A CRITERION FOR SPLITTING $C^*$-ALGEBRAS IN TENSOR PRODUCTS

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Abstract. The goal of the paper is to prove the following theorem: if $A,D$ are unital $C^*$-algebras, $A$ simple and nuclear, then any $C^*$-subalgebra of the $C^*$-tensor product of $A$ and $D$, which contains the tensor product of the unit of $D$, splits in the $C^*$-tensor product of $A$ with some $C^*$-subalgebra of $D$.

Using essentially a result of J.B. Conway on numerical range and certain sets considered by J. Dixmier in type III factors, L. Ge and R.V. Kadison proved in [G-K]: for $R;Q;M$ $W^*$-algebras, $R$ factor, satisfying
\[ R \hat{\otimes} 1_Q \subset M \subset R \hat{\otimes} Q, \]
we have $M = R \hat{\otimes} P$ with $P$ some $W^*$-subalgebra of $Q$.

Making use of the generalization of Conway’s result for global $W^*$-algebras, due independently to H. Halpern ([Hlp]) and Ş. Strâtilă and L. Zsidó ([S-Z1]), as well as of an extension of Tomita’s Commutation Theorem to tensor products over commutative von Neumann subalgebras, it was subsequently proved in [S-Z2]: for $R;Q;M$ $W^*$-algebras with
\[ R \hat{\otimes} 1_Q \subset M \subset R \hat{\otimes} Q, \]
$M$ is generated by $R \hat{\otimes} 1_Q$ and $M \cap (Z(R) \hat{\otimes} Q)$, where $Z(R)$ stands for the centre of $R$.

Using a result from [H-Z], the $C^*$-algebraic counterpart of the above cited results from [Hlp] and [S-Z1], as well as a slice map theorem for nuclear $C^*$-algebras due to S. Wassermann ([W], Prop. 10), we shall prove here:

Theorem. Let $A,D,C$ be unital $C^*$-algebras, $A$ simple and nuclear, such that
\[ A \otimes 1_D \subset C \subset A \otimes_{\min} D. \]
Then
\[ C = A \otimes_{\min} B \]
for some $C^*$-subalgebra $B \subset D$.

The next result is essentially contained in [H-Z], Corollary of Theorem 4:

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**Approximation Lemma.** Let $A$ be a simple unital $C^*$-algebra without tracial state. Denoting by $I_A$ the closure of the convex hull

$$F_A = \text{conv} \{ A \ni a \mapsto uau^* \in A; u \in A \text{ unitary} \}$$

with respect to the topology of the pointwise norm convergence in the Banach algebra of all bounded linear operators on $A$, and by $S(A)$ the state space of $A$, we have

$$S(A)1_A = \{ \Phi \in I_A; \Phi(A) \subset \mathbb{C}1_A \}.$$

**Proof.** Clearly, $\{ \Phi \in I_A; \Phi(A) \subset \mathbb{C}1_A \} = K \cdot 1_A$ with $K \subset S(A)$ convex. $K$ is also weak* closed. Indeed, if $K \ni \psi_i \to \varphi \in S(A)$ in the weak* topology, then $I_A \ni \psi_i \cdot 1_A \to \varphi \cdot 1_A$ in the topology of pointwise norm convergence; hence $\varphi \cdot 1_A \in I_A$, i.e. $\varphi \in K$.

Let us denote for any $a \in A$

$$C_A(a) = \overline{F_A(a) \cap 1_A}.$$

By [H-Z], Corollary of Theorem 4,

$$C_A(a) \neq \emptyset$$

for all $a \in A$

and, plainly,

$$C_A(\Phi(a)) \subset C_A(a), \ \Phi \in F_A, a \in A.$$

Using the idea of the proof of Lemma 4 in [D], Ch. III, §5, it is easy to see that, for every

$$a \in A, \lambda \cdot 1_A \in C_A(a), \ \lambda_1, \ldots, \lambda_n \in A, \ \varepsilon > 0$$

there exists $\Phi \in F_A$ with

$$\| \Phi(a) - \lambda \cdot 1_A \| < \varepsilon,$$

$$\| \Phi(a_j) - \lambda_j \cdot 1_A \| < \varepsilon$$

for some $\lambda_j \in \mathbb{C}, 1 \leq j \leq n$.

Fix some $a \in A$ and $\lambda 1_A \in C_A(a)$. Since the closure of

$$F_{a_1, \ldots, a_n; \varepsilon} = \left\{ \Phi^{**}; \begin{array}{l} \Phi \in F_A, \| \Phi(a) - \lambda 1_A \| < \varepsilon, \\ d(\Phi(a_j), \mathbb{C}1_A) < \varepsilon \text{ for } 1 \leq j \leq n \end{array} \right\} \neq \emptyset$$

with respect to the topology of pointwise $\sigma(A^{**}, A^*)$-convergence is compact,

$$\bigcap_{a_1, \ldots, a_n \in A} F_{a_1, \ldots, a_n; \varepsilon}$$

contains some $\Psi$. Clearly,

$$\Psi(a) = \lambda 1_A.$$

Fix now also some $b \in A$. For every $\varphi_1, \ldots, \varphi_n \in A^*$ and $\varepsilon > 0$ there is some

$$\Theta \in F_{b; \varepsilon}$$

such that

$$|\varphi_j(\Psi(b) - \Theta(b))| < \varepsilon \| \varphi_j \|, 1 \leq j \leq n,$$

and then some

$$\mu \in \mathbb{C}.$$
with 
\[ \| \Theta(b) - \mu \cdot 1_A \| < \varepsilon. \]
Then 
\[ |\mu| < \| b \| + \varepsilon \text{ and } |\varphi_j(\Psi(b) - \mu \cdot 1_A)| < 2\varepsilon \| \varphi_j \|, \ 1 \leq j \leq n. \]
It follows that the downward directed compact sets
\[ \left\{ \mu \in \mathbb{C}; |\mu| \leq \| b \| + \varepsilon, \right. \]
\[ \left. |\varphi_j(\Psi(b) - \mu \cdot 1_A)| \leq 2\varepsilon \| \varphi_j \| \text{ for } 1 \leq j \leq n \right\}, \]
\[ \varphi_i, \ldots, \varphi_n \in A^*, \ \varepsilon > 0, \]
are not empty; hence their intersection contains some \( \mu(b) \in \mathbb{C} \). Then 
\[ \Psi(b) = \mu(b) \cdot 1_A \in C1_A. \]
By the above, 
\[ \Psi(A) \subset C1_A. \]
Moreover, since \( \Psi|A \) takes values in \( C1_A \subset A \), it belongs to the pointwise \( \sigma(A, A^*) \)-closure of the convex set 
\[ \{ \Phi^{**}|A; \Phi \in \mathcal{F}_A \} = \mathcal{F}_A, \]
which is the pointwise norm closure \( \mathcal{I}_A \) of \( \mathcal{F}_A \).
We conclude: for any \( a \in A \) and any \( \lambda \cdot 1_A \in C_A(a) \) there exists \( \Phi \in \mathcal{I}_A \) with \( \Phi(A) \subset C1_A \) and \( \Phi(a) = \lambda \cdot 1_A \). In other words,
\[ \lambda = \psi(a) \text{ for some } \psi \in K. \]
By the Hahn-Banach theorem \( K = S(A) \) follows if we show that for every \( a^* = a \in A \) with
\[ (*) \]
\[ \psi(a) \leq \lambda_0 \text{ for all } \psi \in K \]
we have 
\[ \varphi(a) \leq \lambda_0 \text{ for all } \varphi \in S(A). \]
But, according to \([\text{I}E\text{Z}], \) Corollary of Theorem 4, we have for every \( \varphi \in S(A) \)
\[ \varphi(a) \cdot 1_A \in C_A(a), \]
so, by the above,
\[ \varphi(a) = \psi(a) \text{ for some } \psi \in K \]
and \( (*) \) yields 
\[ \varphi(a) \leq \lambda_0. \]
Now we prove the main ingredient for the proof of the announced theorem:

**Invariance Lemma.** Let $A, D, C$ be unital $C^*$-algebras, $A$ simple, such that
\[ A \otimes 1_D \subset C \subset A \otimes_{\min} D. \]
Then, for any state $\varphi$ on $A$,
\[ (\varphi \cdot 1_A \otimes id_D)(C) \subset C; \]
hence
\[ (\varphi \cdot 1_A \otimes id_D)(C) = C \cap (1_A \otimes D). \]

**Proof.** First we reduce the proof to the case in which $A$ has no tracial state.

Let $A_0$ be any simple unital $C^*$-algebra without tracial state (e.g. the Calkin algebra or a type III factor of countable type), and $\varphi_0$ any state on $A_0$. Then $A_0 \otimes_{\min} A$ is a simple unital $C^*$-algebra without tracial state and
\[ (A_0 \otimes_{\min} A) \otimes 1_D \subset A_0 \otimes_{\min} C \subset (A_0 \otimes_{\min} A) \otimes_{\min} D. \]
If we assume that in this case
\[ (\varphi_0 \cdot 1_{A_0} \otimes \varphi \cdot 1_A \otimes id_D)(A_0 \otimes_{\min} C) \subset A_0 \otimes_{\min} C, \]
then
\[ 1_{A_0} \otimes (\varphi \cdot 1_A \otimes id_D)(C) = (\varphi_0 \cdot 1_{A_0} \otimes \varphi \cdot 1_A \otimes id_D)(1_{A_0} \otimes C) \subset A_0 \otimes_{\min} C, \]
\[ (\varphi \cdot 1_A \otimes id_D)(C) \subset C. \]

Now let us assume that $A$ has no tracial state. According to the Approximation Lemma there exists a net $(\Phi_i)_i$ in $\mathcal{F}_A$ such that
\[ \|\Phi_i(a) - \varphi(a) \cdot 1_A\| \to 0 \quad \text{for all } a \in A; \]
hence
\[ \|(\Phi_i \otimes id_D)(a \otimes d) - (\varphi \cdot 1_A \otimes id_D)(a \otimes d)\| \to 0 \quad \text{for all } a \in A \text{ and } d \in D. \]
Since every $\Phi_i \otimes id_D$ is contractive, it follows that
\[ \|(\Phi_i \otimes id_D)(x) - (\varphi \cdot 1_A \otimes id_D)(x)\| \to 0 \quad \text{for all } x \in A \otimes_{\min} D. \]
But every $\Phi_i \otimes id_D$ is convex combination of operators of the form $Ad(u \otimes 1_D)$ with $u \otimes 1_D \in A \otimes 1_D \subset C$, so it leaves $C$ invariant. Consequently also their pointwise norm limit $\varphi_1 A \otimes id_D$ leaves $C$ invariant.

**Proof of the theorem.** $C \cap (1_A \otimes D)$ is of the form $1_A \otimes B$ for some $C^*$-subalgebra $B \subset D$ and we have to prove that the $C^*$-subalgebras $A \otimes_{\min} B \subset C$ of $A \otimes_{\min} D$ coincide.

By the Invariance Lemma
\[ C \cap (1_A \otimes D) = \{(\varphi \cdot 1_A \otimes id_D)(x); \varphi \in S(A), x \in C\}, \]
so
\[ B = \{E_\varphi(x); \varphi \in S(A), x \in C\}, \]
where $E_\varphi : A \otimes_{\min} D \to D$ is the slice map defined by
\[ 1_A \otimes E_\varphi(x) = (\varphi \cdot 1_A \otimes id_D)(x), \quad x \in A \otimes_{\min} D. \]
In other words,

\[ E_\varphi(x) \in B \text{ for all } x \in C \text{ and } \varphi \in S(A). \]

Since \( A \) is nuclear, we can apply S. Wassermann’s slice map theorem ([W], Prop. 10) and conclude that \( C \subseteq A \otimes_{\min} B \).

We notice that in the above proof the nuclearity of \( A \) was used only by applying Wassermann’s slice map theorem to \( A \). It is an open question whether this slice map theorem holds assuming \( A \) exact, that is, if

\[ A \text{ exact } C^*\text{-algebra, } B \subseteq D \text{ } C^*\text{-algebras,} \]

imply \( x \in A \otimes_{\min} D, \ E_\varphi(x) \in B \) for all \( \varphi \in S(A) \). In the case of positive answer it would be enough to assume in our theorem \( A \) simple and exact.

Since the closure of the union of every upward directed family of nuclear \( C^*\)-subalgebras is still a nuclear \( C^*\)-subalgebra (see e.g. [M], Th. 6.3.10), the Zorn lemma implies that any nuclear \( C^*\)-subalgebra is contained in a maximal nuclear \( C^*\)-subalgebra.

**Corollary.** Let \( D \) be a unital \( C^*\)-algebra, \( 1_D \in B \subseteq D \) a maximal nuclear \( C^*\)-subalgebra, and \( A \) a nuclear, simple, unital \( C^*\)-algebra. Then \( A \otimes_{\min} B \) is a maximal nuclear \( C^*\)-subalgebra of \( A \otimes_{\min} D \).

**Proof.** The \( C^*\)-algebra \( A \otimes_{\min} B (= A \otimes_{\max} B) \) is clearly nuclear.

Now let \( A \otimes_{\min} B \subseteq C \subseteq A \otimes_{\min} D \) be an arbitrary nuclear \( C^*\)-subalgebra. By the above theorem

\[ C = A \otimes_{\min} B_0 \]

for some \( C^*\)-subalgebra \( B \subseteq B_0 \subseteq D \). The nuclearity of \( C \) implies the nuclearity of \( B_0 \), and then the maximal nuclearity of \( B \) in \( D \) yields \( B_0 = B \). Thus

\[ C = A \otimes_{\min} B. \]

We notice also that in the case of positive answer to the above slice map question for exact \( C^*\)-algebras, when our theorem would follow for \( A \) only simple and exact, a counterpart of the above corollary would hold for maximal exact \( C^*\)-subalgebras.

**Note added in proof.** After this work was completed, we learned that our theorem was independently obtained also by Joachim Zacharias in his preprint: “A note on a result of Ge and Kadison and its \( C^*\)-algebra version”, 1998. He uses a different way to approximate states on simple unital \( C^*\)-algebras by elementary mappings.

**References**


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