VIEWING AF-ALGEBRAS AS GRAPH ALGEBRAS

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ABSTRACT. Every AF-algebra A arises as the C^*-algebra of a locally finite pointed directed graph in the sense of Kumjian, Pask, Raeburn, and Renault. For AF-algebras, the diagonal subalgebra defined by Strătilă and Voiculescu is consistent with Kumjian's notion of diagonal, and the groupoid arising from a well-chosen Bratteli diagram for A coincides with Kumjian's twist groupoid constructed from a diagonal of A.

1. INTRODUCTION

A Bratteli diagram [2] can be viewed as a particular kind of directed graph, and to each such diagram there can be associated a unique AF C^*-algebra. Recently, Kumjian, Pask, Raeburn, and Renault introduced a more general notion of constructing a C^*-algebra from a directed graph. For certain kinds of graphs, their strategy for doing so was to build from the graph a locally compact r-discrete groupoid, and then use the theory of Renault [8] to construct a C^*-algebra. In this paper, we show that these two procedures are compatible.

Suppose E is a row-finite directed graph with no sinks. In [6], Kumjian, Pask, Raeburn, and Renault construct a locally compact r-discrete groupoid $\mathcal{G}_E$ from E, and study the associated C^*-algebra $C^*(\mathcal{G}_E)$. The class of C^*-algebras that arise in this way includes, up to Morita equivalence, Cuntz-Krieger algebras and AF-algebras. In [5], Kumjian, Pask, and Raeburn define the C^*-algebra $C^*(E)$ of any row-finite directed graph E, and note that $C^*(E)$ coincides with $C^*(\mathcal{G}_E)$ in case E has no sinks. They then show that $C^*(E)$ is an AF-algebra if and only if E has no (directed) loops. The main result of this paper is that every AF-algebra arises in this way, at least if we are willing to point the graph. That is, every AF-algebra A is the C^*-algebra of a row-finite (pointed) directed graph. Not surprisingly, that directed graph will be a Bratteli diagram for A. An arbitrary Bratteli diagram for A, however, may not produce the desired result, so we start in Section 2 by giving a procedure for building appropriate Bratteli diagrams. Section 3 recalls the basics from [6] and then proves the main result.

In Section 4, we explore the notion of the diagonal subalgebra. This is defined in [9] for unital AF-algebras and in [4] for general C^*-algebras. We show that every AF-algebra (unital or not) contains a subalgebra which is diagonal in the sense of [4] and that, not surprisingly, the two notions of diagonal coincide in the AF case.

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Further, our construction of a groupoid from a suitable Bratteli diagram for A is compatible with Kumjian's construction in [4] of a groupoid from a diagonal of A.

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2. Preliminaries

Our conventions for AF-algebras come primarily from [3]. Given a sequence of finite-dimensional C^*-algebras $\{A_n\}$, and injective $*$-homomorphisms $\phi_n : A_n \hookrightarrow A_{n+1}$, we define $\lim_\to (A_n, \phi_n)$ to be the C^*-algebraic direct limit of the inductive system

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots .$$

The C^*-algebra $\lim_\to (A_n, \phi_n)$ is called an *AF-algebra*.

Given an AF-algebra

$$A = \varinjlim (A_n, \phi_n),$$

let p_n be the number of simple summands in the algebra A_n, and let $[n, 1], [n, 2], \ldots,$ $[n, p_n]$ denote the sizes of the corresponding matrix algebras. That is, for each $n \in \mathbf{N}$, $A_n = \bigoplus_{i=1}^{p_n} A_i^n$, with $A_i^n \cong M_{[n,i]}$. We will denote by $[K^{(n)}]$ the $p_{n+1} \times p_n$ matrix of multiplicities corresponding to the map $\phi_n : A_n \to A_{n+1}$. That is, the multiplicity of the embedding $\phi_{i,j}^n : A_j^n \longrightarrow A_i^{n+1}$ is $k_{i,j}^{(n)}$. For $m < n$, define $\phi_{n,m} = \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m : A_m \longrightarrow A_n$.

Given this set-up, we construct a Bratteli diagram G for A as follows: the vertex set V for G is defined by $V = \bigcup_{i=1}^\infty V^n$ and $V^n = \{v_1^n, v_2^n, \ldots, v_{p_n}^n\}$. For every $n \in N$, and for every $i \leq p_{n+1}$ and $j \leq p_n$, put $k_{i,j}^{(n)}$ edges from v_j^n to v_i^{n+1}. G is now a row-finite (locally finite, in fact) directed graph in the sense of [6]. There are, of course, many Bratteli diagrams associated with a particular AF-algebra, but we will call the one constructed above the *Bratteli diagram corresponding to the system* (A_n, ϕ_n). One more bit of notation: for $n > 1, i \leq p_n$, put

$$\sigma_i^n = [n, i] - \sum_{j=1}^{p_{n-1}} k_{i,j}^{(n-1)}[n-1, j],$$

and put $\sigma_i^1 = [1, i]$ for $i \leq p_1$. Think of σ_i^n as the "slack" at vertex i of level n, the extent to which the embeddings of the algebras at level $n - 1$ fail to fill up the algebra A_i^n. Of course σ_i^n is a nonnegative integer for all n, i; it will also be convenient in what follows to arrange things so that $\sigma_i^n \leq 1$. The idea of the following lemma is to replace a Bratteli diagram which has vertices with too much slack with one that corresponds to the same AF-algebra, but which has limited slack at all of its vertices.

Lemma 1. *Every AF-algebra $A = \lim_\to (A_n, \phi_n)$ is isomorphic to an AF-algebra $D = \lim_\to (D_n, \psi_n)$ which satisfies $D_1 = \mathbf{C}$ and*

(1) $\sigma_i^n \leq 1$ *for all $n > 1$ and all $i \leq p_n$.*

Before we proceed with the proof, we present an example to guide the reader. Suppose we are confronted with the following undesirable Bratteli diagram:

$$3 \longrightarrow 11 \longrightarrow 21$$
$$5 \xrightarrow{\quad 2 \quad} 10 \longrightarrow 21$$

First we fix the beginning:

$$1 \xrightarrow{\ 3\ } 3 \longrightarrow 11 \longrightarrow 21$$
$$5 \searrow$$
$$5 \xrightarrow{\quad 2 \quad} 10 \longrightarrow 21$$

Then we add a layer:

$$1 \xrightarrow{\ 3\ } 3 \longrightarrow 10 \longrightarrow 11 \longrightarrow 21$$
$$5$$
$$5 \xrightarrow{\quad 2 \quad} 10 \longrightarrow 10 \longrightarrow 21$$

And another:

$$1 \xrightarrow{\ 3\ } 3 \longrightarrow 9 \longrightarrow 10 \longrightarrow 11 \longrightarrow 21$$
$$5$$
$$5 \xrightarrow{\quad 2 \quad} 10 \longrightarrow 10 \longrightarrow 10 \longrightarrow 21$$

Further, we will fix the maps along the way so that, for instance, the map $M_5 \hookrightarrow M_{11}$ in the original diagram is the same as the composition of the three maps $M_5 \hookrightarrow M_9 \hookrightarrow M_{10} \hookrightarrow M_{11}$ in the final diagram.

Proof. Given A with $A_1 \neq \mathbf{C}$, define a new inductive system (B_n, ρ_n) as follows. Let $B_1 = \mathbf{C}$, and let $B_n = A_{n-1}$ for $n > 1$. Define $\rho_n = \phi_{n-1}$ for $n > 1$, and let ρ_1 be the unique unital homomorphism from $\mathbf{C}$ to B_2. If B is defined to be $\lim_{\to}(B_n, \rho_n)$, then it is clear that $B \cong A$. So we may assume, without loss of generality, that $A_1 = \mathbf{C}$. We now proceed to modify A so that it satisfies condition (1).

First let n be the smallest integer for which there is an $i \leq p_n$ with $\sigma_i^n > 1$. Define $D_j = A_j$ for $j = 1, \dots, n-1$ and let $\psi_j = \phi_j$ for $j = 1, \dots, n-2$. Now define D_n as follows. For $k = 1, \dots, p_n$, if $\sigma_k^n \leq 1$, put $D_k^n = A_k^n$. If $\sigma_k^n > 1$, choose D_k^n such that $\bigcup_{j=1}^{p_{n-1}} \text{range}(\phi_{k,j}^{n-1}) \subset D_k^n \subset A_k^n$ and $D_k^n \cong M_{[n,k]-1}$. Put $D_n = \bigoplus_{k=1}^{p_n} D_k^n$.

Let $\psi_{n-1} = \phi_{n-1}$, but viewed as a map from A_{n-1} to D_n. Now let $D_{n+1} = A_n$ and let $\psi_n : D_n \hookrightarrow D_{n+1}$ denote the inclusion map. It is clear that $\phi_{n-1} = \psi_n \circ \psi_{n-1}$.

Now define $D_{n+j} = A_{n+j-1}$ for $j \geq 2$ and $\psi_{n+j} = \phi_{n+j-1}$ for $j \geq 1$. This gives a new inductive system, and thus a new AF-algebra. Now repeat this process as many times as necessary (possibly countably many) and denote the resulting system also by $D = \lim_{\to}(D_n, \psi_n)$.

It is clear that D satisfies the required condition, so it remains to show that $D \cong A$. To that end, we note that $D = \lim_{\rightarrow}(D_{n_k}, \psi_{n_{k+1},n_k})$ for any subsequence $\{n_k\}$. By construction, $A_1 = D_1$. Inductively, for $k > 1$, we can choose an $n_k > n_{k-1}$ such that $A_k = D_{n_k}$. Because of the way the ψ_n's were constructed, the sequence $(D_{n_k}, \psi_{n_{k+1},n_k})$ coincides with the sequence (A_k, ϕ_k). Thus $A \cong D$. $\square$

3. The main result

Before we state the main result, we collect some notation and basic facts from [6]. Given a directed graph E, let E^0 denote the set of vertices and E^1 the set of edges. For $e \in E^1$, let $s(e)$ and $r(e)$ denote the source and range of e, so $s, r : E^1 \longrightarrow E^0$. For $n \in \mathbf{N}$, let $E^n = \{e_1 e_2 \ldots e_n \,|\, e_i \in E^1, s(e_i) = r(e_{i-1}), i = 2, \ldots, n\}$, the set of all paths in E of length n. Let $E^* = \bigcup_{n \in \mathbf{N}} E^n$, the set of all finite paths, and $E^\infty = \{e_1 e_2 \ldots \,|\, s(e_i) = r(e_{i-1})\}$, the infinite path space of E.

For $x, y \in E^\infty$, $k \in \mathbf{Z}$, say $x \sim_k y$ if and only if $x_i = y_{i+k}$ for large i. Define $\mathcal{G}_E$, the path groupoid of E, by $\mathcal{G}_E = \{(x, k, y) \,|\, x \sim_k y\}$. The groupoid operations in $\mathcal{G}_E$ are as follows:

- $(x, k, y)^{-1} = (y, -k, x)$;
- $(x, k, y) \cdot (y, l, z) = (x, k + l, z)$.

For $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$, define $Z(\alpha, \beta) = \{(x, k, y) \,|\, x = \alpha z, y = \beta z, k = |\alpha| - |\beta|\}$. The $Z(\alpha, \beta)$'s are compact open G-sets in the topology they generate. Note that $r(x, k, y) = (x, 0, x)$, so we identify E^∞ with $\mathcal{G}_E^0$, the unit space of $\mathcal{G}_E$. The relative topology on E^∞ as a subset of $\mathcal{G}_E$ agrees with the product topology.

Since the $Z(\alpha, \beta)$'s are compact and open and generate the topology of $\mathcal{G}_E$, we have that $C^*(\mathcal{G}_E)$ is generated by the characteristic functions of these sets. That is, span$\{1_{Z(\alpha,\beta)} \,|\, \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}$ is dense in $C^*(\mathcal{G}_E)$. Note how two of these characteristic functions multiply in $C^*(\mathcal{G}_E)$:

$$1_{Z(\alpha,\beta)} * 1_{Z(\gamma,\delta)} = \begin{cases} 1_{Z(\alpha\epsilon,\delta)} & \text{if } \gamma = \beta\epsilon, \\ 1_{Z(\alpha,\delta\epsilon)} & \text{if } \beta = \gamma\epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

that is, the product is 0 unless one of β, γ continues the other.

One more piece of notation: suppose S is any subset of E^0. Considering only those paths whose source is in S determines an open subset of $\mathcal{G}_E^0$. We will refer to the reduction of the groupoid $\mathcal{G}_E$ to this open set as $\mathcal{G}_E|_S$, slightly abusing the notation. Distinguishing a set of vertices of the graph is known as *pointing* the graph. In what follows, we will be dealing with path groupoids reduced to such subsets of the unit space. Because S is open, the Haar system for $\mathcal{G}_E$ restricts to a Haar system for $\mathcal{G}_E|_S$. Also, $\{Z(\alpha, \beta) \,|\, s(\alpha), s(\beta) \in S\}$ is a base of compact open G-sets for the topology of $\mathcal{G}_E|_S$, so the span of the characteristic functions of those sets is dense in $C^*(\mathcal{G}_E|_S)$.

We now state the main result:

Theorem 1. *Given an AF-algebra*

$$A = \varinjlim(A_n, \phi_n),$$

there is a Bratteli diagram E for A and a set S of vertices such that $C^*(\mathcal{G}_E|_S) \cong A$.

Proof. First use the above procedure (if necessary) to construct a new inductive system for the AF-algebra such that the Bratteli diagram corresponding to that

inductive system satisfies condition (1). That is, build a Bratteli diagram for A which has no excess slack. Denote by E the Bratteli diagram corresponding to this inductive system. Next, define S to be the set of all vertices where the slack is 1, along with the vertex at the top level, and for each n and i define F_i^n to be the set of all finite paths in E that start in S and end at the vertex v_i^n. That is, $S := \{v_i^n \,|\, \sigma_i^n = 1\} \cup \{v_1^1\}$ and $F_i^n := \{\alpha \in E^* \,|\, s(\alpha) \in S, r(\alpha) = v_i^n\}$. Let $|F_i^n|$ denote the cardinality of F_i^n.

Now we claim that for every integer n,

(2) $$|F_i^n| = [n, i] \text{ for every } i \le p_n.$$

The proof is by induction. First note that the case $n = 1$ is taken care of by the construction of A. That is, since $p_1 = 1$ and $[1, 1] = 1$, the set F_1^1 has only one element, the zero-length path v_1^1. Now suppose that (2) is true for $n = l$. Fix $i \le p_{l+1}$. If $\sigma_i^{l+1} = 0$, then v_i^{l+1} is not an element of S, so every path starting in S and ending at v_i^{l+1} must have come through some vertex in level l. There are p_l such vertices, the jth of which, by the induction hypothesis, is the range of $[l, j]$ paths emanating from S. There are $k_{i,j}^{(l)}$ different ways to get from v_j^l to v_i^{l+1}, so we must have

$$|F_i^{l+1}| = \sum_{j=1}^{p_l} k_{i,j}^{(l)} [l, j] = [l + 1, i] - \sigma_i^{l+1} = [l + 1, i],$$

as desired. If $\sigma_i^{l+1} = 1$, then the calculations are similar: we end up with $[l+1, i] - 1$ paths which pass through the preceding level. Adding in the zero-length path v_i^{l+1}, which is in S, brings us to the right number. This establishes the claim.

Now define, for any positive integer n and any $i \le p_n$, $B_i^n := \text{span}\{1_{Z(\alpha,\beta)} \,|\, \alpha, \beta \in F_i^n\} \subset C^*(\mathcal{G}_E|_S)$. Note that, given two generators of B_i^n, say $1_{Z(\alpha,\beta)}$ and $1_{Z(\gamma,\delta)}$, because β and γ end at the same vertex, one cannot continue the other unless they coincide, so we have:

$$1_{Z(\alpha,\beta)} * 1_{Z(\gamma,\delta)} = \begin{cases} 1_{Z(\alpha,\delta)} & \text{if } \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Also, $1_{Z(\alpha,\beta)}^* = 1_{Z(\beta,\alpha)}$. So the elements of B_i^n behave like matrix units, and there are, by the above claim, $[n, i]^2$ of them, so $B_i^n \cong M_{[n,i]}$. Further, if $i \ne j$, $B_i^n B_j^n = 0$ because if a path ends at v_i^n, it cannot be a continuation of (or be continued by) a path ending at v_j^n. Thus, $B_n := \text{span } B_i^n$ coincides with the direct sum $\bigoplus_{i=1}^{p_n} B_i^n$, and we have

$$B_n \cong \bigoplus_{i=1}^{p_n} M_{[n,i]} \cong A_n.$$

Note now that each of these algebras embeds naturally into the next; for instance,

$$1_{Z(\alpha,\beta)} = \sum_{\{e \in E^1 \,|\, s(e) = r(\alpha)\}} 1_{Z(\alpha e, \beta e)}$$

gives the inclusion ι_n of B_n into B_{n+1}. The multiplicity of the embedding of B_j^n into B_i^{n+1} is the number of edges from v_j^n to v_i^{n+1}, which is $k_{i,j}^{(n)}$. Thus the Bratteli diagram corresponding to (B_n, ι_n) is the same as the Bratteli diagram corresponding to (A_n, ϕ_n). Hence, by [2, Theorem 2.7],

$$B := \varinjlim (B_n, \iota_n) \cong A.$$

Since $\bigcup_{n=1}^{\infty} B_n = \text{span}\{1_{Z(\alpha,\beta)} \,|\, \alpha, \beta \in F_i^n \text{ for some } n, i\}$ generates $C^*(\mathcal{G}_E|_S)$, it follows that $A \cong C^*(\mathcal{G}_E|_S)$. $\qquad\square$

Note that it is necessary to correctly modify and correctly point the Bratteli diagram, as the following simple example shows: M_3 can be thought of as an AF-algebra with the following Bratteli diagram:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 3 \longrightarrow \cdots .$$

If we turn this into a directed graph in the obvious way and fail to point it, we obtain the following graph:

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots$$

of which the C^*-algebra is $\mathcal{K}$, the compact operators. If we point the graph, for instance, at only one vertex, we obtain the graph:

$$* \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots$$

whose C^*-algebra is $\mathbf{C}$. Applying the procedure of Section 2 to the Bratteli diagram in question would produce the following pointed graph:

$$* \longrightarrow * \longrightarrow * \longrightarrow \cdot \longrightarrow \cdot$$

which has the desired algebra, M_3, as a C^*-algebra.

Note that there are other procedures for modifying and pointing the Bratteli diagram that would yield the same result. One such (communicated to us by Alex Kumjian) would be to add a vertex to every level below which there is a vertex with slack greater than 0, then add the appropriate number of edges from the new vertex to the ones with slack. That is, if $\sigma_{n,i} > 0$ for some $i \leq p_n$, add a vertex to level $n-1$ and add $\sigma_{n,i}$ edges from the new vertex to the vertex v_i^n. Then point this Bratteli diagram at the added vertices. This approach is somewhat more natural because every pointed vertex is a source, but it makes for unnatural Bratteli diagrams.

Also note that, in case A is unital, it is possible to construct a Bratteli diagram for A which has no slack at all and which has a single vertex at the top level. In this case, it is only necessary to point the top vertex.

4. Connection with the diagonal

In [4], Kumjian defines the notion of a diagonal subalgebra of a C^*-algebra A as follows. An abelian subalgebra B of A is said to be *diagonal in A* if it contains a positive element which is strictly positive in A and if there exists a faithful conditional expectation $P : A \longrightarrow B$ such that $N_f(B)$ spans a dense subset of $\ker P$. Here, $N_f(B)$ denotes the *free normalizers* of B in A, namely the set of all $a \in A$ such that $a^* B a \cup a B a^* \subset B$ and $a^2 = 0$. He defines a *twist* as a proper $\mathbf{T}$-groupoid Γ such that $\Gamma/\mathbf{T}$ is an r-discrete equivalence relation, and shows that there is a one-to-one correspondence between twists and diagonal pairs of C^*-algebras. We briefly recall the basics here.

Suppose Γ is a twist. Define

$$E(\Gamma) = \{f \in C_c(\Gamma) \,|\, f(t\gamma) = tf(\gamma) \text{ for all } t \in \mathbf{T}, \gamma \in \Gamma\}$$

and

$$D(\Gamma) = \{f \in E(\Gamma) \,|\, \text{supp}\, f \subset \mathbf{T}\Gamma^0\},$$

where $\mathbf{T}\Gamma^0$ denotes the isotropy group bundle of Γ.

$E(\Gamma)$ is then a *-algebra with a distinguished abelian subalgebra $D(\Gamma)$. Further, $D(\Gamma) \cong C_c(\Gamma^0)$. $E(\Gamma)$ becomes a pre-Hilbert $D(\Gamma)$-module; the completion, which Kumjian denotes by $\mathcal{H}(\Gamma)$, is a Hilbert $C_0(\Gamma^0)$-module.

Kumjian then constructs a *-homomorphism $\pi : E(\Gamma) \longrightarrow L(\mathcal{H}(\Gamma))$ such that $\pi(f)g = fg$ (the convolution product) for all $f, g \in E(\Gamma)$ and defines $A(\Gamma)$ to be the closure of $\pi(E(\Gamma))$ and $B(\Gamma)$ to be the closure of $\pi(D(\Gamma))$. It turns out that $B(\Gamma)$ is diagonal in $A(\Gamma)$, and this shows that every twist gives rise to a diagonal pair of C^*-algebras.

Conversely, given C^*-algebras A and B with B diagonal in A, Kumjian constructs a twist Γ such that $A \cong A(\Gamma)$ and $B \cong B(\Gamma)$, giving a bijective correspondence between twists and diagonal pairs.

The following theorem is perhaps known to experts, but we could not find it in the literature.

Theorem 2. *Let $\mathcal{G}$ be a Hausdorff, amenable, r-discrete equivalence relation. Then $(C^*(\mathcal{G}), C_0(\mathcal{G}^0))$ is a diagonal pair in Kumjian's sense, and the associated twist is the trivial twist over $\mathcal{G}$.*

Proof. Given $\mathcal{G}$, we construct the trivial twist Γ_1 over $\mathcal{G}$. That is, let $\Gamma_1 = \mathcal{G} \times \mathbf{T}$. The $\mathbf{T}$-groupoid operations on Γ_1 are as follows:

- $(g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1 t_2)$;
- $(g, t)^{-1} = (g^{-1}, \overline{t})$;
- $s(g, t) = (g, t)^{-1}(g, t) = (s(g), 1)$ (hence we identify the unit space of Γ_1 with the unit space of $\mathcal{G}$);
- $t_1(g, t_2) = (g, t_1 t_2)$.

Note that $\mathcal{G}$ can be viewed as a subgroupoid of Γ_1 via $g \mapsto (g, 1)$. Thus, given a continuous equivariant function f with compact support in Γ_1 (i.e. an element of Kumjian's $E(\Gamma_1)$), we can define $\hat{f} \in C_c(\mathcal{G})$ by $\hat{f} = f|_{\mathcal{G}}$. It is easily checked that $\hat{\ } : E(\Gamma_1) \longrightarrow C_c(\mathcal{G})$ is an isomorphism of *-algebras.

Thus, since $C_c(\mathcal{G})$ is dense in both $A(\Gamma_1)$ and $C^*(\mathcal{G})$, in order to see that $A(\Gamma_1) \cong C^*(\mathcal{G})$, it suffices to show:

$$||f||_{A(\Gamma_1)} = ||f||_{C^*(\mathcal{G})} \qquad \text{for every } f \in C_c(\mathcal{G}).$$

Because $\mathcal{G}$ is amenable, $C^*(\mathcal{G})$ coincides with $C_r^*(\mathcal{G})$ [1, Proposition 6.1.5], so $||f||_{C^*(\mathcal{G})} = \sup_{u \in \mathcal{G}^0} ||\mathrm{Ind}\epsilon_u(f)||$, where $\mathrm{Ind}\epsilon_u$ is the representation of $C_c(\mathcal{G})$ on $L^2(\nu_u^{-1})$ given by $\mathrm{Ind}\epsilon_u(f)\xi(x) = \int f(y)\xi(y^{-1}x)d\lambda^{r(x)}(y)$. Here, ν_u denotes the measure on $\mathcal{G}$ induced by the measure ϵ_u on $\mathcal{G}^0$, given by $\nu_u = \int_{v \in \mathcal{G}^0} \lambda^v d\epsilon_u(v)$. The measure ν_u^{-1} is given by $\int \phi(x)d\nu_u^{-1}(x) = \int \phi(x^{-1})d\nu_u(x)$.

Now, for $u \in \mathcal{G}^0$, consider the representation $\delta_u : C_0(\mathcal{G}^0) \longrightarrow \mathbf{C}$ given by $\delta_u(f) = f(u)$. Since $\mathcal{H}(\Gamma_1)$ is a (right) Hilbert $C_0(\mathcal{G}^0)$-module with a left action of $A(\Gamma_1)$ by adjointable operators, we can induce the representation δ_u up to a representation $\mathrm{Ind}\delta_u$ of $A(\Gamma_1)$ on the Hilbert space $\mathcal{H}(\Gamma_1) \otimes_{C_0(\mathcal{G}^0)} \mathbf{C}$. We claim that for every $u \in \mathcal{G}^0$, $\mathrm{Ind}\delta_u$ is unitarily equivalent to $\mathrm{Ind}\epsilon_u$.

Define $v : \mathcal{H}(\Gamma_1) \odot \mathbf{C} \longrightarrow L^2(\nu_u^{-1})$ by $v(\sum g_i \otimes \lambda_i) = \sum \lambda_i g_i$. We have

$$
\begin{aligned}
\langle \sum_i g_i \otimes \lambda_i, \sum_j h_j \otimes \mu_j \rangle_{\mathcal{H}(\Gamma_1) \otimes \mathbf{C}} &= \sum_{i,j} \langle g_i \otimes \lambda_i, h_j \otimes \mu_j \rangle \\
&= \sum_{i,j} \overline{\lambda_i} \mu_j g_i^* h_j(u) \\
&= \sum_{x \in \mathcal{G}^u} \sum_{i,j} \overline{\lambda_i} \mu_j \overline{g_i(x^{-1})} h_j(x^{-1}) \\
&= \sum_{x \in \mathcal{G}^u} \overline{v(\sum_i g_i \otimes \lambda_i)(x^{-1})} v(\sum_j h_j \otimes \mu_j)(x^{-1}) \\
&= \langle v(\sum_i g_i \otimes \lambda_i), v(\sum_j h_j \otimes \mu_j) \rangle_{L^2(\nu_u^{-1})}.
\end{aligned}
$$

Thus v can be extended to a unitary (also denoted by v) from $\mathcal{H}(\Gamma_1) \otimes_{C_0(\mathcal{G}^0)} \mathbf{C}$ to $L^2(\nu_u^{-1})$. Also, for all $f \in C_c(\mathcal{G})$,

$$
\begin{aligned}
v \operatorname{Ind}\delta_u(f)(\sum g_i \otimes \lambda_i) &= v(\sum f g_i \otimes \lambda_i) \\
&= \sum \lambda_i f g_i \\
&= \sum \lambda_i \operatorname{Ind}\epsilon_u(f) g_i \\
&= \operatorname{Ind}\epsilon_u(f) \, v(\sum g_i \otimes \lambda_i).
\end{aligned}
$$

Thus $\operatorname{Ind}\delta_u$ is unitarily equivalent to $\operatorname{Ind}\epsilon_u$ for every $u \in \mathcal{G}^0$.

Since $\{\delta_u\}$ is a separating family of representations of $C_0(\mathcal{G}^0)$, $\{\operatorname{Ind}\delta_u\}$ is a separating family of representations of $A(\Gamma_1)$. Thus, for $f \in C_c(\mathcal{G})$,

$$
||f||_{A(\Gamma_1)} = \sup_{u \in \mathcal{G}^0} ||\operatorname{Ind}\delta_u(f)|| = \sup_{u \in \mathcal{G}^0} ||\operatorname{Ind}\epsilon_u(f)|| = ||f||_{C^*(\mathcal{G})},
$$

as desired.

Now, by [4, Theorem 2.9], it follows that $C_0(\mathcal{G}^0)$ is diagonal in $C^*(\mathcal{G})$. Since the correspondence between diagonal pairs and twists is bijective (up to isomorphism), we must have that Γ, the twist constructed from this diagonal pair, is isomorphic to Γ_1, the trivial twist over $\mathcal{G}$. $\square$

Corollary 1. *Every AF-algebra contains a subalgebra which is diagonal in Kumjian's sense.*

Proof. By Theorem 1, every AF-algebra A is the C^*-algebra of a Hausdorff, r-discrete equivalence relation, which is amenable by [6, Corollary 5.5]. Thus, by Theorem 2, A contains a diagonal subalgebra. $\square$

In [9], Strătilă and Voiculescu defined a notion of a diagonal subalgebra for AF-algebras, and it is not surprising that the two notions of diagonal coincide in the AF case. We briefly review the Strătilă-Voiculescu set-up.

Given a unital AF-algebra $A = \overline{\bigcup_{n=0}^{\infty} A_n}$, where it is assumed that $A_0 \cong \mathbf{C}$, Strătilă and Voiculescu inductively define an ascending sequence of abelian C^*-subalgebras $\{C_n\}$ as follows:

- $C_0 = A_0$;
- given C_n, define $C_{n+1} = \operatorname{span} C_n D_{n+1}$, where D_{n+1} is an arbitrary maximal abelian self-adjoint subalgebra of $A_{n+1} \cap A_n'$.

The diagonal subalgebra C is defined to be the closure of the union of the C_n's.

Strătilă and Voiculescu's AF masa led to Renault's notion of Cartan subalgebra which led in turn to Kumjian's notion of diagonal. The following result is similar to [8, III.1.15] and is also related to [4, 6.16]. In [9], only unital AF-algebras were considered, but the definition of C makes sense in the nonunital case, and it is in that more general setting that the following theorem is proved.

Theorem 3. *Let A be an AF-algebra. Then Strătilă and Voiculescu's subalgebra C is diagonal in Kumjian's sense.*

Proof. In light of Theorem 1, we can view A as $C^*(\mathcal{G}_E|_S)$ for some Bratteli diagram E for A and some subset S of vertices of E. We claim that the D_n's can be chosen such that for each n, $C_n = \operatorname{span}\{1_{Z(\alpha,\alpha)} \mid r(\alpha) \in V^n\}$ (recall that V^n denotes the nth level of vertices of the Bratteli diagram).

Suppose $C_0, C_1, \ldots, C_n$ are as in the above claim. For each edge e with $r(e) \in V^{n+1}$, define $f_e = \sum 1_{Z(\alpha,\alpha)}$, where the sum runs over all paths α whose last edge is e (we also make the implicit assumption, here and from now on, that all paths start in the distinguished set S). Also, for every $v \in V^{n+1} \cap S$, define $f_v = 1_{Z(v,v)}$. Now define

$$D_{n+1} = \operatorname{span}\{f_e, f_v \mid r(e) \in V^{n+1}, v \in V^{n+1} \cap S\}.$$

Lemma 2. D_{n+1} *is a MASA in $A_{n+1} \cap A_n'$.*

Proof. Clearly D_{n+1} is abelian. To show that it is a MASA in $A_{n+1} \cap A_n'$, we will show that $x \in A_{n+1} \cap A_n' \cap D_{n+1}'$ implies $x \in D_{n+1}$. So fix $x \in A_{n+1} \cap A_n' \cap D_{n+1}'$. We can write $x = \sum \lambda_{\alpha,\beta} 1_{Z(\alpha,\beta)}$, where the sum runs over all α, β with $r(\alpha) = r(\beta) \in V^{n+1}$ and each pair appears only once.

First suppose that there exist $\alpha_0 \neq \beta_0$ such that $\lambda_{\alpha_0,\beta_0} \neq 0$. Then one of the following three cases holds:

Case 1: One of α_0 or β_0 is a zero-length path.

Case 2: α_0 and β_0 have the same last edge (hence differ somewhere before the last edge).

Case 3: α_0 and β_0 have different last edges.

For Case 1, assume (without loss of generality) that $|\alpha_0| = 0$ and let $y = 1_{Z(\alpha_0,\alpha_0)} \in D_{n+1}$. For Case 2, write $\alpha_0 = \gamma_0 e$ where e is an edge, and let $y = 1_{Z(\gamma_0,\gamma_0)} \in A_n$. For Case 3, denote by e_0 the last edge of α_0, and let $y = f_{e_0} \in D_{n+1}$.

In all three cases, x and y should commute, but the reader may tediously verify that $1_{Z(\beta_0,\alpha_0)} xy 1_{Z(\beta_0,\alpha_0)} = 0$ and $1_{Z(\beta_0,\alpha_0)} yx 1_{Z(\beta_0,\alpha_0)} = \lambda_{\alpha_0,\beta_0} 1_{Z(\beta_0,\alpha_0)} \neq 0$. Thus x has no "off-diagonal" entries. So we can write $x = \sum \lambda_\alpha 1_{Z(\alpha,\alpha)}$.

Now suppose there exist $\alpha_1 \neq \alpha_2$ such that α_1, α_2 have the same last edge, but $\lambda_{\alpha_1} \neq \lambda_{\alpha_2}$. For $i = 1, 2$, write $\alpha_i = \gamma_i e$ where e is an edge, and set $y = 1_{Z(\gamma_1,\gamma_2)} \in A_n$. The reader may check that $1_{Z(\alpha_1,\alpha_1)} xy \neq 1_{Z(\alpha_1,\alpha_1)} yx$. Hence $xy \neq yx$, violating the assumption that $x \in A_n'$.

Thus $x = \sum \lambda_\alpha 1_{Z(\alpha,\alpha)} \in A_{n+1} \cap A_n' \cap D_{n+1}'$ implies that $\lambda_{\alpha_1} = \lambda_{\alpha_2}$ whenever the last edges of α_1 and α_2 agree, which implies that $x \in D_{n+1}$. $\square$

Now, $C_{n+1} = \operatorname{span} C_n D_{n+1} \subset \operatorname{span}\{1_{Z(\alpha,\alpha)} \mid r(\alpha) \in V^{n+1}\}$. To show the reverse inclusion, fix α with $r(\alpha) \in V^{n+1}$. If $|\alpha| = 0$, $1_{Z(\alpha,\alpha)} \in D_{n+1} \subset C_{n+1}$. If not, write $\alpha = \gamma e$ where e is an edge and note that $1_{Z(\alpha,\alpha)} = f_e 1_{Z(\gamma,\gamma)} \in \operatorname{span}\{C_n, D_{n+1}\}$.

Thus, with the D_n's chosen in this way, $C_n = \text{span}\,\{1_{Z(\alpha,\alpha)}\,|\,r(\alpha) \in V^n\}$ for every n, so $C = \overline{\text{span}}\,\{1_{Z(\alpha,\alpha)}\} = C_0((\mathcal{G}_E|_S)^0)$, which is diagonal in Kumjian's sense.

Now we have shown that the C_n's can be chosen in such a way that C is diagonal. Now, suppose $\tilde{C}$ is another Strătilă-Voiculescu diagonal. Then there exists an automorphism α of A which maps C to $\tilde{C}$. In fact, this automorphism can be chosen to be approximately inner (i.e. there exists a sequence $\{u_k\}$ of unitaries in A such that $\alpha(a) = \lim_{k\to\infty} u_k a u_k^*$; see [7]). Defining $\tilde{P} = \alpha \circ P \circ \alpha^{-1}$, the reader may check that $\tilde{P}$ is a faithful conditional expectation from A onto $\tilde{C}$, and that $\ker \tilde{P} = \alpha(\ker P) = \overline{\text{span}}\,N_f(\tilde{C})$. Thus $\tilde{C}$ is diagonal in Kumjian's sense. $\square$

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