

A SCALING EQUATION WITH ONLY NON-MEASURABLE ORTHOGONAL SOLUTIONS

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ABSTRACT. In this paper we construct a non-measurable scaling function for the scaling equation

$$\phi(x) = \phi(3x) + \phi(3x - 2) + \phi(3x - 4),$$

using the Axiom of Choice. We prove that –apart from being non-measurable– it satisfies the classical conditions for a scaling function to lead to orthonormal wavelets.

While non-measurable functions are not directly useful for numerical calculations, the example given here explains the possible anomalous behaviour of numerical methods. Indeed the origin of this paper lies in the solution of the scaling equation above, calculated on a finite grid consisting of the points $i/3^k$, $0 \leq i < 2 \cdot 3^k$. The result is

$$\phi\left(\frac{i}{3^k}\right) = \begin{cases} 0 & i \text{ odd,} \\ 1 & i \text{ even.} \end{cases}$$

This finite approximation seemingly satisfies the conditions for a scaling equation but, as we show here, any extension is either non-measurable or not orthogonal. While in this case, and for the chosen grid, the strange behaviour is apparent from the graph of the approximating function which looks like a comb, there is of course no guarantee that this will be clear in different situations.

INTRODUCTION

In the sequel, any finite sequence of scalars is supposed to be padded at left and at right with zeros. If e.g. the sequence (a_0, \dots, a_k) is given, this implies that $a_i = 0$ for $i < 0$, as well as for $i > k$. If no boundaries for a summation are, implicitly or explicitly, given the index range is \mathbb{Z} .

Let m , an integer greater than 1, be the so-called *scaling factor*. A sequence (a_0, \dots, a_n) of complex numbers is called a set of scaling coefficients iff it satisfies

$$(0.1) \quad \sum_i a_i = m,$$

$$(0.2) \quad \sum_i a_i \overline{a_{i-mk}} = m\delta_{0k}, \quad \forall k \in \mathbb{Z}.$$

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Given a set of scaling coefficients a function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ is called a scaling function iff it has compact support and satisfies

- (1) $\phi(x) = \sum_i a_i \phi(mx - i)$,
- (2) $\int_0^1 (\sum_i \phi(x + i)) dx = 1$,
- (3) $\int_{-\infty}^{+\infty} \phi(x) \overline{\phi(x - k)} = 0$, for all non-zero $k \in \mathbb{Z}$.

Usually (2) is stated in the form $\int_{-\infty}^{+\infty} \phi(x) dx = 1$, which does not make sense for non-measurable functions. Moreover, if ϕ is an L_2 function (which it certainly is if it satisfies the conditions (1)-(3) and is bounded and measurable as well) it is possible to construct an orthonormal basis of $L_2(\mathbb{R})$ of wavelets from it (this is a classical way of obtaining wavelets, see e.g. [2]). Formal application of the method of building wavelets to the non-measurable scaling functions given below leads to non-measurable functions.

W. Lawton proved in [1] that, if $m = 2$, almost all sets of scaling coefficients lead to orthonormal wavelets. The argument is easily extended to $m > 2$. Here we take one of the exceptional cases and 'construct' scaling functions satisfying (1)-(3) which are not measurable. The construction can be generalised to some, but not all, exceptional cases, and it is an open question whether a scaling function satisfying (1)-(3) exists for all possible scaling coefficients. The result of this construction is the following:

Theorem 0.1. *There exist functions ϕ , zero outside the interval $[0, 2]$, satisfying:*

- (1) $\phi(x) = \phi(3x) + \phi(3x - 2) + \phi(3x - 4)$.
- (2) $\phi(x) + \phi(x + 1) = 1$, almost everywhere in $[0, 1]$.
- (3) $\int_{-\infty}^{+\infty} \phi(x) \overline{\phi(x - 1)} = 0$.

None of them is measurable.

1. CONSTRUCTING ϕ

The functions satisfying (1)-(3) constructed here will be characteristic functions of a subset of $[0, 2]$. A suitable subset will be constructed using the Axiom of Choice.

For arbitrary x in the half-open interval $[0, 2[$ there is exactly one element in the set $\{3x, 3x - 2, 3x - 4\} \cap [0, 2[$. Denoting this element as $y(x)$, (1) can be rewritten as

$$(1.1) \quad \phi(x) = \phi(y(x)).$$

The aim of the construction is now to build a set S such that $y(x)$ is in S if and only if x itself is, and such that either $x + 1$ or x is in S , but not both. These conditions should hold for x almost everywhere in $[0, 1]$.

To comply with the first condition, S must be the union of equivalence classes for the equivalence relation \sim on $[0, 2[$ defined by

$$x \sim t \iff \exists m, n, l \in \mathbb{Z} : 3^m x - 3^n t + 2l = 0.$$

This indeed is an equivalence relation: symmetry and reflexivity are obvious. If both $3^m x - 3^n t + 2l = 0$ and $3^p t - 3^q s + 2k = 0$ (where we can assume n and q are positive), then $3^{m+p} x - 3^{q+n} s + 2(3^p l + 3^n k) = 0$. This proves transitivity. The equivalence class containing x will be denoted by $[x]$.

Now we take separately the set

$$R = \{x : x \sim x + 1\}.$$

Lemma 1.1. *R is countable. Moreover it is a union of equivalence classes.*

Proof. The set of triples (m, n, l) is \mathbb{Z}^3 , which is obviously countable. Writing $U(m, n, l)$ for $\{x : 3^m x - 3^n(x + 1) + 2l = 0\}$, the definition of R can be rewritten as

$$R = \bigcup_{(m,n,l) \in \mathbb{Z}^3} U(m, n, l).$$

If we prove that, for each triple in \mathbb{Z}^3 , $U(m, n, l)$ contains at most one element, it is clear that R is countable. Notice that the equation for elements of $U(m, n, l)$ can be rewritten as $(3^m - 3^n)x - 3^n + 2l = 0$. There are two cases:

- $m = n$. The coefficient of x in the equation becomes zero, but 3^n is never an even integer, so the equation is of the form $0x + b = 0$, where $b \neq 0$, and has no solutions.
- $m \neq n$. The coefficient of x is non-zero, and the equation has exactly one solution (which can be outside $[0, 2]$).

To prove that R is the union of equivalence classes, notice that, in general, $x \sim t$ implies $x + 1 \sim t + 1$. Indeed, in the equation $3^m x - 3^n t + 2l = 0$ we can take both m and n non-negative, and then we obtain $3^m(x + 1) - 3^n(t + 1) + 2l - 3^m + 3^n = 0$. But $3^n - 3^m$ is obviously an even integer, proving that $x + 1 \sim t + 1$. So, let x be in R . Then for any $t \in [x]$, $t \sim x \sim x + 1 \sim t + 1$, and so $t \in R$. □

Take now all pairs of equivalence classes of the form $([x], [x + 1])$, where $x \notin R$, as well as the pair (R, \emptyset) . Notice the pair $([x], [x + 1])$ is independent from the element in the equivalence class chosen: if $[x] = [t]$, and both are smaller than one, then $[x + 1] = [t + 1]$. Using the Axiom of Choice take V containing exactly one element of each pair, and define

$$S = \bigcup_{A \in V} A.$$

Then, as we shall presently prove, $\phi = \chi_S$ satisfies the conditions.

Proof of Theorem 0.1. That $\phi = \chi_S$ satisfies (1) follows directly from the construction, because S is a union of equivalence classes. Take $x \in [0, 2[$ arbitrary: the element $y(x)$ is equivalent to x because of the definition of the equivalence classes, and so (1.1) is satisfied, since ϕ is, by construction, constant on equivalence classes.

For (2), the equation $\phi(x) + \phi(x + 1) = 1$ is obviously true for all x with $0 \leq x \leq 1$ unless $x \in R$, but R is a countable set having measure zero.

Finally (3). Obviously $\phi(x)\phi(x - 1) = 0$ for $x \notin R$, and R has measure zero, so the integral is zero. Notice that the function $\phi(x)\phi(x - 1)$ is measurable, even if $\phi(x)$ itself does not satisfy this condition.

For the second part assume that a function ϕ satisfies (1), (2) and (3). Then we shall prove, by contradiction, that ϕ is not measurable.

Assume that ϕ is measurable. We first prove that it follows that ϕ is a function in $L_2([0, 2])$. The integral in (3) can be rewritten (using the fact that ϕ equals zero

outside $[0, 2]$, and condition (2)) as

$$\begin{aligned} \int_{-\infty}^{+\infty} \phi(x) \overline{\phi(x-1)} &= \int_0^1 \overline{\phi(x)} \phi(x+1) \\ &= \int_0^1 \overline{\phi(x)} (1 - \phi(x)) \\ &= \int_0^1 \overline{\phi(x)} - |\phi(x)|^2. \end{aligned}$$

Split $[0, 1]$ into the set $A = \{x \in [0, 1] : |\phi(x)| \geq 2\}$ and its complement $B = [0, 1] \setminus A$. The integral over B of $\overline{\phi} - |\phi|^2$ is obviously finite, and the integral over A equals this (up to sign), and must be finite too. On A we have the estimate

$$-\Re(\overline{\phi(x)} - |\phi(x)|^2) \geq |\phi(x)|^2 - |\phi(x)| \geq \frac{1}{2} |\phi(x)|^2.$$

Therefore $\int_A |\phi(x)|^2$ is finite. Since $\int_B |\phi(x)|^2 \leq 4$ obviously is finite, we have that ϕ is L_2 on the interval $[0, 1]$ and, using condition (2), it then must be L_2 on $[1, 2]$.

We now show ϕ is not measurable. Take the function $\psi = (1/2)\chi_{[0,2]}$, which is in $L_2(\mathbb{R})$. We shall show that from the assumption that ϕ is measurable it follows that ϕ and ψ are equal in the L_2 -sense. This however implies that $\phi(x) = \psi(x)$ a.e. which is absurd, because ψ does not satisfy condition (3).

Consider ϕ and ψ as elements of the dual of $L_2(\mathbb{R})$. To prove that $\phi = \psi$ in the L_2 -sense, it is sufficient to prove that $\langle \psi, f \rangle = \langle \phi, f \rangle$ for any $f \in L_2(\mathbb{R})$, with the pairing

$$\langle g, f \rangle = \int_{-\infty}^{+\infty} \overline{g(x)} f(x).$$

As a matter of fact, by continuity, it is sufficient to prove this for the dense subspace of step functions, and for this it is sufficient to prove it for characteristic functions of intervals, and even of those intervals $[a, b]$ for which $0 \leq a < b < 2$ (if $f = \chi_{[a,b]}$ with $a > 2$ or $b < 0$, then obviously $\langle \psi, f \rangle = \langle \phi, f \rangle = 0$). Assume this holds.

Notice that $\phi(x) = 1 - \phi(x + 3^{-n})$ a.e. in $[0, 2 - 3^{-n}]$ because $x + 3^{-n} \sim x + 1$, and so, for $3^{-n} < 2 - b$

$$\begin{aligned} \langle \phi, f \rangle &= \int_a^b \phi(x) dx \\ &= \int_{a+3^{-n}}^{b+3^{-n}} (1 - \phi(x)) dx \\ &= (b - a) - \int_{a+3^{-n}}^{b+3^{-n}} \phi(x) dx. \end{aligned}$$

But, since $\lim_{n \rightarrow \infty} \int_{a+3^{-n}}^{b+3^{-n}} \phi(x) dx = \int_a^b \phi(x) dx$ we obtain $\langle \phi, f \rangle = (b - a) - \langle \phi, f \rangle$, and finally $\langle \phi, f \rangle = (1/2)(b - a) = \langle \psi, f \rangle$. \square

2. NUMERICAL APPROXIMATIONS

Lawton proved only a sufficient criterion for a scaling equation to lead to orthonormal wavelets, and one might be tempted to try with a numerical method

whether a suitable scaling function exists. The construction above gives an example where a numerical approximation of a scaling function seems to indicate orthonormal wavelets can be constructed, while this is actually not true.

Indeed, any numerical method can only result in a value of ϕ in a set A containing a finite number of points (and therefore intersecting a finite number of equivalence classes). Therefore, if A does not contain any of the exceptional points for which $x \sim x + 1$, a function ϕ on A can be found satisfying the finite approximations of conditions (1)–(3) of Theorem 0.1.

The most straightforward choice for A in this case is the finite grid consisting of the points $i/3^k$, $0 \leq i < 2 \cdot 3^k$, for some fixed positive integer k . The result of numerical approximation is

$$\phi\left(\frac{i}{3^k}\right) = \begin{cases} 0 & i \text{ odd,} \\ 1 & i \text{ even,} \end{cases}$$

extended with $\phi(i/3^k) = 0$ for $i < 0$ or $i \geq 2 \cdot 3^k$. Since, if i is odd (even) also $3i$, $3i - 2 \cdot 3^k$ and $3i - 4 \cdot 3^k$ are odd (even), while $i + 3^k$ and $i - 3^k$ are even (odd) it is immediate that the finite approximate conditions

$$\phi\left(\frac{i}{3^k}\right) = \phi\left(\frac{3i}{3^k}\right) + \phi\left(\frac{3i - 2 \cdot 3^k}{3^k}\right) + \phi\left(\frac{3i - 4 \cdot 3^k}{3^k}\right),$$

$$\phi\left(\frac{i}{3^k}\right) + \phi\left(\frac{i + 3^k}{3^k}\right) = 1,$$

$$\sum_{i=0}^{2 \cdot 3^k} \phi\left(\frac{i}{3^k}\right) \phi\left(\frac{i - 3^k}{3^k}\right) = 0$$

hold.

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