Higher order symmetric spaces and the roots of the identity in a Lie group

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Abstract. Let $r_k(G)$ denote the set of all $k$-roots of the identity in a Lie group $G$. We show that $r_k(G)$ is always an embedded submanifold of $G$, having the conjugacy classes of its elements as open submanifolds. These conjugacy classes are examples of $k$-symmetric spaces and we show, more generally, that every $k$-symmetric space of a Lie group $G$ is a covering manifold of an embedded submanifold $\text{Orb}$ of $G$. We compute also the Hessian of the inclusions of $r_k(G)$ and $\text{Orb}$ into $G$, relative to the natural connection on the domain and to the symmetric connection on $G$.

1. Introduction

There are several examples of homogeneous manifolds of a Lie group $G$ that can be realized equivariantly as connected components of the set $r_k(G)$ of all $k$-roots of the identity $e$ of $G$, with $G$ acting by conjugation, in particular as conjugation classes of elements of $G$: If $E$ is a Hermitian vector space, the Grassmann manifold $Gr_p(E)$, whose elements are the $p$-dimensional vector subspaces, can be realized as a connected component of $r_2(U(E))$, as observed by Uhlenbeck [6], and, in an analogous way, if $E$ is a Euclidean space, $Gr_p(E)$ admits a connected component of $r_2(O(E))$ as an equivariant model; More generally, if $E$ is a Hermitian space, the connected components of $r_k(U(E))$ are models of the flag manifolds $G_{p_1,\ldots,p_k}(E)$, whose elements are the systems $(F_1,\ldots,F_k)$ of mutually orthogonal subspaces, with dimensions $p_1,\ldots,p_k$, whose direct sum is $E$ [3]; If $E$ is a Euclidean space, the connected components of $r_3(O(E))$ are models of the manifolds $F_p(E)$, whose elements are the $p$-dimensional partially complex structures, i.e. the couples $(F,J)$, where $J$ is a compatible complex structure on the $2p$-dimensional real subspace $F \subset E$ [4], with the exception of the extreme case $\dim(E) = 2p$, where $F_p(E)$ is the union of two connected components.

With the previous examples in mind, we prove that, for a general Lie group $G$, $r_k(G)$ is always a submanifold of $G$, in general with variable dimension, having the conjugation classes of its elements as open submanifolds (we will use always the word “submanifold” with the meaning “embedded submanifold”).

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The conjugation classes of elements of \( r_k(G) \) are examples of \( k \)-symmetric manifolds; they are of the form \( G/G' \), where \( G' \) is the fixed point subgroup of a smooth automorphism \( \tau: G \to G \), satisfying \( \tau^k = I_{\text{ad}} \). The structure and classification of \( k \)-symmetric manifolds have been extensively studied in \([3]\) and it has been known for a long time (cf. \([2]\)) that every \( 2 \)-symmetric manifold of the form \( G/G' \) can be one-to-one immersed into the Lie group \( G \), by associating \( g\tau(g^{-1}) \) with the class of an element \( g \in G \), a fact that can be generalized trivially to \( k \)-symmetric spaces. We will prove, more precisely, that this one-to-one immersion is always an embedding, and hence that the \( k \)-symmetric space \( G/G' \) admits a model \( \text{Orb} \) that is a submanifold of \( G \). This fact was established in \([1]\), for the special case where \( G \) is compact, where the name “Cartan embedding” is used (in fact, Burstall uses the inverse \( \tau(g)g^{-1} \) instead of \( g\tau(g^{-1}) \) but this makes no essential difference). Of course, for general \( k \)-symmetric spaces, those of the form \( G/H \), with \( H \) an open subgroup of \( G' \), all we can guarantee is that they are covering manifolds of the manifold \( \text{Orb} \).

Every \( k \)-symmetric space is reductive in a canonical way and, as such, it has a canonical connection, and it is known \([2]\) that, for \( k = 2 \), its embedding into the Lie group \( G \), considered with its symmetric connection, is totally geodesic. This fact led us to compute, for general \( k \), the Hessian of the inclusions of \( r_k(G) \) and \( \text{Orb} \) into \( G \).

In the next section we will prove a basic lemma that guarantees that, under certain conditions, the image of a smooth map is a smooth manifold. This lemma will be used in section 3 in order to prove that \( \text{Orb} \) is a submanifold but it is possible that it may present some independent interest.

2. A BASIC LEMMA

**Lemma 1.** Let \( X, Y \) and \( Z \) be manifolds and \( f: X \to Y \) and \( \varphi, \psi: Y \to Z \) be smooth maps such that \( \varphi \circ f = \psi \circ f \). Let \( x_0 \in X \) be such that every vector \( v \in T_{f(x_0)}(Y) \) verifying the condition \( D\varphi_{f(x_0)}(v) = D\psi_{f(x_0)}(v) \) is in the image of \( Df_{x_0}: T_{x_0}(X) \to T_{f(x_0)}(Y) \). Then the set \( B = \{ y \in Y \mid \varphi(y) = \psi(y) \} \) is a submanifold of \( Y \) at \( f(x_0) \), with the image of \( Df_{x_0} \) as tangent space, and \( f(X) \) is a neighborhood of \( f(x_0) \) in \( B \); in particular \( f(X) \) is also a submanifold of \( Y \) at \( f(x_0) \), with the same tangent space.

**Proof.** The question being local, we may assume that \( X, Y \) and \( Z \) are open in finite dimensional spaces \( E, F \) and \( G \) and that \( x_0 = 0 \) and \( f(x_0) = 0 \). Let us fix norms in these vector spaces, let \( H \subset F \) be a direct sum complement of the vector subspace \( Df_{x_0}(E) \) and let \( g = \psi - \varphi: Y \to G \). The fact that the linear map \( Dg_0: F \to G \) is one-to-one in \( H \) allows us to consider \( \delta > 0 \) such that, for each \( w \in H \), \( \|Dg_0(w)\| \geq \delta \|w\| \) (if \( H \) is not trivial, let \( \delta \) be the minimum of \( \|Dg_0(w)\| \), for \( w \in H \) with \( \|w\| = 1 \)). Let \( \varepsilon > 0 \) be such that, for each \( y \in F \) with \( \|y\| < \varepsilon \), we have \( \|Dg_0 - Dg_0|| \leq \delta/2 \). By the mean value theorem, if \( \|y'\| < \varepsilon \) and \( \|y''\| < \varepsilon \), then \( \|g(y') - g(y'') - Dg_0(y' - y'')\| \leq \delta/2\|y' - y''\| \). Let \( U \subset X \) and \( W \subset H \) be open sets, with \( 0 \in U \) and \( 0 \in W \), such that, for each \( x \in U \) and \( w \in W \), \( \|f(x) + w\| < \varepsilon \) and let us remark that, for \( x \in U \) and \( w \in W \), \( f(x) + w \in B \) if and only if \( w = 0 \). In fact, one of the implications is trivial and, for the other, if \( f(x) + w \in B \), then

\[
\delta \|w\| \leq \|Dg_0(w)\| = \|g(f(x) + w) - g(f(x)) - Dg_0(w)\| \leq \frac{\delta}{2} \|w\|;
\]
hence \( w = 0 \). The derivative at \((0,0)\) of the map \( X \times H \to F, (x,w) \mapsto f(x) + w \), maps \((u,v)\) onto \( Df_0(u) + v \) and is hence onto, so that a standard result about submersions guarantees the existence of an open set \( V \), with \( 0 \in V \subset Y \), and of smooth maps \( \sigma_1: V \to U \subset E \) and \( \sigma_2: V \to W \subset H \), satisfying \( \sigma_1(0) = 0 \), \( \sigma_2(0) = 0 \) and \( f(\sigma_1(y)) + \sigma_2(y) = y \), for each \( y \in V \). By derivation, we have

\[
(1) \quad Df_0(\sigma_{10}(v)) + D\sigma_{20}(v) = v;
\]

hence \( D\sigma_{20} \) is the projection from \( F \) onto \( H \) associated to the direct sum, in particular is onto. As we proved above, for \( y \in V \), we have \( y \in B \) if, and only if, \( \sigma_2(y) = 0 \); hence \( B \) is a submanifold of \( Y \) at 0 and \( \mathcal{T}_0(B) \) is the kernel of the linear map \( D\sigma_{20}: F \to H \), so that, by (1), \( Df_0: E \to \mathcal{T}_0(B) \) is onto. This implies that \( f(X) \) is indeed a neighborhood of 0 in \( B \).

Although we will not apply it, we cannot resist stating the following trivial consequence of the previous lemma:

**Corollary 1.** Let \( X \) be a manifold and \( f: X \to X \) be a smooth map such that \( f \circ f = f \). Then \( f(X) = \{ y \in X \mid f(y) = y \} \) is a submanifold of \( X \) and

\[
\mathcal{T}_y(f(X)) = Df_y(\mathcal{T}_y(X)) = \{ u \in \mathcal{T}_y(X) \mid Df_y(u) = u \}.
\]

3. Embedding a \( k \)-symmetric space

In this section we will fix an integer \( k \geq 2 \), a Lie group \( G \) and a smooth automorphism \( \tau: G \to G \), such that \( \tau^k = Id_G \), and we will consider the corresponding \( k \)-symmetric space \( G/G^\tau \), where \( G^\tau = \{ g \in G \mid \tau(g) = g \} \) is the fixed point subgroup. For each \( g \in G \), we will denote by \([h]\) the corresponding class in \( G/G^\tau \).

We will denote \( \mathcal{G} = T_e(G) \) the Lie algebra of \( G \) and \( \theta = D\tau_e: \mathcal{G} \to \mathcal{G} \) the corresponding Lie algebra automorphism, that satisfies \( \theta^k = Id_{\mathcal{G}} \). Of course, the Lie algebra of the subgroup \( G^\tau \) is \( T_e(G^\tau) = \mathcal{G}^\theta = \{ u \in \mathcal{G} \mid \theta(u) = u \} \). The equality

\[
(Id - \theta) \circ (Id + \theta + \cdots + \theta^{k-1}) = 0,
\]

with commuting factors having trivial intersection kernels, implies that \( \mathcal{G} = \mathcal{H}_c \oplus \mathcal{M}_c \), where

\[
\mathcal{H}_c = \mathcal{G}^\theta = \ker(Id - \theta) = \{ u + \theta(u) + \cdots + \theta^{k-1}(u) \}_{u \in \mathcal{G}},
\]

\[
\mathcal{M}_c = \ker(Id + \theta + \cdots + \theta^{k-1}) = \{ u - \theta(u) \}_{u \in \mathcal{G}}.
\]

If \( g \in G^\tau \), the fact that the conjugation automorphism \( c_g \) commutes with \( \tau \) implies that the Lie algebra automorphism \( Ad_g \) commutes with \( \theta \) and hence that the direct sum \( \mathcal{G} = \mathcal{H}_c \oplus \mathcal{M}_c \) is \( Ad_g \)-invariant. We have hence a well-defined structure of reductive homogeneous space on \( G/G^\tau \), the one that will be considered implicitly. We remark that this is the same reductive structure in \( G/G^\tau \) that has been defined in [3], using the eigenspaces of the complexification of \( \theta \); however the direct approach will be useful later.

**Proposition 1.** Let us consider the smooth action of \( G \) in \( G \) defined by \( g \cdot h = gh\tau(g^{-1}) \) and let \( \text{Orb} = \{ g\tau(g^{-1}) \}_{g \in \mathcal{G}} \) be the orbit of \( e \) for this action. Let \( B \subset G \) be the set

\[
B = \{ h \in G \mid h\tau(h) \cdots \tau^{k-1}(h) = e \}.
\]
Then Orb is a submanifold of $G$, open in $B$, and there is an equivariant diffeomorphism $f : G/G^* \to Orb$ defined by $f([g]) = g\tau(g^{-1})$. Moreover, $T_e(Orb) = \mathcal{M}_e = \mathcal{M}_c$.

**Proof.** It is straightforward to verify that we have a well defined one-to-one smooth equivariant map $f : G/G^* \to G$, $[g] \mapsto g\tau(g^{-1})$, whose image is Orb, so that all we have to prove is that Orb is a submanifold of $G$, open in $B$. The fact that $B$, like Orb, is invariant by the action of $G$ reduces us to proving that Orb is a neighborhood of $e$ in $B$ and that $B$ is a submanifold of $G$ at the point $e$. To simplify notations, let us denote by $\tilde{f} : G \to G$ the smooth map $g \mapsto g\tau(g^{-1})$, whose image is Orb. Let $\varphi : G \to G$ be the smooth map defined by $\varphi(h) = h\tau(h) \cdots \tau^{k-1}(h)$. We have $\varphi(\tilde{f}(g)) = e$, for each $g \in G$; in particular Orb $\subseteq B$. By differentiating at $e$, we obtain $D\tilde{f}_e(u) = u - \theta(u)$ and $D\varphi_e(v) = v + \theta(v) + \cdots + \theta^{k-1}(v)$, so that by what was discussed above, both the image of $D\tilde{f}_e$ and the kernel of $D\varphi_e$ are equal to $\mathcal{M}_c$. Applying Lemma 1, with a constant map as $\psi$, ends the proof.

The fact that Orb is a reductive homogeneous manifold of the Lie group $G$ gives it a natural $G$-invariant connection. One method of characterizing this connection is to compute the Hessian of the inclusion of Orb into $G$, when we consider in $G$ its natural symmetric connection. That is what we do now, limiting our computation to what happens at $e \in Orb$, because the general formula can be obtained through the left and right invariance of the connection of $G$. Following the formalism of [2], we compute first the Maurer-Cartan form $\beta_e : T_e(Orb) \to \mathcal{M}_e \subseteq \mathcal{G}$. We recall that $\beta_e$ is the inverse of the restriction to $\mathcal{M}_e$ of the derivative at $e$, $\rho_e : \mathcal{G} \to T_e(Orb)$, of the map $G \to Orb$, $g \mapsto g \cdot e = g\tau(g^{-1})$, a linear map that is hence defined by $\rho_e(u) = u - \theta(u)$.

**Lemma 2.** The Maurer-Cartan form $\beta_e : T_e(Orb) = \mathcal{M}_e \to \mathcal{M}_e$ is defined by

$$\beta_e(v) = -\frac{1}{k} \sum_{j=1}^{k-1} \theta^j(v).$$

**Proof.** All we have to prove is that the linear map $\beta_e$, defined above, maps $\mathcal{M}_e$ into $\mathcal{M}_e$ and verifies $\rho_e \circ \beta_e = Id_{\mathcal{M}_e}$ and this is a straightforward calculation if we recall that $\theta^{k} = Id\mathcal{G}$ and the characterization of $\mathcal{M}_e$ as a kernel.

**Proposition 2.** The Hessian $h_e : \mathcal{M}_e \times \mathcal{M}_e \to \mathcal{G}$, of the inclusion Orb $\to G$ at $e$, is given by

$$h_e(u,v) = [\beta_e(u) - \frac{1}{2} u, v - \theta(v)].$$

**Proof.** Let us denote by $\nabla$ and $\nabla^G$ the covariant derivatives associated to the connections we are considering in Orb and in $G$. Let $u, v \in \mathcal{M}_e$ and let $Y$ be the vector field on Orb associated to $v$ and to the action of $G$ on Orb, that is defined by $Y_g = D\tilde{R}_{g e}(v) - D\tilde{L}_{g e}(\theta(v))$, where $R_g$ and $L_g$ denote the right and left translations by $g$. Let us denote also by $Y$ the vector field on $G$ defined by the same formula. Then

$$\nabla^G Y_e(u) = -\frac{1}{2} [u, v + \theta(v)]$$

and

$$\nabla Y_e(u) = -\rho_e([\beta_e(u), v]) = -[u, \theta(v)] - [\beta_e(u), v - \theta(v)],$$

and the result is now a consequence of the formula $h_e(u,v) = \nabla^G Y_e(u) - \nabla Y_e(u)$. 


For $k = 2$, we have $\beta_k(u) = \frac{1}{2} u$; hence $h_e(u, v) = 0$, and we retrieve the conclusion that $\text{Orb}$ is a totally geodesic submanifold.

**Remark 1.** There is another equivariant embedding of the $k$-symmetric space $G/G^\tau$ that, for $k = 2$, coincides with the previous one: The product group $G^k$ acts transitively on the manifold $G^{k-1}$ by

$$(g_1, \ldots, g_k) \cdot (h_1, \ldots, h_{k-1}) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_3^{-1}, \ldots, g_{k-1} h_{k-1} g_k^{-1})$$

and $G^{k-1}$ is then a $k$-symmetric manifold, with the permutation automorphism $\tau: G^k \to G^k$, $\tau(g_1, \ldots, g_k) = (g_2, \ldots, g_k, g_1)$ associated to the base point $(e, \ldots, e)$ in $G^{k-1}$. The isotropy subgroup is the diagonal $\{(g_1, \ldots, g_k) \in G^k \mid g_1 = \cdots = g_k\}$, the corresponding Lie algebra is $H(e, \ldots, e) = \{(u_1, \ldots, u_k) \in G^k \mid u_1 = \cdots = u_k\}$ and the corresponding direct complement is $M(e, \ldots, e) = \{(u_1, \ldots, u_k) \mid u_1 + \cdots + u_k = 0\}$. We have hence an associated connection on $G^{k-1}$. We can consider the smooth morphism $\psi: G \to G^k$, $\psi(g) = (g, \tau(g), \ldots, \tau^{k-1}(g))$ and we have then a $\psi$-equivariant and $\psi$-reductive map $\Psi: G/G^\tau \to G^{k-1}$,

$$\Psi([g]) = (g \tau(g^{-1}), \tau(g) \tau^2(g^{-1}), \ldots, \tau^{k-2}(g) \tau^{k-1}(g^{-1})).$$

This map is hence totally geodesic and, by looking to the first coordinate, we conclude that it is an embedding of $G/G^\tau$ into a submanifold of $G^{k-1}$.

4. **The manifold $r_k(G)$**

In this section we will fix an integer $k \geq 2$ and a Lie group $G$ and we will denote by $r_k(G)$ the set of $k$-roots of the identity in $G$,

$$r_k(G) = \{g \in G \mid g^k = e\}.$$

The group $G$ acts on $r_k(G)$ by conjugation: $h \cdot g = c_h(g) = h g h^{-1}$. For each $g \in r_k(G)$, we will denote by $\text{Orb}_g = \{h g h^{-1}\}_{h \in G}$ the orbit of $g$ for this action. We will denote by $R_g$ and $L_g$ the right and left translations by $g$.

**Proposition 3.** The set $r_k(G)$ is a closed submanifold of $G$ and the orbits $\text{Orb}_g$, with $g \in r_k(G)$, are open in $r_k(G)$. Moreover, for each $g \in r_k(G)$,

$$T_g(r_k(G)) = DR_{g_e}(M_g) = DL_{g_e}(M_g) = \{DR_{g_e}(u) - DL_{g_e}(u)\}_{u \in G},$$

where $M_g \subset G$ is defined by

$$M_g = \ker(\text{Id} + Ad_g + \cdots + Ad_g^{k-1}) = \{u - Ad_g(u)\}_{u \in G}.$$

**Proof.** Let $g \in r_k(G)$. Then $c_g: G \to G$ is a smooth automorphism, such that $c_g^k = \text{Id}_G$ and the corresponding Lie algebra automorphism is $Ad_g: G \to G$. By Proposition 1, we conclude that $\text{Orb}_{(g)} = \{h c_g(h^{-1})\}_{h \in G}$ is a submanifold of $G$, open in

$$B_{(g)} = \{h \in G \mid h c_g(h) \cdots c_g^{k-1}(h) = e\}$$

and having $M_g$ as tangent space at $e$. Considering the diffeomorphism $R_g: G \to G$, we conclude that $R_g(\text{Orb}_{(g)})$ is a submanifold of $G$ open in $R_g(B_{(g)}) = r_k(G)$ and that $T_g(r_k(G)) = T_g(\text{Orb}_{(g)}) = DR_{g_e}(M_g)$. The other characterizations of $T_g(r_k(G))$ follow from the equality $DL_{g_e} = DR_{g_e} \circ Ad_g$ and from the $Ad_g$-invariance of $M_g$. 

\[\square\]
Each orbit $Orb_g = \{gh^{-1}\}_{h \in G}$ is a homogeneous manifold of $G$ and the restriction of the right translation to the homogeneous manifold $Orb_{(g)} = \{hc_g(h^{-1})\}_{h \in G}$, associated to the automorphism $c_g : G \rightarrow G$, is an equivariant diffeomorphism $R_g : Orb_{(g)} \rightarrow Orb_g$. By transport through this equivariant diffeomorphism, $Orb_g$ becomes a reductive homogeneous space, such that, for each $g' = gh^{-1} \in Orb_g$, the horizontal space $M_{g'} \subset G$ is equal to the horizontal space of $Orb_{(g)}$ at $hc_g(h^{-1})$; hence

$$M_{g'} = Ad_h(M_c) = \{Ad_h(u) - Ad_h(Ad_g(u))\}_{u \in G} = \{v - Ad_{g'}(v)\}_{v \in G},$$

which is compatible with the notation used in the previous proposition and proves, in particular, that the reductive structure in $Orb_g$ does not depend of the choice of $g$ in the orbit.

The reductive homogeneous structure of the submanifolds $Orb_g$ entitles them to, and hence $r_k(G)$, with an associated connection. The fact that $R_g : Orb_{(g)} \rightarrow Orb_g$ and $R_g : G \rightarrow G$ are totally geodesic maps allows us to deduce the following proposition from Proposition 2:

**Proposition 4.** For each $g \in r_k(G)$, the Hessian $h_g : T_g(r_k(G)) \times T_g(r_k(G)) \rightarrow T_g(G)$, of the inclusion $r_k(G) \rightarrow G$ at $g$, is defined by

$$h_g(\text{DR}_{g_c}(u), \text{DR}_{g_c}(v)) = \text{DR}_{g_c}([\beta_g(u) - \frac{1}{2}u, v - Ad_g(v)]),$$

for $u, v \in M_g$, where $\beta_g(u) = -\frac{1}{k} \sum_{j=1}^{k-1} j Ad_g^j(u)$.

Again, for $k = 2$, we have $\beta_g(u) = \frac{1}{2}u$, hence $h_g = 0$, and we retrieve the conclusion that $r_2(G)$ is a totally geodesic submanifold of $G$.

**References**


