

TRUDINGER TYPE INEQUALITIES IN \mathbf{R}^N AND THEIR BEST EXPONENTS

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ABSTRACT. We study Trudinger type inequalities in \mathbf{R}^N and their best exponents α_N . We show for $\alpha \in (0, \alpha_N)$, $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ (ω_{N-1} is the surface area of the unit sphere in \mathbf{R}^N), there exists a constant $C_\alpha > 0$ such that

$$(*) \quad \int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_{L^N(\mathbf{R}^N)}} \right)^{\frac{N}{N-1}} \right) dx \leq C_\alpha \frac{\|u\|_{L^N(\mathbf{R}^N)}^N}{\|\nabla u\|_{L^N(\mathbf{R}^N)}^N}$$

for all $u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}$. Here $\Phi_N(\xi)$ is defined by

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!} \xi^j.$$

It is also shown that (*) with $\alpha \geq \alpha_N$ is false, which is different from the usual Trudinger's inequalities in bounded domains.

0. INTRODUCTION

In this note, we study the limit case of Sobolev's inequalities; suppose $N \geq 2$ and let $D \subset \mathbf{R}^N$ be an open set. We denote by $W_0^{1,N}(D)$ the usual Sobolev space, that is, the completion of $C_0^\infty(D)$ with the norm $\|u\|_{W_0^{1,p}(D)} = \|\nabla u\|_p + \|u\|_p$. Here

$$\|u\|_p = \left(\int_D |u|^p dx \right)^{1/p}.$$

It is well-known that

$$\begin{aligned} W_0^{1,p}(D) &\subset L^{\frac{pN}{N-p}}(D) & \text{if } 1 \leq p < N, \\ W_0^{1,p}(D) &\subset L^\infty(D) & \text{if } N < p. \end{aligned}$$

The case $p = N$ is the limit case of these imbeddings and it is known that

$$\begin{aligned} W_0^{1,N}(D) &\subset L^q(D) & \text{for } N \leq q < \infty, \\ W_0^{1,N}(D) &\not\subset L^\infty(D). \end{aligned}$$

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This case is studied by Trudinger [14] more precisely and he showed for bounded domains $D \subset \mathbf{R}^N$

$$(0.1) \quad \int_D \exp \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx \leq C |D|$$

for $u \in W_0^{1,N}(D) \setminus \{0\}$, where the constants α, C are independent of u and D .

Trudinger's result is extended into two directions; the first one is to find the best exponents in (0.1). Moser [8] proved that (0.1) holds for $\alpha \leq \alpha_N$ but not for $\alpha > \alpha_N$, where

$$(0.2) \quad \alpha_N = N \omega_{N-1}^{1/(N-1)}$$

and ω_{N-1} is the surface area of the unit sphere in \mathbf{R}^N . See also Adams [1]. We also refer to [3], [5], [7], [13] for the attainability of

$$\sup \left\{ \int_D \exp \left(\alpha_N \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx; u \in W_0^{1,N}(D) \setminus \{0\} \right\}.$$

The second direction is to extend Trudinger's result for unbounded domains and for Sobolev spaces of higher order and fractional order. We refer to D. R. Adams [1], R. A. Adams [2], Ogawa [9], Ogawa-Ozawa [10], Ozawa [11], Strichartz [12].

In this paper, we study a version of Trudinger inequalities in \mathbf{R}^N and their best exponents; we show

$$(0.3) \quad \int_{\mathbf{R}^N} \exp \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) - \sum_{j=0}^{N-2} \frac{1}{j!} \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right)^j dx \leq C \frac{\|u\|_N^N}{\|\nabla u\|_N^N}$$

for $u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}$, where $\alpha, C > 0$ are independent of u , and we also find the best exponents α for (0.3).

In [9], [11], [2], (0.3) and related inequalities are obtained without studying their best exponents; Ogawa [9] obtained (0.3) for $N = 2$ and Ozawa [11] extended it for functions in the Sobolev space $H^{N/p,p}(\mathbf{R}^N) = (1 - \Delta)^{-N/2p} L^p(\mathbf{R}^N)$ of fractional order. See also [10]. Adams [2] studied a different version of (0.3); however the dependence in u of the right-hand side is not given explicitly.

The main purpose of this paper is to study the best exponents α in (0.3) as well as to give a simplified proof of (0.3).

To simplify notation, we use

$$(0.4) \quad \Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!} \xi^j.$$

With this notation, (0.3) becomes

$$(0.5) \quad \int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx \leq C \frac{\|u\|_N^N}{\|\nabla u\|_N^N}.$$

One of the virtues of the inequality (0.5) is its scale-invariance; for $u \in W^{1,N}(\mathbf{R}^N)$ and $\lambda > 0$, we set

$$(0.6) \quad u_\lambda(x) = u(\lambda x).$$

We can easily see that $\|\nabla u_\lambda\|_N = \|\nabla u\|_N$ and

$$(0.7) \int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u_\lambda(x)|}{\|\nabla u_\lambda\|_N} \right)^{\frac{N}{N-1}} \right) dx = \lambda^{-N} \int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx,$$

$$(0.8) \qquad \|u_\lambda\|_N^N = \lambda^{-N} \|u\|_N^N.$$

Thus (0.5) is invariant under the scaling (0.6) and we believe the best exponents α in (0.5) are of interest.

Our main result is the following.

Theorem 0.1. *Suppose $N \geq 2$. Then for any $\alpha \in (0, \alpha_N)$ (α_N is given in (0.2)), there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx \leq C_\alpha \frac{\|u\|_N^N}{\|\nabla u\|_N^N} \quad \text{for } u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}.$$

We remark that the restriction $\alpha < \alpha_N$ is optimal. The limit exponent α_N is excluded for (0.5). It is quite different from Moser’s result for (0.1).

Theorem 0.2. *For $\alpha \geq \alpha_N$, there exists a sequence $(u_k(x))_{k=1}^\infty \subset W^{1,N}(\mathbf{R}^N)$ such that $\|\nabla u_k\|_N = 1$ and*

$$(0.9) \qquad \begin{aligned} & \frac{1}{\|u_k\|_N^N} \int_{\mathbf{R}^N} \Phi_N \left(\alpha \left(\frac{|u_k(x)|}{\|\nabla u_k\|_N} \right)^{\frac{N}{N-1}} \right) dx \\ & \geq \frac{1}{\|u_k\|_N^N} \int_{\mathbf{R}^N} \Phi_N \left(\alpha_N \left(\frac{|u_k(x)|}{\|\nabla u_k\|_N} \right)^{\frac{N}{N-1}} \right) dx \rightarrow \infty \end{aligned}$$

as $k \rightarrow \infty$.

Remark 0.3. Even if we consider (0.5) in a bounded domain D , i.e.,

$$(0.10) \int_D \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}} \right) dx \leq C_\alpha \frac{\|u\|_N^N}{\|\nabla u\|_N^N} \quad \text{for } u \in W^{1,N}(D) \setminus \{0\},$$

the limit exponent α_N is still excluded. It is because of the scale-invariance (0.7)–(0.8). See Remark 2.1 below.

As to the proof of the inequality (0.5), following the original idea of Trudinger, [9], [10], [11] made use of a combination of the power series expansion of the exponential function and sharp multiplicative inequalities:

$$(0.11) \qquad \|u\|_q \leq C(N, q) \|u\|_N^{N/q} \|\nabla u\|_N^{1-N/q}.$$

For multiplicative inequalities of type (0.11) and their applications, we refer to Edmunds-Ilyin [4] and Kozono-Ogawa-Sohr [6]. We also remark that in Ozawa [11] multiplicative inequalities for functions $H^{N/p,p}(\mathbf{R}^N)$ are given and they are applied to obtain Brezis-Gallouet-Wainger type inequalities.

We give proofs of Theorems 0.1 and 0.2 in the following sections. We take a different approach from [9], [10], [11], we use Moser’s idea; we take symmetrization of functions and we reduce (0.5) to one-dimensional inequality.

1. PROOF OF THEOREM 0.1

To prove Theorem 0.1, we use an idea of Moser [8]. By means of symmetrization, it suffices to show the desired inequality (0.5) for functions $u(x) = u(|x|)$, which are non-negative, compactly supported, radially symmetric, and $u(|x|) : [0, \infty) \rightarrow \mathbf{R}$ are decreasing.

Following Moser's argument, we set

$$(1.1) \quad w(t) = N^{\frac{N-1}{N}} \omega_{N-1}^{\frac{1}{N}} u\left(e^{-\frac{t}{N}}\right), \quad |x|^N = e^{-t}.$$

Then $w(t)$ is defined on $(-\infty, \infty)$ and satisfies

$$(1.2) \quad w(t) \geq 0 \quad \text{for } t \in \mathbf{R},$$

$$(1.3) \quad \dot{w}(t) \geq 0 \quad \text{for } t \in \mathbf{R},$$

$$(1.4) \quad w(t_0) = 0 \quad \text{for some } t_0 \in \mathbf{R}.$$

Moreover we have

$$(1.5) \quad \int_{\mathbf{R}^N} |\nabla u|^N dx = \int_{-\infty}^{\infty} |\dot{w}(t)|^N dt,$$

$$(1.6) \quad \int_{\mathbf{R}^N} \Phi_N\left(\alpha u^{\frac{N}{N-1}}\right) dx = \frac{\omega_{N-1}}{N} \int_{-\infty}^{\infty} \Phi_N\left(\frac{\alpha}{\alpha_N} w(t)^{\frac{N}{N-1}}\right) e^{-t} dt,$$

$$(1.7) \quad \int_{\mathbf{R}^N} |u(x)|^N dx = \frac{1}{N^N} \int_{-\infty}^{\infty} |w(t)|^N e^{-t} dt.$$

Thus, to prove Theorem 0.1, it suffices to show that for any $\beta \in (0, 1)$ there exists a constant $C_\beta > 0$ such that

$$(1.8) \quad \int_{-\infty}^{\infty} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} dt \leq C_\beta \int_{-\infty}^{\infty} |w(t)|^N e^{-t} dt$$

for all functions $w(t)$ satisfying (1.2)–(1.4) and

$$(1.9) \quad \int_{-\infty}^{\infty} |\dot{w}(t)|^N dt = 1.$$

Proof of Theorem 0.1. Let $w(t)$ be a function satisfying (1.2)–(1.4) and (1.9). We set

$$T_0 = \sup\{t \in \mathbf{R}; w(t) \leq 1\} \in (-\infty, \infty].$$

We decompose the integral on the left-hand side of (1.8) according to the decomposition $(-\infty, \infty) = (-\infty, T_0] \cup [T_0, \infty)$.

For $t \in (-\infty, T_0]$, we have $w(t) \in [0, 1]$. We can find a constant $m_N > 0$ such that

$$\Phi_N(\xi) \leq m_N \xi^{N-1} \quad \text{for } \xi \in [0, 1].$$

Thus we have

$$(1.10) \quad \int_{-\infty}^{T_0} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} dt \leq m_N \int_{-\infty}^{T_0} w(t)^N e^{-t} dt.$$

Next we consider the integral over $[T_0, \infty)$. Since $w(T_0) = 1$, we have for $t \geq T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^t \dot{w}(\tau) d\tau \\ &\leq w(T_0) + (t - T_0)^{\frac{N-1}{N}} \left(\int_{T_0}^{\infty} \dot{w}(\tau)^N d\tau \right)^{\frac{1}{N}} \\ &\leq 1 + (t - T_0)^{\frac{N-1}{N}}. \end{aligned}$$

We remark that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$1 + s^{\frac{N-1}{N}} \leq ((1 + \varepsilon)s + C_\varepsilon)^{\frac{N-1}{N}} \quad \text{for all } s \geq 0.$$

Thus, we have

$$|w(t)|^{\frac{N}{N-1}} \leq (1 + \varepsilon)(t - T_0) + C_\varepsilon \quad \text{for } t \geq T_0.$$

Since $\beta \in (0, 1)$, we can choose $\varepsilon > 0$ small so that $\beta(1 + \varepsilon) < 1$. Thus we have

$$\begin{aligned} \int_{T_0}^{\infty} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}} \right) e^{-t} dt &\leq \int_{T_0}^{\infty} \exp \left(\beta w(t)^{\frac{N}{N-1}} - t \right) dt \\ &\leq \int_{T_0}^{\infty} \exp \left((\beta(1 + \varepsilon) - 1)(t - T_0) + \beta C_\varepsilon - T_0 \right) dt \\ (1.11) \qquad \qquad \qquad &= \frac{1}{1 - \beta(1 + \varepsilon)} e^{\beta C_\varepsilon} e^{-T_0}. \end{aligned}$$

On the other hand,

$$(1.12) \qquad \int_{T_0}^{\infty} |w(t)|^N e^{-t} dt \geq \int_{T_0}^{\infty} e^{-t} dt = e^{-T_0}.$$

Therefore it follows from (1.11) and (1.12) that

$$(1.13) \qquad \int_{T_0}^{\infty} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}} \right) e^{-t} dt \leq \frac{e^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)} \int_{T_0}^{\infty} |w(t)|^N e^{-t} dt.$$

Thus, setting $C_\beta = \max\{m_N, \frac{e^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)}\}$, we obtain (1.8). □

2. PROOF OF THEOREM 0.2

It suffices to show Theorem 0.2 for $\alpha = \alpha_N$. We use the idea of Moser again. Repeating the argument of the previous section, it suffices to find a sequence of functions $w_k(t) : \mathbf{R} \rightarrow \mathbf{R}$ which satisfies (1.1)–(1.4), (1.9) and

$$(2.1) \qquad \int_{-\infty}^{\infty} |w_k(t)|^N e^{-t} dt \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

$$(2.2) \qquad \int_{-\infty}^{\infty} \Phi_N \left(w_k(t)^{\frac{N}{N-1}} \right) e^{-t} dt \geq \frac{1}{2} \quad \text{for large } k.$$

If we define a sequence of functions $(u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbf{R}^N)$ from $(w_k(t))_{k=1}^{\infty}$ through the relation (1.1), it follows from (1.5)–(1.7), (1.9), (2.1) and (2.2) that $\|\nabla u_k\|_N = 1$ and (0.9). Thus $(u_k)_{k=1}^{\infty}$ has a desired property in Theorem 0.2.

Here we give an example of $(w_k(t))_{k=1}^\infty$ explicitly. We set

$$w_k(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ k^{\frac{N-1}{N}} \frac{t}{k} & \text{for } 0 \leq t \leq k, \\ k^{\frac{N-1}{N}} & \text{for } k \leq t. \end{cases}$$

Such functions appeared in [8] to show that the integral on the left-hand side of (0.1) can be made arbitrarily large for $\alpha > \alpha_N$. It is easily seen that $w_k(t)$ satisfies (1.2)–(1.4) and (1.9).

First we verify (2.1).

$$\begin{aligned} \int_{-\infty}^\infty |w_k(t)|^N e^{-t} dt &= \int_0^k \left(k^{\frac{N-1}{N}} \frac{t}{k}\right)^N e^{-t} dt + \int_k^\infty k^{N-1} e^{-t} dt \\ &\leq \frac{1}{k} \int_0^\infty t^N e^{-t} dt + k^{N-1} e^{-k} \\ &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Next we deal with (2.2).

$$\begin{aligned} &\int_{-\infty}^\infty \Phi_N \left(w_k(t)^{\frac{N}{N-1}}\right) e^{-t} dt \\ &= \int_0^k \Phi_N \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right) e^{-t} dt + \int_k^\infty \Phi_N(k) e^{-t} dt \\ &= \int_0^k \left(\exp \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right) - \sum_{j=0}^{N-2} \frac{1}{j!} \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right)^j\right) e^{-t} dt \\ &\quad + \Phi_N(k) e^{-k} \\ &= \int_0^k \exp \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}} - t\right) dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{j}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t} dt \\ &\quad + \left(e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j\right) e^{-k} \\ &\geq \int_0^k e^{-t} dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{j}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t} dt \\ &\quad + \left(e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j\right) e^{-k} \\ &\rightarrow 1 - 1 + 1 = 1 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Thus we obtain (2.1) and (2.2). This completes the proof of Theorem 0.2. □

Remark 2.1. The function $u_k(x)$ corresponding $w_k(t)$ has a compact support, i.e., $\text{supp } u_k(x) \subset \{x \in \mathbf{R}^N; |x| \leq 1\}$. Thus (0.10) with $\alpha = \alpha_N$ is false for $D = \{x \in \mathbf{R}^N; |x| < 1\}$. If we set for $a > 0$

$$w_{a,k}(t) = w_k(t + N \log a),$$

then corresponding $u_{a,k}(x)$ has a compact support, i.e., $\text{supp } u_{a,k}(x) \subset \{x \in \mathbf{R}^N; |x| \leq a\}$ and satisfies $\|\nabla u_{a,k}\|_N = 1$ and (0.9). Since we can choose $a > 0$ arbitrarily small, (0.10) with $\alpha = \alpha_N$ is false for any domain D .

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