TRUDINGER TYPE INEQUALITIES IN $\mathbb{R}^N$
AND THEIR BEST EXPOSANTS

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Abstract. We study Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents $\alpha_N$. We show for $\alpha \in (0, \alpha_N)$, $\alpha_N = N \omega_{N-1}^{1/(N-1)}$ ($\omega_{N-1}$ is the surface area of the unit sphere in $\mathbb{R}^N$), there exists a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{\|u(x)\|}{\|\nabla u\|_{L^N(\mathbb{R}^N)}} \right)^p \right) \, dx \leq C_\alpha \frac{\|u\|_{L^N(\mathbb{R}^N)}}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}$$

for all $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$. Here $\Phi_N(\xi)$ is defined by

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!} \xi^j.$$

It is also shown that $(\ast)$ with $\alpha \geq \alpha_N$ is false, which is different from the usual Trudinger’s inequalities in bounded domains.

0. Introduction

In this note, we study the limit case of Sobolev’s inequalities; suppose $N \geq 2$ and let $D \subset \mathbb{R}^N$ be an open set. We denote by $W_0^{1,N}(D)$ the usual Sobolev space, that is, the completion of $C_0^\infty(D)$ with the norm $\|u\|_{W_0^{1,p}(D)} = \|\nabla u\|_p + \|u\|_p$. Here

$$\|u\|_p = \left( \int_D |u|^p \, dx \right)^{1/p}.$$

It is well-known that

$$W_0^{1,p}(D) \subset L^{\frac{pN}{N-p}}(D) \quad \text{if } 1 \leq p < N,$$
$$W_0^{1,p}(D) \subset L^\infty(D) \quad \text{if } N < p.$$

The case $p = N$ is the limit case of these imbeddings and it is known that

$$W_0^{1,N}(D) \subset L^q(D) \quad \text{for } N \leq q < \infty,$$
$$W_0^{1,N}(D) \not\subset L^\infty(D).$$

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This case is studied by Trudinger [14] more precisely and he showed for bounded domains $D \subset \mathbb{R}^N$

\[(0.1) \quad \int_D \exp \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^\frac{N}{N-1} \right) \, dx \leq C |D| \]

for $u \in W^{1,N}_0(D) \setminus \{0\}$, where the constants $\alpha$, $C$ are independent of $u$ and $D$.

Trudinger’s result is extended into two directions; the first one is to find the best exponents in (0.1). Moser [8] proved that (0.1) holds for $N$ but not for $N > N$, where

\[(0.2) \quad \alpha_N = N \omega_{N-1}^{1/(N-1)} \]

and $\omega_{N-1}$ is the surface area of the unit sphere in $\mathbb{R}^N$. See also Adams [1]. We also refer to [3], [5], [7], [13] for the attainability of

\[
\sup \left\{ \int_D \exp \left( \alpha_N \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^\frac{N}{N-1} \right) \, dx; \ u \in W^{1,N}_0(D) \setminus \{0\} \right\}.
\]

The second direction is to extend Trudinger’s result for unbounded domains and for Sobolev spaces of higher order and fractional order. We refer to D. R. Adams [1], R. A. Adams [2], Ogawa [9], Ogawa-Ozawa [10], Ozawa [11], Strichartz [12].

In this paper, we study a version of Trudinger inequalities in $\mathbb{R}^N$ and their best exponents; we show

\[(0.3) \quad \int_{\mathbb{R}^N} \exp \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^\frac{N}{N-1} \right) \, dx \leq C \|u\|_N^{\frac{N}{N-1}} \]

for $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, where $\alpha$, $C > 0$ are independent of $u$, and we also find the best exponents $\alpha$ for (0.3).

In [9], [11], [2], (0.3) and related inequalities are obtained without studying their best exponents; Ogawa [9] obtained (0.3) for $N = 2$ and Ozawa [11] extended it for functions in the Sobolev space $H^{N/p; p}(\mathbb{R}^N) = (1 - \Delta)^{-N/2p}L^p(\mathbb{R}^N)$ of fractional order. See also [10]. Adams [2] studied a different version of (0.3); however the dependence in $u$ of the right-hand side is not given explicitly.

The main purpose of this paper is to study the best exponents $\alpha$ in (0.3) as well as to give a simplified proof of (0.3).

To simplify notation, we use

\[(0.4) \quad \Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!} \xi^j. \]

With this notation, (0.3) becomes

\[(0.5) \quad \int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( \frac{|u(x)|}{\|\nabla u\|_N} \right)^\frac{N}{N-1} \right) \, dx \leq C \|u\|_N^{\frac{N}{N-1}}. \]

One of the virtues of the inequality (0.5) is its scale-invariance; for $u \in W^{1,N}(\mathbb{R}^N)$ and $\lambda > 0$, we set

\[(0.6) \quad u_\lambda(x) = u(\lambda x). \]
We can easily see that \( k \| u \|_{N} = \| \nabla u \|_{N} \) and

\[
\int_{\mathbb{R}^{N}} \Phi_{N} \left( \alpha \left( \frac{|u(x)|}{\| \nabla u \|_{N}} \right)^{N \alpha} \right) dx = \lambda^{-N} \int_{\mathbb{R}^{N}} \Phi_{N} \left( \alpha \left( \frac{|u(x)|}{\| \nabla u \|_{N}} \right)^{N \alpha} \right) dx,
\]

(0.7)

\[
\| u \|_{N}^{N} = \lambda^{-N} \| u \|_{N}^{N}.
\]

Thus (0.5) is invariant under the scaling (0.6) and we believe the best exponents \( \alpha \) in (0.5) are of interest.

Our main result is the following.

**Theorem 0.1.** Suppose \( N \geq 2 \). Then for any \( \alpha \in (0, \alpha_N) \) (\( \alpha_N \) is given in (0.2)), there exists a constant \( C_\alpha > 0 \) such that

\[
\int_{\mathbb{R}^{N}} \Phi_{N} \left( \alpha \left( \frac{|u(x)|}{\| \nabla u \|_{N}} \right)^{N \alpha} \right) dx \leq C_\alpha \frac{\| u \|_{N}^{N}}{\| \nabla u \|_{N}^{N}} \quad \text{for } u \in W^{1,N}(\mathbb{R}^{N}) \setminus \{0\}.
\]

We remark that the restriction \( \alpha < \alpha_N \) is optimal. The limit exponent \( \alpha_N \) is excluded for (0.5). It is quite different from Moser’s result for (0.1).

**Theorem 0.2.** For \( \alpha \geq \alpha_N \), there exists a sequence \( (u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbb{R}^{N}) \) such that \( \| \nabla u_k \|_{N} = 1 \) and

\[
\frac{1}{\| u_k \|_{N}^{N}} \int_{\mathbb{R}^{N}} \Phi_{N} \left( \alpha \left( \frac{|u_k(x)|}{\| \nabla u_k \|_{N}} \right)^{N \alpha} \right) dx \geq \frac{1}{\| u_k \|_{N}^{N}} \int_{\mathbb{R}^{N}} \Phi_{N} \left( \alpha_N \left( \frac{|u_k(x)|}{\| \nabla u_k \|_{N}} \right)^{N \alpha} \right) dx \to \infty
\]

as \( k \to \infty \).

**Remark 0.3.** Even if we consider (0.5) in a bounded domain \( D \), i.e.,

\[
\int_{D} \Phi_{N} \left( \alpha \left( \frac{|u(x)|}{\| \nabla u \|_{N}} \right)^{N \alpha} \right) dx \leq C_\alpha \frac{\| u \|_{N}^{N}}{\| \nabla u \|_{N}^{N}} \quad \text{for } u \in W^{1,N}(D) \setminus \{0\},
\]

the limit exponent \( \alpha_N \) is still excluded. It is because of the scale-invariance (0.7)–(0.8). See Remark 2.1 below.

As to the proof of the inequality (0.5), following the original idea of Trudinger, \[9\], \[10\], \[11\] made use of a combination of the power series expansion of the exponential function and sharp multiplicative inequalities:

\[
\| u \|_{q} \leq C(N, q) \| u \|_{N}^{N/q} \| \nabla u \|_{N}^{1-1-N/q}.
\]

(0.11)

For multiplicative inequalities of type (0.11) and their applications, we refer to Edmunds-Ilyin \[4\] and Kozono-Ogawa-Sohr \[6\]. We also remark that in Ozawa \[11\] multiplicative inequalities for functions \( H^{N/p;p}(<\mathbb{R}^{N}) \) are given and they are applied to obtain Brezis-Gallouet-Wainger type inequalities.

We give proofs of Theorems 0.1 and 0.2 in the following sections. We take a different approach from \[9\], \[10\], \[11\], we use Moser’s idea; we take symmetrization of functions and we reduce (0.5) to one-dimensional inequality.
1. Proof of Theorem 0.1

To prove Theorem 0.1, we use an idea of Moser [8]. By means of symmetrization, it suffices to show the desired inequality (0.5) for functions \( u(x) = u(|x|) \), which are non-negative, compactly supported, radially symmetric, and \( u(|x|) : [0, \infty) \to \mathbb{R} \) are decreasing.

Following Moser’s argument, we set

\[
W(t) = N^{-\frac{N-1}{2}} \frac{1}{\omega_{N-1}} u \left( e^{-\frac{t}{N}} \right), \quad |x|^N = e^{-t}.
\]

Then \( W(t) \) is defined on \((0, \infty)\) and satisfies

\[
\begin{align*}
W(t) &\geq 0 \quad \text{for } t \in \mathbb{R}, \\
\dot{W}(t) &\geq 0 \quad \text{for } t \in \mathbb{R}, \\
W(t_0) &= 0 \quad \text{for some } t_0 \in \mathbb{R}.
\end{align*}
\]

Moreover we have

\[
\begin{align*}
\int_{\mathbb{R}^N} |\nabla u|^N \, dx &= \int_{-\infty}^\infty |\dot{W}(t)|^N \, dt, \\
\int_{\mathbb{R}^N} \Phi_N \left( \alpha u^{\frac{N-1}{N-\beta}} \right) \, dx &= \frac{\omega_{N-1}}{N} \int_{-\infty}^\infty \Phi_N \left( \frac{\alpha}{\omega_{N-1}} w(t)^{\frac{N}{N-1}} \right) e^{-t} \, dt, \\
\int_{\mathbb{R}^N} |u(x)|^N \, dx &= \frac{1}{N^N} \int_{-\infty}^\infty |w(t)|^N e^{-t} \, dt.
\end{align*}
\]

Thus, to prove Theorem 0.1, it suffices to show that for any \( \beta \in (0, 1) \) there exists a constant \( C_{\beta} > 0 \) such that

\[
\int_{-\infty}^\infty \Phi_N \left( \beta w(t)^{\frac{N}{N-\beta}} \right) e^{-t} \, dt \leq C_{\beta} \int_{-\infty}^\infty |w(t)|^N e^{-t} \, dt
\]

for all functions \( w(t) \) satisfying (1.2)–(1.4) and

\[
\int_{-\infty}^\infty |\dot{w}(t)|^N \, dt = 1.
\]

Proof of Theorem 0.1. Let \( w(t) \) be a function satisfying (1.2)–(1.4) and (1.9). We set

\[
T_0 = \sup \{ t \in \mathbb{R}; w(t) \leq 1 \} \in (-\infty, \infty].
\]

We decompose the integral on the left-hand side of (1.8) according to the decomposition \((-\infty, \infty) = (-\infty, T_0] \cup [T_0, \infty).\)

For \( t \in (-\infty, T_0] \), we have \( w(t) \in [0, 1] \). We can find a constant \( m_N > 0 \) such that

\[
\Phi_N(\xi) \leq m_N \xi^{N-1} \quad \text{for } \xi \in [0, 1].
\]

Thus we have

\[
\int_{-\infty}^{T_0} \Phi_N \left( \beta w(t)^{\frac{N}{N-\beta}} \right) e^{-t} \, dt \leq m_N \int_{-\infty}^{T_0} w(t)^N e^{-t} \, dt.
\]
Next we consider the integral over \([T_0, \infty)\). Since \(w(T_0) = 1\), we have for \(t \geq T_0\)
\[
w(t) = w(T_0) + \int_{T_0}^{t} \dot{w}(\tau) d\tau
\leq w(T_0) + (t - T_0)^{\frac{N-1}{N}} \left( \int_{T_0}^{\infty} \dot{w}(\tau)^N d\tau \right)^{\frac{1}{N}}
\leq 1 + (t - T_0)^{\frac{N-1}{N}}.
\]
We remark that for any \(\varepsilon > 0\) there exists a constant \(C_\varepsilon > 0\) such that
\[1 + s^{\frac{N}{N-1}} \leq ((1 + \varepsilon)s + C_\varepsilon)^{\frac{N}{N-1}} \quad \text{for all} \ s \geq 0.
\]
Thus, we have
\[|w(t)|^{\frac{N}{N-1}} \leq (1 + \varepsilon)(t - T_0) + C_\varepsilon \quad \text{for} \ t \geq T_0.
\]
Since \(\beta \in (0, 1)\), we can choose \(\varepsilon > 0\) small so that \(\varepsilon(1 + \varepsilon) < 1\). Thus we have
\[
\int_{T_0}^{\infty} \Phi_N \left( \beta w(t)^{\frac{N}{N-1}} \right) e^{-t} dt \leq \int_{T_0}^{\infty} \exp \left( \beta w(t)^{\frac{N}{N-1}} - t \right) dt
\leq \int_{T_0}^{\infty} \exp \left( (\beta(1 + \varepsilon) - 1)(t - T_0) + \beta C_\varepsilon - T_0 \right) dt
= \frac{1}{1 - \beta(1 + \varepsilon)} e^{\beta C_\varepsilon e^{-T_0}}.
\]
On the other hand,
\[
\int_{T_0}^{\infty} |w(t)|^N e^{-t} dt \geq \int_{T_0}^{\infty} e^{-t} dt = e^{-T_0}.
\]
Therefore it follows from (1.11) and (1.12) that
\[
\int_{T_0}^{\infty} \Phi_N \left( \beta w(t)^{\frac{N}{N-1}} \right) e^{-t} dt \leq \frac{e^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)} \int_{T_0}^{\infty} |w(t)|^N e^{-t} dt.
\]
Thus, setting \(C_\beta = \max\{m_N, \frac{e^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)}\}\), we obtain (1.8).

2. Proof of Theorem 0.2

It suffices to show Theorem 0.2 for \(\alpha = \alpha_N\). We use the idea of Moser again. Repeating the argument of the previous section, it suffices to find a sequence of functions \(w_k(t) : \mathbb{R} \to \mathbb{R}\) which satisfies (1.1)–(1.4), (1.9) and
\[
\int_{-\infty}^{\infty} |w_k(t)|^N e^{-t} dt \to 0 \quad \text{as} \ k \to \infty,
\]
\[
\int_{-\infty}^{\infty} \Phi_N \left( w_k(t)^{\frac{N}{N-1}} \right) e^{-t} dt \geq \frac{1}{2} \quad \text{for large} \ k.
\]
If we define a sequence of functions \((u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbb{R}^N)\) from \((w_k(t))_{k=1}^{\infty}\) through the relation (1.1), it follows from (1.5)–(1.7), (1.9), (2.1) and (2.2) that \(\|
\nabla u_k\|_N = 1\) and (0.9). Thus \((u_k)_{k=1}^{\infty}\) has a desired property in Theorem 0.2.
Here we give an example of \((w_k(t))_{k=1}^{\infty}\) explicitly. We set
\[
w_k(t) = \begin{cases} 
0 & \text{for } t \leq 0, \\
k^{N-1} \frac{t}{k} & \text{for } 0 \leq t < k, \\
k^{N-1} \frac{1}{N} & \text{for } k \leq t.
\end{cases}
\]
Such functions appeared in [8] to show that the integral on the left-hand side of (0.1) can be made arbitrarily large for \(\alpha > \alpha_N\). It is easily seen that \(w_k(t)\) satisfies (1.2)–(1.4) and (1.9).

First we verify (2.1).
\[
\int_{-\infty}^{\infty} |w_k(t)|^N e^{-t} dt = \int_0^k \left( k^{\frac{N-1}{N}} \frac{t}{k} \right)^N e^{-t} dt + \int_k^{\infty} k^{N-1} e^{-t} dt \\
\leq \frac{1}{k} \int_0^{\infty} t^N e^{-t} dt + k^{N-1} e^{-k} \\
\rightarrow 0 \text{ as } k \rightarrow \infty.
\]

Next we deal with (2.2).
\[
\int_{-\infty}^{\infty} \Phi_N \left( w_k(t)^{\frac{N}{N-1}} \right) e^{-t} dt \\
= \int_0^k \Phi_N \left( \left( \frac{t}{k} \right)^{\frac{N}{N-1}} \right) e^{-t} dt + \int_k^{\infty} \Phi_N(k) e^{-t} dt \\
= \int_0^k \exp \left( \left( \frac{t}{k} \right)^{\frac{N}{N-1}} \right) - \sum_{j=0}^{N-2} \frac{1}{j!} \left( \left( \frac{t}{k} \right)^{\frac{N}{N-1}} \right)^j e^{-t} dt \\
+ \Phi_N(k) e^{-k} \\
= \int_0^k \exp \left( \left( \frac{t}{k} \right)^{\frac{N}{N-1}} - t \right) dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{N}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t} dt \\
+ \left( e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j \right) e^{-k} \\
\geq \int_0^k e^{-t} dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{N}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t} dt \\
+ \left( e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j \right) e^{-k} \\
\rightarrow 1 - 1 + 1 = 1 \text{ as } k \rightarrow \infty.
\]

Thus we obtain (2.1) and (2.2). This completes the proof of Theorem 0.2. \(\square\)

Remark 2.1. The function \(u_k(x)\) corresponding \(w_k(t)\) has a compact support, i.e., \(\text{supp } u_k(x) \subset \{ x \in \mathbb{R}^N ; |x| \leq 1 \}\). Thus (0.10) with \(\alpha = \alpha_N\) is false for \(D = \{ x \in \mathbb{R}^N ; |x| < 1 \}\). If we set for \(a > 0\)
\[
w_a,k(t) = w_k(t + N \log a),
\]
then corresponding $u_{a,k}(x)$ has a compact support, i.e., $\text{supp} u_{a,k}(x) \subset \{ x \in \mathbb{R}^N : |x| \leq a \}$ and satisfies $\| \nabla u_{a,k} \|_N = 1$ and (0.9). Since we can choose $a > 0$ arbitrarily small, (0.10) with $\alpha = \alpha_N$ is false for any domain $D$.

References


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