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UNIVERSAL UNIFORM EBERLEIN COMPACT SPACES

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ABSTRACT. A universal space is one that continuously maps onto all others of its own kind and weight. We investigate when a universal Uniform Eberlein compact space exists for weight κ. If $\kappa = 2^{<\kappa}$, then they exist whereas otherwise, in many cases including $\kappa = \omega_1$, it is consistent that they do not exist. This answers (for many κ and consistently for all κ) a question of Benyamini, Rudin and Wage of 1977.

1. INTRODUCTION

Let $\mathcal{C}$ be a class of compact Hausdorff spaces. The **weight** of a space X is the least cardinality of a base for X. A **universal element** of $\mathcal{C}$ for weight κ is a space $X \in \mathcal{C}$ with weight κ such that every $Y \in \mathcal{C}$ of weight $\leq \kappa$ is a continuous image of X. In topology, the two most famous examples of universal elements are: the Cantor space 2^ω is a universal compact space for weight ω and, under the Continuum Hypothesis, $\beta\omega \setminus \omega$ is a universal compact space for weight ω_1. Regarding the second example, we mention that Shelah [Sh84], [Sh90] has shown that it is consistent that there is no universal boolean algebra (one that contains copies of all others as subalgebras) of size ω_1; hence by duality, it follows that it is consistent that there is no universal compact space of weight ω_1.

An Eberlein compact space (abbreviated EC) is one that is homeomorphic to a weakly compact subspace of a Banach space. A Uniform Eberlein compact space (abbreviated UEC) is one that is homeomorphic to a weakly compact subspace of a Hilbert space. EC spaces have been extensively studied by analysts and topologists; for a good introduction see the survey article of Negrepontis [Ng84]. Benyamini, Rudin and Wage [BRW77] raised the questions: for which κ do there exist universal EC and universal UEC spaces for weight κ? Argyros and Benyamini [AB87] have shown that if $\kappa^\omega = \kappa$ or $\kappa = \omega_1$, then a universal EC for weight κ does not exist, whereas if κ is a strong limit of countable cofinality, then it does exist. Hence under GCH, they exist iff $\mathrm{cf}(\kappa) = \omega$. In section 3 we prove that if $\kappa = 2^{<\kappa}$, then a universal UEC space exists for weight κ (hence under GCH, universal UEC spaces exist for all weights κ). In section 4 we note that it follows from results of Shelah [Sh84], [Sh90] on universal graphs that it is consistent that universal UEC spaces

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(in fact, more classes of spaces) for weight κ do not exist for various cardinals κ including $\kappa = \omega_1$.

We denote the clopen algebra of all clopen subsets of X by $\boldsymbol{CO(X)}$. A **boolean** space is a compact Hausdorff space such that $CO(X)$ is a basis. If B is a boolean algebra, then $\mathrm{st}(B)$ is the space of all ultrafilters of B. The weight of $\mathrm{st}(B)$ is the cardinality or size of B. When we say "duality", we refer to the well-known equivalences of boolean spaces and continuous maps and boolean algebras and homomorphisms.

2. Uniform Eberleins and c-algebras

Let $\alpha\kappa = \kappa \cup \{\infty\}$ be the one-point compactification of the discrete space κ and let $\alpha\kappa^\omega$ be the ωth power of $\alpha\kappa$. The following theorem is a key to our investigation.

Theorem 2.1 (Benyamini, Rudin and Wage [BRW77]). *X is a UEC space of weight $\leq \kappa$ iff X is a Hausdorff continuous image of a closed subspace of $\alpha\kappa^\omega$.*

So, if there is a universal UEC space for weight κ, then there is one that is a closed subspace of $\alpha\kappa^\omega$. Moreover, to construct a universal UEC for weight κ, it suffices to find a closed subspace of $\alpha\kappa^\omega$ that continuously maps onto all other closed subspaces. We decide to work in the realm of boolean algebras.

A **c-algebra** is a boolean algebra B which has a generating set $\bigcup\limits_{n<\omega} B_n$ satisfying

a) each B_n consists of elements pairwise disjoint and $B_n \cap B_m = \emptyset$ for $n \neq m$ and

b) the *nice* property that $\bigvee F \neq 1$ for each finite $F \subset \bigcup\limits_{n<\omega} B_n$. When we mention a c-algebra B, we always have in mind a fixed generating set which we denote by $\mathbf{gen}(\boldsymbol{B}) = \bigcup\limits_{\boldsymbol{n<\omega}} \boldsymbol{B_n}$ and which satisfies a) and b) and we let $\pi : \mathrm{gen}(B) \to \omega$ be defined by $\boldsymbol{\pi(x) = n}$ where $x \in B_n$.

The next theorem was, in essence, communicated to the author by Petr Simon. We say that a family $\mathcal{B}$ of subsets of a space X is $\boldsymbol{T_0}$**-separating** if whenever $x \neq y$ are points of X, then there exists $B \in \mathcal{B}$ such that B contains exactly one of x,y. By duality, a generating set G of a boolean algebra B corresponds to a T_0-separating family of clopen subsets of $X = \mathrm{st}(B)$.

Theorem 2.2. *X is homeomorphic to a closed subspace of $\alpha\kappa^\omega$ iff X is a boolean space and $CO(X)$ is a c-algebra of size at most κ.*

Proof. Assume that X is a closed subspace of $\alpha\kappa^\omega$. For each n and for each $\alpha < \kappa$, put $B_n^\alpha = \{f \in X : f(n) = \alpha\}$. The family $\mathcal{B} = \{B_n^\alpha : \text{all } n \text{ and all } \alpha\}$ generates $CO(X)$. Fix a point $x \in X$. For each n such that $x(n) = \infty$ put $B_{3n} = \{B_n^\alpha : \alpha < \kappa\}$ and for each n such that $x(n) \neq \infty$ put $B_{3n+1} = \{X \setminus B_n^\alpha : x(n) = \alpha\}$ and $B_{3n+2} = \{B_n^\beta : \beta \neq \alpha\}$. Then $\bigcup\limits_{n<\omega} B_n$ generates $CO(X)$, has the nice property, each B_n consists of elements pairwise disjoint, and we can thin each B_n to get $\{B_n : n < \omega\}$ a pairwise disjoint family; so, $CO(X)$ is a c-algebra of size at most κ.

Assume X is a boolean space and that $CO(X)$ is a c-algebra of size at most κ and let $\bigcup\limits_{n<\omega} B_n$ be a generating set of $CO(X)$ witnessing this. For each n, list B_n as $\{B_n^\alpha : \alpha \in S_n \subset \kappa\}$. Define $\phi : X \to \alpha\kappa^\omega$ by $\phi(x)(n) = \infty$ if $x \notin \bigcup\limits_{\alpha \in S_n} B_n^\alpha$

and $\phi(x)(n) = \alpha$ if $x \in B_n^\alpha$. ϕ is an embedding because $\bigcup_{n<\omega} B_n$ is a T_0-separating family. □

We require some c-algebra notions. A c-algebra A is a **c-subalgebra** of B, denoted by $\boldsymbol{A < B}$, if A is a boolean subalgebra of B and $A_n \subset B_n$ for each n. A function $i : A \to B$ between two c-algebras is a **c-embedding** if i is an embedding of boolean algebras and $i[A_n] \subset B_n$ for each n; if, in addition, $i[A_n] = B_n$ for each n, then i is a **c-isomorphism**. U is a **universal** c-algebra of size κ if every c-algebra of size $\leq \kappa$ can be c-embedded into U.

A **nut** N of a c-algebra B is an element of B that is given by a description denoted by $\boldsymbol{(N^+, N^-)}$ so that $N = \bigwedge N^+ - \bigvee N^-$, where N^+ and N^- are disjoint finite subsets of gen(B). Since gen(B) generates B, knowledge of precisely which nuts are 0 completely determines (up to c-isomorphism) the c-algebra B. A fact that we will use in section 3 is that since a c-algebra B has the nice property, if N is a nut of B and $N = 0$, then $N^+ \neq \emptyset$.

We say that A is a **closed** c-subalgebra of a c-algebra B, denoted by $\boldsymbol{A <_c B}$, if $A < B$ and

1. whenever N is a nut of A and F is a finite subset of gen(B) and $N \leq \bigvee F$, then there exists a finite subset G of gen(A) such that $N \leq \bigvee G$ and $\pi[G] = \pi[F]$, and
2. whenever N is a nut of A and F is a finite subset of gen(B) and $N \wedge \bigwedge F \neq 0$, then there exists a finite subset G of gen(A) such that $N \wedge \bigwedge G \neq 0$ and $\pi[G] = \pi[F]$.

To use closed c-subalgebras (i.e. elementarity) in our proof of Lemma 3.1 was suggested to the author by Alan Dow. Three important properties of $A <_c B$ are:

- $A < B$ imply that there exists $C <_c B$ such that $A < C$ and $|C| = |A|$.
- $A <_c B$ and $A < C < B$ imply that $A <_c C$.
- if $A_\alpha < A_\beta$ for $\alpha < \beta < \kappa$ and for each $\alpha < \kappa$, $A_\alpha <_c B$, then $\bigcup_{\alpha<\kappa} A_\alpha <_c B$.

To get a universal UEC space for certain weights κ, we will construct a universal c-algebra U of size κ, then, by Theorem 2.2 and Theorem 2.1, st(U) will be our universal UEC. Whether this is necessary for the existence of universal UEC spaces of weight κ is not known. Recent research of Dzamonja [Dz98] gives sufficient conditions for the non-existence of universal c-algebras of weight κ (in the absence of GCH).

3. Existence of universal uniform Eberleins

It is convenient and economical to decree that $\emptyset$ is a c-algebra and that for any c-algebra B, $\emptyset <_c B$ and $\emptyset : \emptyset \to B$ is a c-embedding.

Lemma 3.1 (Amalgamation Lemma)**.** *Let B and C be c-algebras. If $A <_c C$ and $i : A \to B$ is a c-embedding, then there exists a c-algebra E such that $B < E$, $|E| \leq |B| + |C|$, and there exists a c-embedding $j : C \to E$ such that $j \restriction A = i$.*

Proof. Without loss of generality we assume that i is an inclusion, so $A < B$ (therefore, $A_n \subset B_n$ for each n) and we assume that $(\text{gen}(C) \setminus \text{gen}(A)) \cap B = \emptyset$. Define, for each n, $E_n = (C_n \setminus A_n) \cup B_n$. Our goal is to define a c-algebra structure E with gen$(E) = \bigcup_{n<\omega} E_n$ which simultaneously extends the c-algebra structure of

B and C. Formally, we will use a quotient algebra to do this. Let D be the c-subalgebra of C generated by $\bigcup_{n<\omega}(C_n \setminus A_n)$. Let $B * D$ be the free product of the boolean algebras B and D, i.e., $B * D$ is isomorphic to $CO(\mathrm{st}(B) \times \mathrm{st}(D))$. Let I be the ideal in $B * D$ generated by $I_1 \cup I_2$ where $I_1 = \{(x, y) : x \in B_n \setminus A_n$ and $y \in C_n \setminus A_n$, for some $n\}$ and $I_2 = \{(N, M) : N$ is a nut of A and M is a nut of D and $N \wedge M = 0$ in $C\}$. Put $E = B * D/I$ and let $\phi : B * D \to B * D/I$ be the quotient homomorphism. E is generated by $\bigcup_{n<\omega} E_n$ where $E_n = \{\phi(x, 1) : x \in B_n\} \cup \{\phi(1, y) : y \in C_n \setminus A_n\}$. I_1 ensures that, for each n, E_n consists of elements pairwise disjoint.

Claim 1. E is a c-algebra.

Proof of Claim. We must check that E has the nice property. Striving for a contradiction, assume that $\bigvee_{i<n}\phi(x_i, 1) \vee \bigvee_{j<m}\phi(1, y_j) = 1$ where $x_i \in \mathrm{gen}(B)$ and $y_j \in \mathrm{gen}(D)$. Choose p,$q < \omega$ such that

$$(*) \qquad \bigvee_{i<n}(x_i, 1) \vee \bigvee_{j<m}(1, y_j) \vee \bigvee_{k<p}(N_k, M_k) \vee \bigvee_{l<q}(z_l, w_l) = 1$$

where each $(N_k, M_k) \in I_2$ and $(z_l, w_l) \in I_1$. Since $N_k \wedge M_k = 0$ in C and C has the nice property, we can choose $r_k \in (N_k \wedge M_k)^+ = N_k^+ \cup M_k^+$. So $(N_k, M_k) \le (r_k, 1)$ if $r_k \in N_k^+$ or $(N_k, M_k) \le (1, r_k)$ if $r_k \in M_k^+$. As C has the nice property, $c = \bigvee_{j<m} y_j \vee \bigvee\{r_k : k < p \text{ and } r_k \in M_k^+\} \vee \bigvee_{l<q} w_l < 1$. As B has the nice property, $b = \bigvee_{i<n} x_i \vee \bigvee\{r_k : k < p \text{ and } r_k \in N_k^+\} \vee \bigvee_{l<q} z_l < 1$. Thus, $(1 - b, 1 - c) \neq 0$ but is disjoint from the left-hand side of equation $(*)$. □

Claim 2. $B < E$ in the form $\{\phi(x, 1) : x \in B\}$.

Proof of Claim. By virtue of the free product, if N is a nut of B and $N = 0$, then $\phi(N, 1) = 0$. We must show that if N is a nut of B and $N \neq 0$, then $\phi(N, 1) \neq 0$. Assume not, and let $N \neq 0$ be a nut of B such that

$$(**) \qquad (N, 1) \le \bigvee_{i<n}(N_i, M_i) \vee \bigvee_{j<m}(x_j, y_j)$$

where each $(N_i, M_i) \in I_2$ and $(x_j, y_j) \in I_1$ and $n + m$ is the least k such that there exist k elements of $I_1 \cup I_2$ whose join is $\ge (P, 1)$ for some nut P of B with $P \neq 0$. By minimality of $n + m$, we get that $N \le \bigwedge_{i<n} N_i \wedge \bigwedge_{j<m} x_j$ in B (otherwise, there exists i such that $N - N_i \neq 0$ or $N - x_i \neq 0$ and either possibility yields a non-0 nut P of B such that $(P, 1)$ is $\le$ the join of 1 less element on the right-hand side of $(**)$. Also, $y = \bigvee_{i<n} M_i \vee \bigvee_{j<m} y_j = 1$ in D or else $(N, 1 - y) \neq 0$ but is disjoint from the right-hand side of $(**)$. Since for each $i < n$, $N_i \le M_i'$ (we use $'$ for complement) in C, we get that $\bigwedge_{i<n} N_i \le \bigwedge_{i<n} M_i' \le \bigvee_{j<m} y_j$ in C. As $A <_c C$ and $\bigwedge_{i<n} N_i$ is a nut of A, we can choose a finite $F \subset \mathrm{gen}(A)$ such that $\pi[F] = \pi[\{y_j : j < m\}]$ and $\bigwedge_{i<n} N_i \le \bigvee F$ in A. But, then $0 \neq N \le \bigvee F \wedge \bigwedge_{j<m} x_j$ in B, which is a contradiction since for every $z \in F$ there exists $j < m$ such that z and y_j (and therefore x_j) are in the same column B_n of B with $z \in A_n$ and $x_j \in B_n \setminus A_n$ and therefore $z \cap x_j = 0$ in B; hence $\bigvee F \wedge \bigwedge_{j<m} x_j = 0$ in B. □

Claim 3. $C < E$ in the form $\{\phi(x, 1) : x \in A\} \cup \{\phi(1, y) : y \in D\}$.

Proof of Claim. By virtue of the free product and I_2, if N is a nut of A and M is a nut of D and $N \wedge M = 0$, then $\phi(N, M) = 0$. We must show that if N is a nut of A and M is a nut of D and $N \wedge M \neq 0$, then $\phi(N, M) \neq 0$. Assume not and let N be a nut of A and let M be a nut of D such that $N \wedge M \neq 0$ in C and $(N, M) \leq \bigvee_{i<n} (N_i, M_i) \vee \bigvee_{j<m} (x_j, y_j)$ where each $(N_i, M_i) \in I_2$ and $(x_j, y_j) \in I_1$. Since $N \wedge M \neq 0$ in C and for each $i < n$, $N_i \wedge M_i = 0$ in C, we get that $(N, M) - \bigvee_{i<n} (N_i, M_i) \neq 0$. As

$$(N, M) - \bigvee_{i<n} (N_i, M_i) = \bigvee_{k<p} (P_k, Q_k) \leq \bigvee_{j<m} (x_j, y_j)$$

where each P_k is a nut of A and Q_k is a nut of D, we get that there exists $k < p$ with $(P_k, Q_k) \neq 0$ and $(P_k, Q_k) \leq \bigvee_{j<m} (x_j, y_j)$. So, without loss of generality we can assume that $(N, M) \leq \bigvee_{j<m} (x_j, y_j)$. As before, let m be the least k such that there exist k elements of I_1 whose join is $\geq (P, Q)$ for some nut P of A and for some nut Q of D with $P \wedge Q \neq 0$ in C. By minimality of m, $0 \neq N \wedge M \leq \bigwedge_{j<m} y_j$. Therefore, $N \wedge \bigwedge_{j<m} y_j \neq 0$. As $A <_c C$ and N is a nut of A, we can choose a finite $F \subset \text{gen}(A)$ such that $\pi[F] = \pi[\{y_j : j < m\}]$ and $N \wedge \bigwedge F \neq 0$. But, $N \leq \bigvee_{j<m} x_j$ and $\bigvee_{j<m} x_j \wedge \bigwedge F = 0$ which is a contradiction. □

This completes the proof of our lemma. □

The proofs of the following two results are standard procedures in model theory, but due to the asymmetry of our assumptions (we mix $A < B$ and $A <_c B$) we were unable to find a theorem to quote, and so we present the proofs.

Proposition 3.2. *Let $\lambda = \lambda^{<\kappa}$ and let B be a c-algebra of size $< \kappa$. Then there exists a c-algebra D of size λ such that $B < D$ and such that every c-embedding of any c-algebra A of size $< \kappa$ into D can be extended to a c-embedding of C, if C is any c-algebra of size $< \kappa$ with $A <_c C$.*

Proof. List all triples (X, Y, i) where Y is a c-algebra with $Y_n \subset \kappa \times \{n\}$ for each n, $X <_c Y$ and i is an injection of X into $\lambda \times \omega$ as $(X_\alpha, Y_\alpha, i_\alpha)_{0<\alpha<\lambda}$ so that for each α, $i_\alpha[X_\alpha] \subset \alpha \times \omega$. In our listing we allow the degenerate cases where $X = i = \emptyset$ in order to start our embeddings. We can do this listing because $\lambda = \lambda^{<\kappa}$.

We will build a c-algebra structure D with $\text{gen}(D) = \lambda \times \omega$ by recursively constructing D_α with $\text{gen}(D_\alpha) = \lambda_\alpha \times \omega$ such that for $\alpha < \beta < \lambda$, $\alpha \leq \lambda_\alpha \leq \lambda_\beta$ and $D_\alpha < D_\beta$. To begin, let $\lambda_0 = |B|$ and make D_0 c-isomorphic to B. If α is a limit, put $\lambda_\alpha = \sup\{\lambda_\beta : \beta < \alpha\}$ and $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$. If $\alpha = \beta + 1$ is a successor, then we have D_β with $\text{gen}(D_\beta) = \lambda_\beta \times \omega$. If i_β is not a c-embedding of X_β into D_β, then put $\lambda_\alpha = \lambda_\beta$ and $D_\alpha = D_\beta$. If i_β is a c-embedding of X_β into D_β, then apply the Amalgamation Lemma 3.1 with $A = X_\beta$, $B = D_\beta$, $C = Y_\beta$ and $i = i_\beta$ to yield a c-algebra E such that $D_\beta < E$, $|E| \le |D_\beta| + |Y_\beta|$ and i_β extends to Y_β. Put $D_\alpha = E$ and $\lambda_\alpha =$ the first ordinal $> \lambda_\beta$ that can accomodate the extra elements of $\mathrm{gen}(E)$.

Now, let $D = \bigcup_{\alpha<\lambda} D_\alpha$. If $A <_c C$, $|C| < \kappa$ and $i : A \to D$ is a c-embedding, then we can identify (A, C, i) with a $(X_\beta, Y_\beta, i_\beta)$ for some $\beta < \lambda$ and at stage $\alpha = \beta + 1$, we extended i to $Y_\beta = C$. $\square$

Theorem 3.3. *If $\kappa = 2^{<\kappa}$, then there exists a universal c-algebra U of size κ.*

Proof. If $\kappa = \kappa^{<\kappa}$, then let U be the D of the previous proposition with $\lambda = \kappa$ and $B = \emptyset$.

Otherwise, κ is a singular strong limit cardinal, so let $\kappa = \sup\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ where $\alpha < \beta$ implies $\kappa_\alpha < \kappa_\beta$, α a limit implies $\kappa_\alpha = \sup\{\kappa_\beta : \beta < \alpha\}$, and $\kappa_{\alpha+1} = 2^{\kappa_\alpha}$. By induction on $\alpha < \mathrm{cf}(\kappa)$, we construct c-algebras U_α with, for each n, $U_\alpha^n \subset \kappa_\alpha \times \{n\}$ such that $\alpha < \beta$ implies $U_\alpha < U_\beta$ and α a limit implies $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$. For $\alpha = \beta + 1$, we apply the preceding proposition with $B = U_\beta$ and put $U_\alpha =$ the D of that proposition. Now, let $U = \bigcup_{\alpha<\mathrm{cf}(\kappa)} U_\alpha$.

In either case, to show that our U is universal, let A be a c-algebra of size $\le \kappa$. Write A as $\bigcup_{\alpha<\mathrm{cf}(\kappa)} A_\alpha$ where for each α, $A_\alpha <_c A$, $|A_\alpha| < \kappa$, $\alpha < \beta$ implies $A_\alpha < A_\beta$ and α a limit implies $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. We then have that for $\beta < \alpha$, $A_\beta <_c A_\alpha$. Finally, an induction up to $\mathrm{cf}(\kappa)$, using the preceding proposition, will embed A into U. $\square$

4. Non-existence of universal Uniform Eberleins

$G \subset [X]^2$ (the doubletons of X) is called a **graph** on X. A graph H on a subset Y of X is called a **subgraph** of G if $H = G \cap [Y]^2$. A graph H on a set Y **embeds** into a graph G on a set X if there exists a function $i : Y \to X$ such that for all $x,y \in Y$, $\{x, y\} \in H \iff \{i(x), i(y)\} \in G$. A **universal graph** of size κ is a graph G on a set X of size κ such that every graph on any set of size $\le \kappa$ embeds into G.

Theorem 4.1. *Let $\mathcal{C}$ be a class of compact Hausdorff spaces containing all closed subspaces of $\alpha\kappa \times \alpha\kappa$. Then, if there exists a universal element in $\mathcal{C}$ for weight κ, there exists a universal graph of size κ.*

Proof. Let X be a universal element in $\mathcal{C}$ for weight κ. Let G be the intersection graph on $CO(X)$, i.e., for $x \ne y$ in $CO(X)$, $\{x, y\} \in G$ iff $x \cap y \ne \emptyset$. We claim that G is a universal graph of size κ. To prove this, let H be any graph on an $S \subset \kappa$. Consider the following closed subspace of $\alpha\kappa \times \alpha\kappa$, $Y = \alpha\kappa \times \{\infty\} \cup \{\infty\} \times \alpha\kappa \cup \{(\alpha, \beta), (\beta, \alpha) : \{\alpha, \beta\} \in H\}$. Let $\phi : X \to Y$ be a continuous surjection. For each $\alpha < \kappa$, put $B_\alpha = [\alpha\kappa \times \{\alpha\} \cup \{\alpha\} \times \alpha\kappa] \cap Y$. For each $\alpha < \kappa$, $B_\alpha \in CO(Y)$ and $B_\alpha \cap B_\beta \ne \emptyset$ iff $\{\alpha, \beta\} \in H$. So, the mapping $i : S \to CO(X)$ defined by $i(\alpha) = \phi^{-1}[B_\alpha]$ embeds H into G. $\square$

Corollary 4.2. *If V is a model with $\lambda^{<\lambda} = \lambda < \kappa < \mu$ and P is a Cohen forcing that adds μ Cohen subsets of λ, then in the model V^P there is no universal UEC in the weight κ for any κ with $\lambda < \kappa < \mu$. In particular, if we add ω_2 Cohen reals to V, then there is no universal UEC in the weight ω_1.*

Proof. Shelah [Sh90] has shown that there is no universal graph of size κ in this model. □

An additional result, just following from cardinal arithmetic, is the result of Kojman and Shelah [KS92], that if $\mathrm{cf}(2^\lambda) \leq \kappa < 2^\lambda$, then there does not exist a universal graph of size κ; hence, there would be no universal UEC for this weight κ.

In conclusion, we mention one open problem: Does a universal UEC for weight κ not exist if κ is a singular cardinal which is not a strong limit?

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