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THE PHILLIPS PROPERTIES

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Abstract. A Banach space X has the Phillips property if the canonical projection $p\colon X^{***} \to X^*$ is sequentially weak*-norm continuous, and has the weak Phillips property if p is sequentially weak*-weak continuous. We study both properties in connection with other geometric properties, such as the Dunford-Pettis property, Pelczynski's properties (u) and (V), and the Schur property.

Introduction

By a well-known result of R. Phillips, the canonical projection $p\colon c_0^{***} \to c_0^*$ is sequentially weak*-norm continuous [D1, p. 83]. We will say that a Banach space X has the Phillips property if the canonical projection $p\colon X^{***} \to X^*$ is sequentially weak*-norm continuous, and that X has the weak Phillips property if p is sequentially weak*-weak continuous. If (P) is a Banach space property, we will say that a Banach space X has the hereditary (P) property if every closed subspace of X has property (P).

We will consider the Phillips property, the weak Phillips property, and the hereditary versions of each in connection with other well-known geometric properties such as Pelczynski's properties (u) and (V), as well as the Dunford-Pettis and Schur properties for Banach spaces.

The main results can be summarized as follows. Throughout, X is an infinite-dimensional Banach space.

(a) X has the hereditary Phillips property if and only if X has the hereditary Dunford-Pettis property and does not contain an isomorphic copy of ℓ_1.
(b) X has the hereditary weak Phillips property if and only if X has the hereditary (V) property.
(c) If X has property (V), then X has the weak Phillips property.
(d) A C^*-algebra A has the Phillips property if and only if A^* has the Schur property.
(e) If X has the Phillips property, then X is not complemented in any dual space.
(f) If X is nonreflexive and has the weak Phillips property and X_1^* is weak* sequentially compact, then X is not complemented in any dual space.

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(g) X has the (weak) Phillips property if and only if for every operator $T\colon X^{**} \to c_0$, the restricted operator $T\restriction_X : X \to c_0$ is (weakly) compact.

The main ingredients of the proofs are a result of Knaust and Odell [K-O] saying that if X has the hereditary Dunford-Pettis property, then X has property (u); results due to Pelczynski and his coauthors; results due to Rosenthal; and a result of Pfitzner [Pf], saying that every C^*-algebra has property (V).

1. Notation and background

The notation we use is quite standard. In general, X and Y will denote Banach spaces over the field of complex numbers. The word 'operator' will always mean a bounded linear operator. For any Banach space X, the dual space of bounded linear functionals on X will be denoted by X^*; the closed unit ball of X will be denoted by X_1; and the Banach space of all compact operators on X will be denoted by $K(X)$. We identify X with its image under the canonical embedding of X into X^{**}. For $x \in X$ and $f \in X^*$, we denote by $\langle x, f\rangle$, or $\langle f, x\rangle$, the evaluation of f on x. There exists a canonical projection $p\colon X^{***} \to X^*$ corresponding to the direct sum decomposition $X^{***} = X^* \oplus X^{\perp}$; throughout the paper, p will always mean this projection. Recall that an infinite series $\sum_{n=1}^{\infty} x_n$ of elements of X is said to be weakly unconditionally Cauchy (wuC) if for every $f \in X^*$ one has $\sum_{n=1}^{\infty} |\langle x_n, f\rangle| < \infty$.

We now review the definitions of the main properties used in the paper:

(a) *The Dunford-Pettis property.* A Banach space X has the Dunford-Pettis property if for every weakly null sequence (x_n) in X and every weakly null sequence (f_n) in X^* we have $\lim_{n\to\infty}\langle x_n, f_n\rangle = 0$. For an excellent survey of results related to the Dunford-Pettis property, we refer the reader to [D2], while for the hereditary Dunford-Pettis property, we refer the reader to the paper of Cembranos [C].

(b) *Property* (u). A Banach space X has property (u) if for every weakly Cauchy sequence (y_n) in X there exists a wuC series $\sum_{n=1}^{\infty} x_n$ in X such that the sequence $(y_n - \sum_{j=1}^{n} x_j)$ is weakly null. Pelczynski introduced property (u) in his paper [P]. Note that property (u) is hereditary, i.e., it is inherited by closed subspaces.

(c) *Property* (V). A Banach space X has property (V) if each subset $K \subseteq X^*$ satisfying

$$\lim_{n\to\infty} \sup_{f\in K} |\langle f, x_n\rangle| = 0 \text{ for every wuC series } \textstyle\sum_{n=1}^{\infty} x_n \text{ in } X$$

is relatively weakly compact. Equivalently, for every Banach space Y, every unconditionally converging operator $T\colon X \to Y$ is weakly compact. Property (V) was also introduced by Pelczynski in [P]. As proved there (given Rosenthal's ℓ_1-lemma), if X has property (u) and does not contain an isomorphic copy of ℓ_1, then X has property (V). In fact, such a space has the hereditary (V) property since property (u) is hereditary.

(d) *The Schur property.* A Banach space X has the Schur property if every weakly convergent sequence is norm convergent. A result we make frequent use of is that X^* has the Schur property if and only if X has the Dunford-Pettis property and X does not contain an isomorphic copy of ℓ_1 [D2, Theorem 3].

(e) *Grothendieck spaces.* A Banach space X is a Grothendieck space if every weak* convergent sequence in X^* is weakly convergent.

For any undefined notation, we refer the reader to the book [D1].

2. The Phillips properties

2.1 Definition. Let X be a Banach space.

(a) X has the Phillips property if the canonical projection $p\colon X^{***} \to X^*$ is sequentially weak*-norm continuous.
(b) X has the weak Phillips property if the canonical projection $p\colon X^{***} \to X^*$ is sequentially weak*-weak continuous.
(c) X has the hereditary (weak) Phillips property if every closed subspace of X has the (weak) Phillips property.

We begin by considering the weak Phillips property. An obviously sufficient, although not necessary, condition for X to have the weak Phillips property is that X (or X^{**}) is a Grothendieck space, i.e., weak* convergent sequences in X^* are weakly convergent. We show first that one necessary condition for X to have the weak Phillips property is that X^* is weakly sequentially complete.

2.2 Lemma. *Let X be a Banach space. If X has the weak Phillips property, then X^* is weakly sequentially complete.*

Proof. Let (f_n) be a weakly Cauchy sequence in X^*. It follows that we may define a bounded linear functional $G\colon X^{**} \to \mathbb{C}$ by $G(\phi) = \lim\langle \phi, f_n\rangle$ for all $\phi \in X^{**}$. Clearly, the sequence $(f_n - G)$ is weak* null in X^{***}. Since X has the weak Phillips property, the sequence $(f_n - pG)$ is weakly null in X^*, so that (f_n) is weakly convergent in X^*. Thus X^* is weakly sequentially complete. $\square$

Recall that a Banach space X has the Mazur property if every weak* sequentially continuous functional $\Lambda\colon X^* \to \mathbb{C}$ is weak* continuous, i.e., belongs to X. Note that there exist Banach spaces such as James' space J, which are not reflexive; yet whose second duals are separable and hence have the Mazur property.

2.3 Proposition. *Assume that X^{**} has the Mazur property. Then X has the weak Phillips property if and only if X is reflexive.*

Proof. We need only prove the forward direction. Suppose that X has the weak Phillips property, and let $\phi \in X^{**}$. Let (g_n) be a weak* null sequence in X^{***}. Then (pg_n) is weakly null in X^*, so that

$$\langle g_n, p^*\phi\rangle = \langle pg_n, \phi\rangle \to 0.$$

Since X^{**} has the Mazur property, the map $p^*\phi$ is in X^{**}. It follows that the canonical embedding $\iota\colon X \to X^{**}$ is weakly compact, so that X is reflexive. $\square$

Let (e_n) be the standard basis for ℓ_1. Recall that an operator $T\colon X \to c_0$ is weakly compact if and only if the sequence (T^*e_n) is weakly null in X^* and T is compact if and only if the sequence (T^*e_n) is norm null. Given a Banach space Y and an operator $T\colon X^{**} \to Y$, let $\widehat{T}\colon X \to Y$ denote the restriction of T to X. We use this notation in the following result.

2.4 Theorem. *A Banach space X has the (weak) Phillips property if and only if for every operator $T\colon X^{**} \to c_0$, the operator $\widehat{T}$ is (weakly) compact.*

Proof. Let $T\colon X^{**} \to c_0$ be given, and let (e_n) be the standard basis for ℓ_1. Of course the sequence (T^*e_n) is weak* null, and it is easy to see that $\widehat{T}^*e_n = pT^*e_n$ for all n. Hence if X has the weak Phillips property, then the sequence $(\widehat{T}^*e_n)$ is

weakly null, while if X has the Phillips property, then the sequence $(\widehat{T}^* e_n)$ is norm null. This proves both forward directions.

Now, let (g_n) be a weak* null sequence in X^{***}, and define $T\colon X^{**} \to c_0$ by $T\phi = (\langle \phi, g_j \rangle)_{j=1}^{\infty}$ for all $\phi \in X^{**}$. As in the first part of the proof, we have $g_n = T^* e_n$ for all n, and $pg_n = \widehat{T}^* e_n$. Thus if $\widehat{T}$ is compact, then $pg_n \to 0$ in norm, while if $\widehat{T}$ is weakly compact, then $pg_n \to 0$ weakly, proving both converses. ◻

For the proof of the next theorem, we need the following simple lemma.

2.5 Lemma. *Let Y be a Banach space. Let X be a complemented subspace of Y^*. Then X is complemented in X^{**}.*

Proof. Let $q\colon Y^* \to X \subseteq Y^*$ be a projection with $q(Y^*) = X$, let $\kappa\colon X \to Y^*$ be the inclusion map, and let $p\colon Y^{***} \to Y^*$ be the canonical projection. Then it is easy to check that the mapping $qp\kappa^{**}\colon X^{**} \to X$ is a projection onto X. Thus X is complemented in X^{**}. ◻

Every C^*-algebra has property (V) by a result due to Pfitzner [Pf]. Combining that fact with the next result shows that every C^*-algebra has the weak Phillips property.

2.6 Theorem. *If a Banach space has property* (V)*, then it has the weak Phillips property.*

Proof. We note first that for any Banach space X and any operator $T\colon X^{**} \to c_0$, T is unconditionally converging, for if not, then there exists a closed subspace $E \subseteq X^{**}$, isomorphic to c_0, such that $T\restriction_E\colon E \to c_0$ is an isomorphism. Let $r\colon X^{****} \to X^{**}$ be the canonical projection, and let $i\colon E \to X^{**}$ be the inclusion map. Then the operator $Tri^{**}\colon E^{**} \to c_0$ is weakly compact, since E^{**} is isomorphic to ℓ_∞, whence its restriction to E, which is just $T\restriction_E$, is weakly compact, a contradiction. Thus, if X has property (V) and $T\colon X^{**} \to c_0$ is any operator, then T is unconditionally converging, and hence $\widehat{T}$ is unconditionally converging. Since X has property (V), the operator $\widehat{T}$ is weakly compact. Hence by Theorem 2.4, X has the weak Phillips property. ◻

Property (V) is not a necessary condition for X to have the weak Phillips property —see Example 2.11 below. However, the hereditary versions of the two properties are equivalent.

2.7 Theorem. *Let X be a Banach space. The following are equivalent:*

(a) *X has the hereditary weak Phillips property.*
(b) *For every closed subspace $M \subseteq X$, M^* is weakly sequentially complete.*
(c) *X has property* (u) *and does not contain a copy of ℓ_1.*
(d) *X has the hereditary* (V) *property.*

Proof. The implications (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) follow immediately from Lemma 2.2 and Theorem 2.6, respectively, while the equivalence of (b) and (c) is from [R, Corollary 1.5]. Now, since property (u) is hereditary, (c) $\Rightarrow$ (d) by Rosenthal's ℓ_1-lemma and [P, Proposition 2], so the proof is complete. ◻

We turn now to the Phillips property. Note that if a Banach space X has the Phillips property, then X^* has the Schur property since the map p is the identity on X^*. The next result follows immediately from this fact. In particular, if X has

the Phillips property, then X has the Dunford-Pettis property and does not contain a copy of ℓ_1.

2.8 Proposition. *A Banach space X has the Phillips property if and only if X has the weak Phillips property and X^* has the Schur property.*

Since any closed subspace of a space with an unconditional basis has property (u), we obtain the following corollary.

2.9 Corollary. *Let X be a Banach space which is isomorphic to a closed subspace of a space with an unconditional basis. Then*

(a) *X has the Phillips property if and only if X^* has the Schur property.*
(b) *X has the hereditary weak Phillips property if and only if X does not contain a copy of ℓ_1.*

It follows that if X has the Phillips property, then X has the hereditary weak Phillips property.

We now show that for spaces which fail to contain a copy of ℓ_1, the hereditary Phillips property and the hereditary Dunford-Pettis property are equivalent.

2.10 Theorem. *Let X be a Banach space. The following are equivalent:*

(a) *X has the hereditary Phillips property.*
(b) *For every closed subspace $M \subseteq X$, the dual space M^* has the Schur property.*
(c) *X has the hereditary Dunford-Pettis property and does not contain an isomorphic copy of ℓ_1.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the fact that the dual of any space with the Phillips property has the Schur property, while the implication (b) $\Rightarrow$ (c) holds by [D2, Theorem 3]. Now, assume that (c) holds. Since X has the hereditary Dunford-Pettis property, X has property (u) [K-O, Theorem 2.1]. Let $M \subseteq X$ be a closed subspace of X. Since property (u) is hereditary, M has property (u), and by assumption, M does not contain an isomorphic copy of ℓ_1. It follows that M has property (V) by Rosenthal's ℓ_1-lemma and [P, Proposition 2]. Thus M has the weak Phillips property, and M^* has the Schur property, so that M has the Phillips property, as desired. $\square$

2.11 Example. *A Banach space having the Phillips property, but which fails to have property* (V). Let Y be the Banach space constructed in [B-De]. As shown there, Y^* is isomorphic to ℓ_1, even though Y does not contain a copy of c_0, so that Y^* has the Schur property. Since in addition, $Y^{**} \simeq \ell_\infty$, the space Y^{**} is a Grothendieck space, and hence Y has the Phillips property. Now, Y is not reflexive (though it is 'somewhat' reflexive), and does not contain a copy of c_0, so Y fails to have property (V) by [P, Proposition 8].

The next result stands alone, being of some independent interest.

2.12 Proposition. *Let X and Y be Banach spaces, and assume that X has the Phillips property. Then every operator $T\colon X^{**} \to Y$ which is weak*-weak continuous is compact.*

Proof. Since T is weak*-weak continuous, T is a weakly compact operator. It follows that $\widehat{T}\colon X \to Y$ is also weakly compact, where $\widehat{T}$ denotes the restriction of T to X, so that $\widehat{T}^{**}\colon X^{**} \to Y$. Now, $\widehat{T}^*\colon Y^* \to X^*$ is weakly compact, but since

X^* has the Schur property, the map $\widehat{T}^*$ is compact. Hence $\widehat{T}^{**}$ is also compact, but the weak* density of X_1 in X_1^{**} implies that $T = \widehat{T}^{**}$, so that T is compact, as desired. □

Recall that a subset $B \subseteq X$ of a Banach space X is called limited if for every weak* null sequence (g_n) in X^*, we have $\langle x, g_n \rangle \to 0$ uniformly for $x \in B$. An operator $T\colon X \to Y$ is called limited if the image of the closed unit ball $T(X_1)$ is a limited subset of Y. It is easy to check that an operator T is limited if and only if $T^*\colon Y^* \to X^*$ maps weak* null sequences to norm null sequences. In particular, a Banach space X has the Phillips property if and only if the canonical embedding $\iota\colon X \to X^{**}$ is a limited operator if and only if the unit ball X_1 is a limited subset of X^{**}. It is easy to see that if $B \subseteq X$ is a limited subset of X, and $T\colon X \to Y$ is an operator, then $T(B)$ is a limited subset of Y, since T^* is weak*-weak* continuous.

As is well-known, c_0 is not complemented in its second dual, ℓ_∞. The next result shows that this is true for all (infinite-dimensional) spaces with the Phillips property.

2.13 Theorem. *Let X be an infinite-dimensional Banach space. If X has the Phillips property, then X is not complemented in any dual space.*

Proof. Suppose X is complemented in a dual space and is infinite-dimensional. By Lemma 2.5, X is complemented in X^{**}, so let $q\colon X^{**} \to X$ be a projection onto X. Suppose that X has the Phillips property. By the preceding remarks, $q(X_1) = X_1$ is limited in X. Now, since X has the Phillips property, X does not contain ℓ_1, so by [D1, p. 224], it follows that X_1 is relatively weakly compact, so that X is reflexive. But since X has the Phillips property, X has the Dunford-Pettis property as well. Thus X is finite-dimensional, a contradiction. □

2.14 Corollary. *Every infinite-dimensional dual space fails to have the Phillips property.*

2.15 Remark. An alternative proof of Theorem 2.13 is provided by the main theorem proved in [B-D]. Namely, it is proved (without the terminology) that if an infinite-dimensional Banach space X has the Phillips property, then the canonical embedding $\iota\colon X \to X^{**}$ is strictly cosingular, i.e., if E is any Banach space for which there exist operators $s\colon X \to E$ and $r\colon X^{**} \to E$ such that $s = r\iota$, then E is finite-dimensional. It follows that X cannot be complemented in X^{**}, and so by Lemma 2.5, cannot be complemented in any dual space. In addition, Corollary 2.14 follows immediately from the fact that X^{**} has the Schur property if and only if X is finite-dimensional.

In contrast to the Phillips property, dual spaces can have the weak Phillips property, for example, any von Neumann algebra. The next result gives some qualification of which dual spaces can have the weak Phillips property.

2.16 Theorem. *Let X be a nonreflexive Banach space. Assume the following:*

(a) *X has the weak Phillips property, and*
(b) *X_1^* is weak* sequentially compact.*

Then X is not complemented in any dual space.

Proof. Suppose X is complemented in a dual space. By Lemma 2.5, X is complemented in X^{**}, so let $q\colon X^{**} \to X$ be a projection onto X. Assume that (a) and (b) hold. Since X has the weak Phillips property, $pq^* = \mathrm{id}_{X^*}$ is sequentially

weak*-weak continuous. Now let (f_n) be any sequence in X_1^*. Since X_1^* is weak* sequentially compact, (f_n) has a weak* convergent subsequence. It follows that (f_n) has a weakly convergent subsequence, since $f_n = pq^* f_n$ for all n. Thus X_1^* is weakly compact, so that X is reflexive, a contradiction. □

Recall that if X is weakly compactly generated, then X_1^* is weak* sequentially compact. It follows that any separable nonreflexive space with property (V), such as $K(\ell_p)$, for $1 < p < \infty$, is not complemented in any dual space, in particular in its second dual.

2.17 Corollary. *Every nonreflexive, complemented subspace of a weakly compactly generated dual space fails to have the weak Phillips property. In particular, every nonreflexive separable dual space fails to have the weak Phillips property.*

3. The Phillips properties for C^*-algebras

In this section, we study the Phillips and weak Phillips properties and their hereditary versions for C^*-algebras. Recall that a topological space is called scattered (or dispersed) if it contains no nonempty perfect subset. A C^*-algebra A is called scattered if every positive linear functional on A is the sum of a sequence of pure functionals. Two equivalent properties are that A^* has the RNP, and that A does not contain an isomorphic copy of ℓ_1 (see [Ch, Theorem 3] and [GGMS, Corollary VII-10]).

3.1 Lemma. *Let A be a C^*-algebra. The following are equivalent:*

(a) *A has the Phillips property.*
(b) *A^* has the Schur property.*
(c) *A is scattered and has the Dunford-Pettis property.*

Proof. The equivalence of (b) and (c) is clear by [D2, Theorem 3]. Every C^*-algebra has property (V) by [Pf], so it follows from Theorem 2.6 that every C^*-algebra has the weak Phillips property, thereby proving the equivalence of (a) and (b). □

We now consider the Phillips properties for commutative unital C^*-algebras, i.e., those of the form $\mathcal{C}(K)$ for a compact Hausdorff space K. In this case, we have that $\mathcal{C}(K)$ is scattered if and only if K is scattered [P-S]. Since $\mathcal{C}(K)$ always has the Dunford-Pettis property, it follows from the previous lemma that $\mathcal{C}(K)$ has the Phillips property if and only if K is scattered. For the hereditary versions of the Phillips properties, we have the following result.

3.2 Theorem. *Let K be a compact Hausdorff space. The following are equivalent:*

(a) *$\mathcal{C}(K)$ has the hereditary weak Phillips property.*
(b) *K is scattered and $K^{(\omega)} = \emptyset$.*
(c) *$\mathcal{C}(K)$ has the hereditary Phillips property.*

Proof. Suppose that $\mathcal{C}(K)$ has the hereditary weak Phillips property. By Theorem 2.7, $\mathcal{C}(K)$ has property (u) and does not contain a copy of ℓ_1. By the remarks above, K is scattered. Now, if $K^{(\omega)} \neq \emptyset$, then $\mathcal{C}(K)$ contains an isomorphic copy of $\mathcal{C}(\omega^\omega)$ [D2, p. 29], which fails to have property (u) (cf. [HOR, Proposition 5.3] and [HWW, pp. 210–211]), a contradiction since property (u) is hereditary. This proves the implication (a) ⇒ (b). Now if K is scattered, then $\mathcal{C}(K)$ does not contain a copy of ℓ_1, and if in addition $K^{(\omega)} = \emptyset$, then $\mathcal{C}(K)$ has the hereditary Dunford-Pettis property [P-Sz]. Hence by Theorem 2.10, $\mathcal{C}(K)$ has the hereditary

Phillips property, proving that (b) $\Rightarrow$ (c). Since (c) $\Rightarrow$ (a) holds trivially, the proof is complete. $\square$

Let H be a Hilbert space. For an operator $a \in K(H)$, its Hilbert space adjoint is denoted by $a^* \in K(H)$. Recall that the C^*-algebra $K(H)$ has property (u) and does not contain an isomorphic copy of ℓ_1, since it is an M-ideal in its second dual. Hence by Theorem 2.7, $K(H)$ has the hereditary weak Phillips property. However, obviously $K(H)$ fails to have the Phillips property, so it is natural to ask which closed subspaces of $K(H)$ do have the Phillips property.

3.3 Theorem. *Let X be a closed subspace of $K(H)$. The following are equivalent:*

(a) *X has the hereditary Phillips property.*
(b) *X has the Phillips property.*
(c) *X has the Dunford-Pettis property.*
(d) *X has the hereditary Dunford-Pettis property.*
(e) *X^* has the Schur property.*
(f) *For each $\xi \in H$, the sets $X_1(\xi) = \{a(\xi) : a \in X_1\}$ and $\widetilde{X}_1(\xi) = \{a^*(\xi) : a \in X_1\}$ are relatively compact in H.*

Proof. The equivalence of (e) and (f) is proved in [U] and [B], with the equivalence of (c) and (e) following from the fact that X fails to contain a copy of ℓ_1. The implication (a) $\Rightarrow$ (b) is obvious, and the implication (b) $\Rightarrow$ (c) holds, for if X has the Phillips property, then X^* has the Schur property. Since $K(H)$ does not contain a copy of ℓ_1, (a) and (d) are equivalent by Theorem 2.10. To complete the proof, it is enough to prove the implication (f) $\Rightarrow$ (d). Suppose that (f) holds, and let $E \subseteq X$ be a closed subspace of X. Clearly for each $\xi \in H$, the sets $E_1(\xi) = \{a(\xi) : a \in E_1\}$ and $\widetilde{E}_1(\xi) = \{a^*(\xi) : a \in E_1\}$ are relatively compact in H, so E^* has the Schur property by (e), and so E has the Dunford-Pettis property. Thus X has the hereditary Dunford-Pettis property, completing the proof. $\square$

We end the paper with a result on the group C^*-algebra $C^*(G)$, for G a locally compact group. We refer the reader to [Di] for any pertinent definitions.

3.4 Corollary. *Let G be a locally compact group and set $A = C^*(G)$. The following are equivalent:*

(a) *A has the hereditary Phillips property.*
(b) *A has the Phillips property.*
(c) *A^* has the Schur property.*
(d) *G is compact.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear, while the equivalence (c) $\Leftrightarrow$ (d) is proved in [L-U, Theorem 4.5]. Suppose that G is compact, let μ be the normalized Haar measure on G, and let $H = L^2(\mu)$. $C^*(G)$ is then a C^*-subalgebra of $K(H)$, so that (d) $\Rightarrow$ (a) by Theorem 3.3 above. $\square$

Remark. We do not know if the following assertions are true or not.

(a) There is a Banach space X such that X^* has the Schur property, but such that X fails to have the Phillips property.
(b) More generally, there is a Banach space X such that X^* is weakly sequentially complete, but such that X fails to have the weak Phillips property.

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