

REMARKS ABOUT SCHLUMPRECHT SPACE

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ABSTRACT. Let \mathbf{S} denote the Schlumprecht space. We prove that

- (1) ℓ_∞ is finitely disjointly representable in \mathbf{S} ;
- (2) \mathbf{S} contains an ℓ_1 -spreading model;
- (3) for any sequence (n_k) of natural numbers, \mathbf{S} is isomorphic to the space $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_{\mathbf{S}}$.

Let $(e_i)_{i=1}^\infty$ be the standard basis of the linear space c_{00} , the set of all finitely supported sequences. For $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$, $\text{supp } x$ denotes the set $\{i \in \mathbb{N} : a_i \neq 0\}$. A subset E of \mathbb{N} is said to be an *interval* if there exist a, b such that

$$E = \{c \in \mathbb{N} : a < c < b\}.$$

For finite subsets E, F of \mathbb{N} , $E < F$ means $\max E < \min F$ or E is an empty set. For $x = \sum_{i=1}^\infty a_i e_i$ and a subset E of \mathbb{N} , $E x$ denotes the vector $E x = \sum_{i \in E} a_i e_i$. Let $f : [1, \infty) \rightarrow [1, \infty)$ be the function defined by $f(x) = \log_2(x + 1)$. The Schlumprecht space $\mathbf{S} = (\mathbf{S}, \|\cdot\|)$ is the completion of c_{00} with respect to the norm $\|\cdot\|$ which satisfies the following implicit equation:

$$(1) \quad \|x\| = \max \left\{ \|x\|_\infty, \sup_{E_1 < E_2 < \dots < E_r} \frac{1}{f(r)} \sum_{i=1}^r \|E_i x\| \right\}.$$

Clearly, (e_i) is a 1-unconditional subsymmetric basis of \mathbf{S} . It is easy to see that every normalized block basis (y_i) of (e_i) dominates (e_i) , i.e.

$$\left\| \sum_{i=1}^\infty a_i y_i \right\| \geq \left\| \sum_{i=1}^\infty a_i e_i \right\|$$

for all $(a_i) \in c_{00}$.

Recall that a sequence (x_n) in \mathbf{S} is said to be *successive* if $\text{supp } x_1 < \text{supp } x_2 < \dots$. A sequence of finite subspaces (V_n) is said to be *successive* if $x_j \in V_j \setminus \{0\}$, $x_k \in V_k \setminus \{0\}$ and $j < k$ implies that $\text{supp } x_j < \text{supp } x_k$.

In [S1], Th. Schlumprecht introduced the space \mathbf{S} and he showed that \mathbf{S} is arbitrarily distortable. He also showed that \mathbf{S} has neither ℓ_p -spreading model for any $1 < p < \infty$ nor c_0 -spreading model. He asked the following question.

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Question 1. Does \mathbf{S} contain an ℓ_1 -spreading model?

Recall that a vector $x \in S$ is said to have *character* $p \in \mathbb{N}$ if there exist $E_1 < E_2 < \dots < E_p$ such that

$$\frac{1}{f(p)} \sum_{i=1}^p \|E_i x\| \approx \|x\|.$$

So if x_1, \dots, x_n are n unit vectors in \mathbf{S} with a common character $p \gg n$, then

$$\left\| \sum_{i=1}^n x_i \right\| \geq \frac{nf(p)}{f(np)} \approx n.$$

Here $a \approx b$ means that $|\frac{a}{b} - 1| < \varepsilon$ for some $0 < \varepsilon < 1$ and $a \gg b > 10$ means $\frac{f(a)}{b} > N$ for some $N \in \mathbb{N}$. Recall that two elements y, z in \mathbf{S} are said to have the *same distribution* if there are two strictly increasing sequences $\{n_k\}, \{m_k\}$ of natural numbers and two sequences $\{a_k\}, \{b_k\}$ such that $|a_k| = |b_k|$ for all $k \in \mathbb{N}$, and $y = \sum_{k=1}^{\infty} a_k e_{n_k}, z = \sum_{k=1}^{\infty} b_k e_{m_k}$. Two subspaces V, U are said to have the *same distribution* if there is an isometry from V onto U such that for any $v \in V$, v and $T(v)$ have the same distribution.

For any $n \in \mathbb{N}$, let $u_n = \frac{f(n)}{n} \sum_{k=1}^n e_k$. It is easy to see that for any $E_1 < E_2 < \dots < E_k$,

$$\sum_{i=1}^k \|E_i u_n\| \leq \frac{f(n)}{f(n/k)}.$$

Hence, for any $n \in \mathbb{N}$, $\|u_n\| = 1$ and if u_n has character p , then $f(p) \approx f(n)$. We will show that for any $\varepsilon > 0$, there is a rapidly increasing sequence $(p_i)_{i=1}^n$ such that for any $n \in \mathbb{N}$, there is a finite sequence $(v_j)_{j=1}^n$ in \mathbf{S} such that

- the support of v_j 's are pairwise disjoint;
- v_j and u_{p_j} have the same distribution;
- $\|\sum_{j=1}^n v_j\| \leq 1 + \varepsilon$.

From that fact, it will be easy (see Theorem 6) to show that the answer to Question 1 is affirmative.

E. Odell and Th. Schlumprecht constructed a space which has neither ℓ_p -spreading model for any $1 \leq p < \infty$ nor c_0 -spreading model [OS]. For background information about spreading models see [BL].

A Banach space X is called *minimal* (a notion defined by H. P. Rosenthal) if every infinite dimensional subspace of X contains a further subspace isomorphic to X , and X is called *complementably minimal* (defined by A. Pełczyński) if every infinite dimensional subspace of X contains a subspace which is isomorphic to X and complemented in X . X is called *prime* [LT] if every complemented infinite dimensional subspace of X is isomorphic to X . A. Pełczyński proved that ℓ_p , $1 \leq p < \infty$, and c_0 are prime, while J. Lindenstrauss showed that ℓ_∞ is prime (see e.g. [LT]) and for a long time these were the only known examples. Recently, W. T. Gowers and B. Maurey [GM2] constructed more prime spaces for which all complemented infinite dimensional subspaces are of finite codimension. In [S2], Th. Schlumprecht showed \mathbf{S} is complementably minimal. As was remarked in [S2], the space \mathbf{S} is either prime or fails the Schroeder-Bernstein property (see [C], [G2]). Recall that a Banach space X is injective if for any Banach space Y containing X

as a subspace, there is a bounded linear projection from Y onto X . It is known that ℓ_∞ is injective. It is natural to ask the following two questions.

Question 2. For any sequence (n_k) of natural numbers, does \mathbf{S} contain a complemented subspace which is isomorphic to $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_S$?

Question 3. If the answer of Question 2 is yes, is $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_S$ isomorphic to \mathbf{S} ?

A positive answer to Question 2 easily follows from our construction of ℓ_∞^n and a theorem of Th. Schlumprecht (see Lemma 2 below). A negative answer to Question 3 would have shown that \mathbf{S} is not prime. Unfortunately, in this article, we will show that its answer is positive. Hence, the question whether \mathbf{S} is prime or not is still open. However, our result shows that c_0 is not the only subsymmetric space X which has the property that it is isomorphic to the X -sum of finite dimensional $\ell_\infty^{n_k}$ spaces for an arbitrary sequence of integers (n_k) .

Recall that a nonzero vector $x \in \mathbf{S}$ is said to be an ℓ_{1+}^n -average with constant C if $\frac{x}{\|x\|} = \sum_{i=1}^n x_i$ for some sequence $x_1 < \dots < x_n$ of nonzero elements of \mathbf{S} such that $\|x_i\| \leq Cn^{-1}$ for all $i \leq n$.

We need the following lemmas.

Lemma 1 (Lemma 4, [GM1]). Let $M, n \in \mathbb{N}$ and $C \geq 1$. There exists an N such that for any ℓ_{1+}^N -vector $x \in \mathbf{S}$ with constant C and any sequence $E_1 < \dots < E_n$ of intervals,

$$\sum_{j=1}^M \|E_j x\| \leq C(1 + 2M/N)\|x\|.$$

Lemma 2 (Theorem 3, [S2]). For any $\varepsilon_k > 0$ with $\sum_{k=1}^\infty \varepsilon_k \leq 1$, there exists a constant $C > 1$ such that for any normalized block basis (y_k) of (e_k) with the following properties: there is a sequence $r_k \uparrow \infty$ in \mathbb{N} so that for all $k \in \mathbb{N}$,

$$(2) \quad \sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \dots < E_r}} \sum_{i=1}^r \|E_i(y_k)\| \leq 1 + \varepsilon_k,$$

$$(3) \quad \text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right),$$

we have that (y_k) is C -equivalent to (e_k) .

Theorem 3. For any $\frac{1}{8} > \varepsilon > 0$, there is a rapidly increasing sequence $\{p_k\}$ of natural numbers such that for any n there exist n disjointly supported elements v_j , $1 \leq j \leq n$, such that for any $k \leq n$, v_k and u_{p_k} have the same distribution, and

$$\left\| \sum_{i=k}^n v_j \right\| \leq 1 + \varepsilon.$$

Proof. Let $(p_j)_{j=0}^\infty$ and $(q_j)_{j=0}^\infty$ be two sequences of natural numbers which satisfy the following conditions:

1. $f(q_0) > 20/\varepsilon$ and $p_0 = 1$.
2. If q_j is defined, then $p_{j+1} = p_j q_j^2$.

3. Suppose that p_j is defined. By Lemma 1, there is q_j such that if x is an $\ell_{1+}^{q_j}$ -vector with constant $1 + \varepsilon$, then

$$\sum_{i=1}^p \|E_i x\| \leq (1 + \varepsilon)(1 + 2p/q_j)\|x\| \leq (1 + 2\varepsilon)\|x\|$$

for any $p \leq p_j^{p_j}$ and $E_1 < E_2 < \dots < E_p$.

Let $\{w_{m_1, m_2, \dots, m_j} : j \leq n \text{ and } m_i \leq q_{i=1}^{i-1} \text{ for all } i \leq j\}$ be any sequence of \mathbb{N} such that

$$w_{m_1, m_2, \dots, m_j} < w_{r_1, r_2, \dots, r_\ell}$$

if one of the following conditions holds:

- for some $i \leq \max\{j, \ell\}$, $m_s = r_s$ for all $s < i$ and $m_i < r_i$;
- $j < \ell$ and $m_s = r_s$ for all $s \leq j$.

Finally, let

$$v_j = \frac{f(p_j)}{p_j} \sum_{i_1=1}^{q_0^2} \sum_{i_2=1}^{q_1^2} \dots \sum_{i_j=1}^{q_{j-1}^2} e_{w_{i_1, i_2, \dots, i_j}}.$$

We claim that for any $k \leq n$ and any interval E either $E(v_k) = 0$ or

$$(4) \quad \left\| E \left(\sum_{j=k}^n v_j \right) \right\| \leq (1 + \varepsilon/4 + \varepsilon/2^{k+2})\|(v_k)\|.$$

Clearly, (4) implies our theorem and it holds if $k = n$. Assume that (4) holds for all $k \geq m > 1$. We will show that (4) also holds for $k = m - 1$. We note that:

1. there is a vector y_m such that

$$(a) \quad \|y_m\| = \frac{f(p_{m-1} q_{m-1}^2)}{p_{m-1} f(q_{m-1}^2)} \leq \frac{1 + \varepsilon/4}{p_{m-1}},$$

(b) v_m is the sum of p_{m-1} successive vectors which have the same distribution as y_m .

2. $E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} \sum_{i=1}^j e_{\ell_i}$, for some j and $\ell_1 < \ell_2 < \dots < \ell_j$.

We shall use induction (on j above) to prove (4). First, we summarize some facts that we shall use in our proof.

Fact 1. If $E(v_{m-1}) = 0$, then by the induction hypothesis, we have

$$\left\| E \left(\sum_{k=m}^n v_k \right) \right\| \leq \left(1 + \frac{\varepsilon}{2}\right) \|E(v_m)\| \leq \left(1 + \frac{\varepsilon}{2}\right) \|y_m\| \leq \frac{\sqrt{2}}{p_{m-1}}.$$

If $\|E(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$, then

$$\left\| E \left(\sum_{k=m}^n v_k \right) \right\| \leq (1 + \varepsilon) \|E(v_m)\| \leq 2(1 + \varepsilon) \|y_m\| \leq \frac{4}{p_{m-1}}.$$

Fact 2. We claim that $\|(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$ implies

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

For any $r \geq 2$, and $E_1 < E_2 < \dots < E_r$

$$\begin{aligned} & \frac{1}{f(r)} \sum_{i=1}^r \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \frac{1}{f(r)} \left(\frac{f(p_{m-1}) + 4}{p_{m-1}} + \sum_{E_i \circ E(v_{m-1})=0} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right) \\ & \leq \frac{1}{f(r)} \left(\frac{f(p_{m-1}) + 4}{p_{m-1}} + \frac{4f(r)}{p_{m-1}} \right) < \frac{f(p_{m-1})}{p_{m-1}}. \end{aligned}$$

Therefore,

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \left\| E \left(\sum_{k=m-1}^n v_k \right) \right\|_{\infty} = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

Fact 3. Suppose that $\|E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. It is known that for any $r \geq q_{m-1}^2$, u_r is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon/2$. This implies that $E(v_m)$ is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon/2$ and $E(\sum_{k=m}^n v_k)$ is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon$.

Fact 4. If $\|E(\sum_{k=m-1}^n v_k)\| = \|E(\sum_{k=m-1}^n v_k)\|_{\infty}$, then (4) holds. So we assume that there are $p \geq 2$ and $E_1 < E_2 < \dots < E_p$ such that

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|.$$

We claim that either $E_i \circ E(v_{m-1}) \neq 0$ for all $i \leq p$ or $\|E_i \circ E(v_{m-1})\| \leq \frac{f(p_{m-1})}{p_{m-1}}$ for all $i \leq p$.

First, we note that by the proof of Fact 2, if $E_i \circ E(v_{m-1}) = E(v_{m-1}) \neq 0$, then

$$\frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| < \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|,$$

a contradiction. Hence, $\|E_i \circ E(v_{m-1})\| < \|E(v_{m-1})\|$ for all $i \leq p$. Suppose that our claim is not true. Without loss of generality, we assume that $E_1 \circ E(v_{m-1}) = 0$ and $\|E_2 \circ E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. Let $E_{2,1} < E_{2,2} < \dots < E_{2,p_2}$ be a successive sequence of subintervals of E_2 such that

$$\left\| E_2 \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{1}{f(p_2)} \sum_{j=1}^{p_2} \left\| E_{2,j} \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|.$$

Let ℓ be the smallest number such that $E_{2,\ell} \circ E(v_{m-1}) \neq 0$ and $E'_2 = \bigcup_{j=\ell+1}^{p^2} E_{2,j}$. The proof of Fact 2 shows that

$$\begin{aligned} & \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & < \frac{1}{f(p)} \left(\left\| E_{2,\ell} \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| + \left\| E'_2 \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right. \\ & \qquad \qquad \qquad \left. + \sum_{i=3}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right), \end{aligned}$$

a contradiction.

Fact 5. For any $k \leq p_{m-1}$, $\frac{f(p_{m-1}q_{m-1}^{q_{m-1}^2})}{f(kq_{m-1}^{q_{m-1}^2})} \leq \frac{f(p_{m-1})}{f(k)}$. Therefore, if $E(v_{m-1}) \neq 0$, then

$$\|E(v_m)\| \leq \|E(v_{m-1})\| + \|y_m\| \leq \|E(v_{m-1})\| + \frac{1 + \varepsilon/4}{p_{m-1}}.$$

Fact 2 shows that (4) holds when $j = 1$. So we may assume that $j > 1$. The proof consists of four cases.

Case 1. $1 < p \leq j$. By Fact 4, for all $i \geq p$,

$$E_i \circ E(v_{m-1}) \neq 0 \quad \text{and} \quad \|E_i \circ E(v_{m-1})\| < \frac{jf(p_{m-1})}{p_{m-1}f(j)}.$$

Using the induction hypothesis, we have

$$\begin{aligned} & \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \frac{1 + \varepsilon/4 + \varepsilon/2^{2m+1}}{f(p)} \sum_{i=1}^p \|E_i \circ E(v_{m-1})\| \\ & \leq (1 + \varepsilon/4 + \varepsilon/2^{m+1}) \|E(v_{m-1})\|. \end{aligned}$$

Case 2. $j \leq p \leq q_0$. By Fact 4, for any $i \leq p$ either $E_i \circ E(v_{m-1}) = 0$ or $E_i \circ E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} e_\ell$. Note that $2p \leq 2q_0 \leq \frac{f(p_1)}{q_0} \leq \frac{\varepsilon f(p_1)}{10}$. Hence,

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + \sum_{E_i \circ E(v_{m-1})=0} \left\| E_i \circ E \left(\sum_{k=m}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + \frac{2p}{p_{m-1}} \\ & \leq (1 + \varepsilon/4) \|E(v_{m-1})\|. \end{aligned}$$

Case 3. $q_0 \leq p \leq p_{m-1}^{p_{m-1}}$. By Fact 3, $E(\sum_{k=m}^n v_k)$ is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon$. Therefore,

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_{m-1})\| + \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + (1 + \varepsilon) \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_m)\| \\ & \leq \|E(v_{m-1})\| + \frac{(1 + 2\varepsilon)(1 + \varepsilon) \|E(v_m)\|}{f(p)} \\ & \leq \left(1 + \frac{2}{f(p)} \right) \|E(v_{m-1})\| \leq (1 + \varepsilon/4) \|E(v_{m-1})\|. \end{aligned}$$

Case 4. $p_{m-1}^{p_{m-1}} < p$. In this case, we have $\|E(v_{m-1})\|_1 / f(p) \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}}$.

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_{m-1})\| + \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m}^n v_k \right) \right\| \\ & \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+2}} \right) \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_m)\| \\ & \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+2}} \right) \|E(v_m)\| \\ & \leq \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+1}} \right) \|E(v_{m-1})\|. \end{aligned}$$

The proof is complete. \square

Remark 4. The above theorem was first proved by the second author and independently by the first author with techniques which were later used in [ADKM] for some asymptotic ℓ_1 spaces. Recently, A. Manoussakis has the same result for a more general class of spaces, containing S .

Remark 5. Let the v_j 's be the vectors in the proof of Theorem 3. We have proved that for any $m \leq n$ and $x \in \text{span}\{v_m, \dots, v_n\}$, x is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon$.

By Lemma 1, for any $N, r, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists an m -dimensional space V such that

- $N < V$ and V is $1 + \varepsilon$ equivalent to ℓ_∞^m ;
- For any $x \in V$ and any $E_1 < E_2 < \dots < E_r$, we have

$$\sum_{i=1}^r \|E_i x\| \leq (1 + \varepsilon) \|x\|.$$

Theorem 6. \mathbf{S} has an ℓ_1 -spreading model.

Proof. By Theorem 3, there exists a successive sequence $\{y_k\}$ in the unit ball of \mathbf{S} such that for any k , there exist k disjointly supported elements $\{v_{k,j} : j \leq k\}$ so that for each $j \leq k$, $v_{k,j}, u_{p_j}/2$ have the same distribution and $y_k = \sum_{j=1}^k v_{k,j}$. Hence, if $N < n_1 < n_2 < \dots < n_j$, then

$$\left\| \sum_{i=1}^j y_{n_i} \right\| \geq \left\| \sum_{i=1}^j v_{n_i, N} \right\| \approx \frac{jf(p_N)}{2f(jp_N)}.$$

So $\{y_k : k \in \mathbb{N}\}$ generates an ℓ_1 -spreading model. □

In the sequel we shall use a variant of the refinement of Johnson’s argument for a space with a subsymmetric basis ([BCLT], Proposition 6.3). We shall work with finite dimensional decompositions instead of block-vectors. Fix a sequence (n_k) of natural numbers. Represent any k in the form $k = 2^i(2j - 1)$. Let $m_k = n_j$, i.e. we repeat each n_j infinitely many times. In the following, X, Y and V denote the spaces

$$X = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{n_k-1} \right)_{\mathbf{S}}, \quad Y = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_k} \right)_{\mathbf{S}}, \quad V = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_k} \right)_{\mathbf{S}}.$$

Then

$$Y = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_{2k-1}} \right)_{\mathbf{S}}, \quad V = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_{2k}} \right)_{\mathbf{S}}.$$

So we have

$$V \approx Y \oplus V \quad \text{and} \quad Y = \mathbf{S} \oplus X.$$

Proposition 7. *There exists a complemented subspace U of \mathbf{S} which is 2C-isomorphic to Y .*

Proof. By Theorem 3 and Remark 5, we can construct a sequence (r_k) of natural numbers and a successive sequence (V_k) of the finite dimensional subspaces of \mathbf{S} which satisfy the following conditions:

1. For any $k \in \mathbb{N}$, V_k is a subspace of \mathbf{S} which is $(1 + \varepsilon)$ isomorphic to $\ell_{\infty}^{n_k}$.
2. $V_1 < V_2 < \dots$.
3. For any $k \in \mathbb{N}$, $p \leq r_{k-1}$ and normalized vector $y_k \in V_k$,

$$\sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \dots < E_r}} \sum_{i=1}^r \|E_i(y_k)\| \leq 1 + \varepsilon_k.$$

4. For any $y_k \in V_k$,

$$\text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right).$$

Let U be the space generated by $\bigcup_{k=1}^{\infty} V_k$. By (3), (4) and Lemma 2, for any $y_k \in V_k$ with $\sum_{k=1}^{\infty} y_k \in \mathbf{S}$, we have

$$\left\| \sum_{k=1}^{\infty} \|y_k\| e_k \right\| \leq \left\| \sum_{k=1}^{\infty} y_k \right\| \leq C \left\| \sum_{k=1}^{\infty} \|y_k\| e_k \right\|.$$

Hence U is isomorphic to Y . Now, let F_k be the smallest interval of \mathbb{N} which contains the support of V_k . Since V_k is 2-equivalent to $\ell_{\infty}^{n_k}$ and ℓ_{∞} is an injective

space, there is a projection P_k from $F_k(\mathbf{S})$ onto V_k such that $\|P\| \leq 2$. Define $P : \mathbf{S} \rightarrow U$ by

$$P(x) = \sum_{k=1}^{\infty} P_k(F_k(x)).$$

Since

$$\left\| \sum_{k=1}^{\infty} P_k(F_k(x)) \right\| \leq C \left\| \sum_{k=1}^{\infty} \|P_k(F_k(x))\| e_k \right\| \leq 2C\|x\|,$$

P is a bounded projection from \mathbf{S} onto U . □

Proposition 8. *The space Y is isomorphic to \mathbf{S} .*

Proof. We use Pełczyński's decomposition technique (see [LT] or [BCLT]). By Proposition 7, we may assume that V and Y are complemented subspaces of \mathbf{S} . Let W be the space such that

$$\mathbf{S} \approx V \oplus W.$$

Then

$$Y \oplus \mathbf{S} \approx Y \oplus V \oplus W \approx V \oplus W \approx \mathbf{S}.$$

Evidently, $\mathbf{S} = \mathbf{S} \oplus \mathbf{S}$. Therefore,

$$Y \approx \mathbf{S} \oplus X \approx \mathbf{S} \oplus \mathbf{S} \oplus X \approx \mathbf{S} \oplus Y \approx \mathbf{S}.$$

The proof is complete. □

Let (x_k) be a sequence of successive vectors in \mathbf{S} such that for each $n \in \mathbb{N}$, x_k has the same distribution as u_{n_k} for some $n_k \in \mathbb{N}$. Let Y_1 be the space spanned by $\{x_k : k \in \mathbb{N}\}$. It is known that Y_1 is a complemented subspace of \mathbf{S} ([LT], Theorem 3.a.4) and $Y_1 \oplus \mathbf{S} \approx \mathbf{S}$ ([BCLT], Proposition 6.3). Since \mathbf{S} is complementably minimal, the proof of Proposition 8 shows that $Y_1 \approx \mathbf{S}$. It is also known that if $(x_k)_{k=1}^n$ is a sequence of successive vectors such that for any $k \leq n$, x_k has the same distribution as u_{n_k} , then $(x_k)_{k=1}^n$ is $1 + \frac{3}{n}$ -equivalent to the unit vector basis of ℓ_1^n and for each x in the space of $\{x_k : k \leq n\}$, x is an ℓ_1^n -vector with constant $1 + \frac{3}{n}$. By the proof of Proposition 7, we have the following.

Proposition 9. *Schlumprecht space \mathbf{S} is isomorphic to $(\sum_k \oplus \ell_1^{n_k})_{\mathbf{S}}$ for any sequence of integers (n_k) .*

A Banach space with an unconditional basis is said to have a *unique conditional basis* if any two normalized unconditional bases are equivalent after a permutation; see [BCLT] and for recent results see [CK]. It was remarked in [CK] that the example of Gowers in [G1] has a unique unconditional basis. Proposition 8 (or Proposition 9) implies that S has no unique unconditional basis. This also immediately follows if one uses that S is complementably minimal and combines Proposition 6.3 and Proposition 6.4 of [BCLT].

Clearly, we have by duality that for any sequence (n_k) of natural numbers, \mathbf{S}^* is isomorphic to $(\sum_{k=1}^{\infty} \oplus \ell_1^{n_k})_{\mathbf{S}^*}$ and $(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{n_k})_{\mathbf{S}^*}$. P. Casazza pointed out the following immediate consequence of our observations (in fact, not using their full generality). Recall that the same property for the classical spaces ℓ_p for $p > 2$, $1 < p < 2$ and $p = 1$ was shown in [R], [BDGJN] and [B] respectively.

Corollary 10. *Schlumprecht space \mathbf{S} (resp. its dual \mathbf{S}^*) has a subspace isomorphic to the whole space and not complemented in \mathbf{S} (resp. \mathbf{S}^*).*

Remark 11. Since \mathbf{S} is complementably minimal, most of the above results are valid hereditarily, i.e. in every subspace of \mathbf{S} .

Remark 12. Let Z be a complemented subspace of \mathbf{S} which has a subsymmetric basis. Since Z is isomorphic to its square, then by the decomposition technique we have that Z is isomorphic to \mathbf{S} .

We do not know the answer to the following question:

Question 4. *Does \mathbf{S} have a unique subsymmetric basis?*

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