving for a contradiction, assume that $\bigvee_{i<n} \phi(x_i, 1) \lor \bigvee_{j<m} \phi(1, y_j) = 1$ where $x_i \in \text{gen}(B)$ and $y_j \in \text{gen}(D)$. Choose $p,q < \omega$ such that

\[(*) \quad \bigvee_{i<n} (x_i, 1) \lor \bigvee_{j<m} (1, y_j) \lor \bigvee_{k<p} (N_k, M_k) \lor \bigvee_{l<q} (z_l, w_l) = 1\]

where each $(N_k, M_k) \in I_2$ and $(z_l, w_l) \in I_1$. Since $N_k \cap M_k = 0$ in $C$ and $C$ has the nice property, we can choose $r_k \in (N_k \cap M_k)^+ = N_k^+ \cup M_k^+$. So $(N_k, M_k) \leq (r_k, 1)$ if $r_k \in N_k^+$ or $(N_k, M_k) \leq (1, r_k)$ if $r_k \in M_k^+$. As $C$ has the nice property, $c = \bigvee_{j<m} y_j \lor \bigvee_{k<p} \{r_k : k < p$ and $r_k \in M_k^+\} \lor \bigvee_{l<q} w_l < 1$. As $B$ has the nice property, $b = \bigvee_{i<n} x_i \lor \bigvee_{k<p} \{r_k : k < p$ and $r_k \in N_k^+\} \lor \bigvee_{l<q} z_l < 1$. Thus, $(1 - b, 1 - c) \neq 0$ but is disjoint from the left-hand side of equation $(*)$. 

Claim 2. $B < E$ in the form $\{\phi(x, 1) : x \in B\}$.

Proof of Claim. By virtue of the free product, if $N$ is a nut of $B$ and $N \neq 0$, then $\phi(N, 1) = 0$. We must show that if $N$ is a nut of $B$ and $N \neq 0$, then $\phi(N, 1) \neq 0$. Assume not, and let $N \neq 0$ be a nut of $B$ such that

\[(** \quad (N, 1) \leq \bigvee_{i<n} (N_i, M_i) \lor \bigvee_{j<m} (x_j, y_j)\]

where each $(N_i, M_i) \in I_2$ and $(x_j, y_j) \in I_1$ and $n + m$ is the least $k$ such that there exist $k$ elements of $I_1 \cup I_2$ whose join is $\geq (P, 1)$ for some nut $P$ of $B$ with $P \neq 0$. By minimality of $n + m$, we get that $N \leq \bigwedge_{i<n} N_i \land \bigwedge_{j<m} x_j$ in $B$ (otherwise, there exists $i$ such that $N - N_i \neq 0$ or $N - x_i \neq 0$ and either possibility yields a non-$0$ nut $P$ of $B$ such that $(P, 1)$ is $\leq$ the join of $1$ less element on the right-hand side of $(*)$. Also, $y = \bigvee_{i<n} M_i \lor \bigvee_{j<m} y_j = 1$ in $D$ or else $(N, 1 - y) \neq 0$ but is disjoint from the right-hand side of $(** \quad (N, 1) \leq \bigvee_{i<n} (N_i, M_i) \lor \bigvee_{j<m} (x_j, y_j)\)$. Since each $i < n$, $N_i \leq M_i'$ (we use $'$ for complement) in $C$, we get that $\bigwedge_{i<n} N_i \leq \bigwedge_{i<n} M_i' \leq \bigvee_{j<m} y_j$ in $C$. As $A < C$ and $\bigwedge_{i<n} N_i$ is a nut of $A$, we can choose a finite $F \subseteq \text{gen}(A)$ such that $\pi[F] = \pi[\{y_j : j < m\}]$ and $\bigwedge_{i<n} N_i \leq \bigvee F$ in $A$. But, then $0 \neq N \leq \bigvee F \land \bigwedge_{j<m} x_j$ in $B$, which is a contradiction since for every $z \in F$ there exists $j < m$ such that $z$ and $y_j$ (and therefore $x_j$) are
in the same column $B_n$ of $B$ with $z \in A_n$ and $x_j \in B_n \setminus A_n$ and therefore $z \cap x_j = 0$ in $B$; hence $\bigvee F \land \bigwedge_{j<m} x_j = 0$ in $B$.

\textbf{Claim 3.} $C \leq E$ in the form $\{ \phi(x,1) : x \in A \} \cup \{ \phi(1,y) : y \in D \}$.

\textbf{Proof of Claim.} By virtue of the free product and $I_2$, if $N$ is a nut of $A$ and $M$ is a nut of $D$ and $N \land M = 0$, then $\phi(N,M) = 0$. We must show that if $N$ is a nut of $A$ and $M$ is a nut of $D$ and $N \land M \neq 0$, then $\phi(N,M) \neq 0$. Assume not and let $N$ be a nut of $A$ and let $M$ be a nut of $D$ such that $N \land M \neq 0$ in $C$ and $(N,M) = \bigvee_{i<n} (N_i,M_i) \land \bigvee_{j<m} (x_j,y_j)$ where each $(N_i,M_i) \in I_2$ and $(x_j,y_j) \in I_1$. Since $N \land M \neq 0$ in $C$ and for each $i < n$, $N_i \land M_i = 0$ in $C$, we get that $(N,M) - \bigvee_{i<n} (N_i,M_i) \neq 0$. As

\[ (N,M) - \bigvee_{i<n} (N_i,M_i) = \bigvee_{k<p} (P_k,Q_k) \leq \bigvee_{j<m} (x_j,y_j) \]

where each $P_k$ is a nut of $A$ and $Q_k$ is a nut of $D$, we get that there exists $k < p$ with $(P_k,Q_k) \neq 0$ and $(P_k,Q_k) \leq \bigvee_{j<m} (x_j,y_j)$. So, without loss of generality we can assume that $(N,M) \leq \bigvee_{j<m} (x_j,y_j)$. As before, let $m$ be the least $k$ such that there exist $k$ elements of $I_1$ whose join is $\geq (P,Q)$ for some nut $P$ of $A$ and for some nut $Q$ of $D$ with $P \land Q \neq 0$ in $C$. By minimality of $m$, $0 \neq N \land M \leq \bigwedge_{j<m} y_j$.

Therefore, $N \land \bigwedge_{j<m} y_j \neq 0$. As $A \leq C$ and $N$ is a nut of $A$, we can choose a finite $F \subset \text{gen}(A)$ such that $\pi[F] = \pi[[y_j : j < m]]$ and $N \land \bigwedge_{j<m} F \neq 0$. But, $N \leq \bigvee_{j<m} x_j$ and $\bigvee_{j<m} x_j \land \bigwedge_{j<m} F = 0$ which is a contradiction.

This completes the proof of our lemma.

The proofs of the following two results are standard procedures in model theory, but due to the asymmetry of our assumptions (we mix $A < B$ and $A < c B$) we were unable to find a theorem to quote, and so we present the proofs.

\textbf{Proposition 3.2.} Let $\lambda = \lambda^{<\kappa}$ and let $B$ be a $c$-algebra of size $< \kappa$. Then there exists a $c$-algebra $D$ of size $\lambda$ such that $B < D$ and such that every $c$-embedding of any $c$-algebra $A$ of size $< \kappa$ into $D$ can be extended to a $c$-embedding of $C$, if $C$ is any $c$-algebra of size $< \kappa$ with $A < c C$.

\textbf{Proof.} List all triples $(X,Y,i)$ where $Y$ is a $c$-algebra with $Y_n \subset \kappa \times \{ n \}$ for each $n$, $X \leq c Y$ and $i$ is an injection of $X$ into $\lambda \times \omega$ as $(X_\alpha,Y_\alpha,i_\alpha)_{\alpha < \kappa < \lambda}$ so that for each $\alpha$, $i_\alpha[X_\alpha] \subset \alpha \times \omega$. In our listing we allow the degenerate cases where $X = i = \emptyset$ in order to start our embeddings. We can do this listing because $\lambda = \lambda^{<\kappa}$.

We will build a $c$-algebra structure $D$ with $\text{gen}(D) = \lambda \times \omega$ by recursively constructing $D_\alpha$ with $\text{gen}(D_\alpha) = \lambda_\alpha \times \omega$ such that for $\alpha < \beta < \lambda$, $\alpha \leq \lambda_\alpha \leq \lambda_\beta$ and $D_\alpha < D_\beta$. To begin, let $\lambda_0 = |B|$ and make $D_0$ $c$-isomorphic to $B$. If $\alpha$ is a limit, put $\lambda_\alpha = \sup \{ \lambda_\beta : \beta < \alpha \}$ and $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$. If $\alpha = \beta + 1$ is a successor, then we have $D_\beta$ with $\text{gen}(D_\beta) = \lambda_\beta \times \omega$. If $i_\beta$ is not a $c$-embedding of $X_\beta$ into $D_\beta$, then put $\lambda_\alpha = \lambda_\beta$ and $D_\alpha = D_\beta$. If $i_\beta$ is a $c$-embedding of $X_\beta$ into $D_\beta$, then apply the Amalgamation Lemma [4.1] with $A = X_\beta$, $B = D_\beta$, $C = Y_\beta$ and $i = i_\beta$ to yield
a \( c \)-algebra \( E \) such that \( D_\beta < E, |E| \leq |D_\beta| + |Y_\beta| \) and \( i_\beta \) extends to \( Y_\beta \). Put \( D_\alpha = E \) and \( \lambda_\alpha = \) the first ordinal > \( \lambda_\beta \) that can accommodate the extra elements of \( \text{gen}(E) \).

Now, let \( D = \bigcup_{\alpha < \lambda} D_\alpha \). If \( A <_c C, |C| < \kappa \) and \( i : A \to D \) is a \( c \)-embedding, then we can identify \( (A,C,i) \) with a \( (X_\beta,Y_\beta,i_\beta) \) for some \( \beta < \lambda \) and at stage \( \alpha = \beta + 1 \), we extended \( i \) to \( Y_\beta = C \).

**Theorem 4.1.**

**Proof.** If \( \kappa = \kappa^c \), then there exists a universal \( c \)-algebra \( U \) of size \( \kappa \).

**Proof.** If \( \kappa = \kappa^c \), then let \( U \) be the \( D \) of the previous proposition with \( \lambda = \kappa \) and \( B = \emptyset \).

Otherwise, \( \kappa \) is a singular strong limit cardinal, so let \( \kappa = \text{sup}\{\kappa_\alpha : \alpha < \text{cf}(\kappa)\} \) where \( \alpha < \beta \) implies \( \kappa_\alpha < \kappa_\beta \), \( \alpha \) a limit implies \( \kappa_\alpha = \text{sup}\{\kappa_\beta : \beta < \alpha\} \), and \( \kappa_{\alpha+1} = 2^{\kappa_\alpha} \). By induction on \( \alpha < \text{cf}(\kappa) \), we construct \( c \)-algebras \( U_\alpha \) with, for each \( \alpha, U_\alpha \subseteq \kappa_\alpha \times \{\alpha\} \) such that \( \alpha < \beta \) implies \( U_\alpha < U_\beta \) and \( \alpha \) a limit implies \( U_\alpha = \bigcup_{\beta < \alpha} U_\beta \). For \( \alpha = \beta + 1 \), we apply the preceding proposition with \( B = U_\beta \) and put \( U_\alpha = \) the \( D \) of that proposition. Now, let \( U = \bigcup_{\alpha < \text{cf}(\kappa)} U_\alpha \).

In either case, to show that our \( U \) is universal, let \( A \) be a \( c \)-algebra of size \( \leq \kappa \). Write \( A \) as \( \bigcup_{\alpha < \text{cf}(\kappa)} A_\alpha \) where for each \( \alpha, A_\alpha = A, |A_\alpha| < \kappa, \alpha < \beta \) implies \( A_\alpha < A_\beta \) and \( \alpha \) a limit implies \( A_\alpha = \bigcup_{\beta < \alpha} A_\beta \). We then have that for \( \beta < \alpha, A_\beta <_c A_\alpha \). Finally, an induction up to \( \text{cf}(\kappa) \), using the preceding proposition, will embed \( A \) into \( U \).

4. **Non-existence of universal Uniform Eberleins**

\( G \subseteq [X]^2 \) (the doubletons of \( X \)) is called a **graph** on \( X \). A graph \( H \) on a subset \( Y \) of \( X \) is called a **subgraph** of \( G \) if \( H = G \cap [Y]^2 \). A graph \( H \) on a set \( Y \) **embeds** into a graph \( G \) on a set \( X \) if there exists a function \( i : Y \to X \) such that for all \( x,y \in Y, \{x,y\} \in H \iff \{i(x),i(y)\} \in G \). A **universal graph** of size \( \kappa \) is a graph \( G \) on a set \( X \) of size \( \kappa \) such that every graph on any set of size \( \leq \kappa \) embeds into \( G \).

**Theorem 4.2.** Let \( \mathcal{C} \) be a class of compact Hausdorff spaces containing all closed subspaces of \( \alpha \kappa \times \alpha \kappa \). Then, if there exists a universal element in \( \mathcal{C} \) for weight \( \kappa \), there exists a universal graph of size \( \kappa \).

**Proof.** Let \( X \) be a universal element in \( \mathcal{C} \) for weight \( \kappa \). Let \( G \) be the intersection graph on \( CO(X) \), i.e., for \( x \neq y \) in \( CO(X), \{x,y\} \in G \) iff \( x \cap y \neq \emptyset \). We claim that \( G \) is a universal graph of size \( \kappa \). To prove this, let \( H \) be any graph on an \( S \subseteq \kappa \). Consider the following closed subspace of \( \alpha \kappa \times \alpha \kappa, Y = \alpha \kappa \times \{\infty\} \cup \{\infty\} \times \alpha \kappa \cup \{(\alpha,\beta),\beta,\alpha) : \{\alpha, \beta) \in H \). Let \( \phi : X \to Y \) be a continuous surjection. For each \( \alpha < \kappa \), put \( B_\alpha = [\alpha \kappa \times \{\alpha\} \cup \{\alpha\} \times \alpha \kappa] \cap Y \). For each \( \alpha < \kappa, B_\alpha \subseteq CO(Y) \) and \( B_\alpha \cap B_\beta \neq \emptyset \) iff \( \beta, \alpha \in H \). So, the mapping \( i : S \to CO(X) \) defined by \( i(\alpha) = \phi^{-1}[B_\alpha] \) embeds \( H \) into \( G \).

**Corollary 4.2.** If \( V \) is a model with \( \lambda^{<\lambda} = \lambda < \kappa < \mu \) and \( P \) is a Cohen forcing that adds \( \mu \) Cohen subsets of \( \lambda \), then in the model \( V^P \) there is no universal UEC in the weight \( \kappa \) for any \( \kappa \) with \( \lambda < \kappa < \mu \). In particular, if we add \( \omega_2 \) Cohen reals to \( V \), then there is no universal UEC in the weight \( \omega_1 \).
Proof. Shelah \cite{Sh90} has shown that there is no universal graph of size $\kappa$ in this model.

An additional result, just following from cardinal arithmetic, is the result of Kojman and Shelah \cite{KS92}, that if $\text{cf}(2^\kappa) \leq \kappa < 2^\lambda$, then there does not exist a universal graph of size $\kappa$; hence, there would be no universal UEC for this weight $\kappa$.

In conclusion, we mention one open problem: Does a universal UEC for weight $\kappa$ not exist if $\kappa$ is a singular cardinal which is not a strong limit?

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