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REFINABLE SUBSPACES OF A REFINABLE SPACE

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ABSTRACT. Local refinable finitely generated shift-invariant spaces play a significant role in many areas of approximation theory and geometric design. In this paper we present a new approach to the construction of such spaces. We begin with a refinable function $\psi : \mathbb{R} \to \mathbb{R}^m$ which is supported on $[0, 1]$. We are interested in spaces generated by a function $\phi : \mathbb{R} \to \mathbb{R}^n$ built from the shifts of ψ.

1. INTRODUCTION

Local refinable finitely generated shift-invariant spaces naturally arise in the theory of (multi)wavelets, splines, finite-elements, and subdivision schemes. In this paper we introduce and begin to develop a method for constructing and studying such spaces.

Let L^1_{loc} denote the space of all functions $f : \mathbb{R} \to \mathbb{R}$ which belong to $L^1(\mathbb{R})$ locally; that is, $f \in L^1_{\text{loc}}$ provided that (f is measurable and) $\int_K |f| < \infty$ for every compact $K \subset \mathbb{R}$. This space is topologized by the family of seminorms

$$|f|_N := \int_{[-N,N]} |f|, \qquad N \in \mathbb{N}.$$

We refer to a (row) vector $\phi = [\phi_1, \dots, \phi_n]$ of functions in L^1_{loc} as a *generator*.

A generator $\phi = [\phi_1, \dots, \phi_n]$ is said to be *refinable* if there exists a finitely supported sequence $b : \mathbb{Z} \to \mathbb{R}^{n\times n}$ (called a *mask* for ϕ) for which

$$\phi = \sum_{j\in\mathbb{Z}} \phi(2 \cdot -j)b(j).$$

We begin with a generator $\psi = [\psi_1, \dots, \psi_m]$ supported on $[0, 1]$ that is refinable with a two-term mask:

$$\psi = \psi(2\cdot)a(0) + \psi(2 \cdot -1)a(1); \tag{1.1}$$

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and we intend to construct more useful (read "smoother") refinable generators by using the shifts of ψ. That is, we consider generators of the form

$$\phi = \sum_{j\in\mathbb{Z}} \psi(\cdot - j)c(j),$$

for some sequence $c : \mathbb{Z} \to \mathbb{R}^{m\times n}$. The motivation for this approach is that it is much easier to study the properties of ψ, since it is supported on $[0, 1]$ (hence the shifts of ψ do not "interfere" with each other). The crux is that ϕ constructed in this way will not, in general, be refinable.

Let V be a subspace of $L^1_{\rm loc}$. Then we say V is *shift-invariant* if

$$f \in V \implies f(\cdot \pm 1) \in V;$$

we say V is a *finitely generated shift-invariant (FSI)* space if

$$V = S(\phi) := \operatorname{clos}_{L^1_{\rm loc}} \operatorname{span}\{\phi_i(\cdot - j) \mid i = 1, \dots , n;\ j \in \mathbb{Z}\}$$

for some generator ϕ; and we say an FSI space V is *local* if $V = S(\phi)$ for some compactly supported generator ϕ. Lastly, we say V is *refinable* if

$$f \in V \implies f(\cdot/2) \in V.$$

Evidently, $S(\phi)$ is refinable whenever ϕ is refinable.

The main objective of this paper is: *Given a refinable generator ψ supported on $[0, 1]$ with mask $\big(a(0), a(1)\big)$, characterize all local refinable FSI subspaces $V \subset S(\psi)$.* Under the condition that $a(0)$ is invertible, we achieve this goal in Corollaries 2.4 and 2.6 and Theorem 2.5.

We point out that every local refinable FSI space is a subspace of $S(\psi)$ for some refinable generator ψ supported on $[0, 1]$. For $f \in L^1_{\rm loc}$ and $V \subset L^1_{\rm loc}$, define

$$f_| := f\chi_{[0,1]} \quad \text{and} \quad V_| := \{\, f_| \mid f \in V \,\},$$

where χ_A denotes the characteristic function of the set A. Suppose V is a local refinable FSI space. Then $m := \dim V_|$ is finite. We refer to m as the *local dimension of V*. Let $\psi = [\psi_1, \dots , \psi_m]$ be a basis for $V_|$. Then ψ is refinable and $V \subset S(\psi)$.

This has been observed and exploited already by Jia in [J4], [J5], and [J6], where the author studied a given function ϕ via a basis ψ for $S(\phi)_|$. We point out that ψ is necessarily refinable with a two-term mask, a fact which, though surely known to Jia, was not exploited since his results relate to general shift-invariant spaces as opposed to refinable ones.

The simpler structure due to the small supports of ψ and a in Eq. (1.1) has also been recognized by Micchelli et al. in, for example, [M], [MP1], [MP2], and [MS]. In particular, given a univariate refinable function ϕ with finite mask b, they define $a(0)$ and $a(1)$ by $a(\varepsilon) := [b(\varepsilon+2j-i)]_{i,j}$ and study ϕ via the refinable function having mask a (also see [DL]). Among other things, this was used to provide necessary and sufficient conditions for the convergence of a given subdivision scheme and, in [MS], to provide a fairly thorough study of regularity for refinable function vectors.

Our approach is different in that the mask a and generator ψ come first. In this paper, we identify all local refinable subspaces $S(\phi)$ of $S(\psi)$. The next steps are to provide further characterizations of the properties of $S(\psi)$ in terms of a; to determine when these properties are preserved by a subspace $S(\phi)$; and to put these ideas together to construct desirable refinable generators.

2. Results

Throughout this paper, we assume that $\big(a(0), a(1)\big)$ is a mask for a refinable generator $\psi = [\psi_1, \ldots, \psi_m]$ supported on $[0,1]$ (in particular, each ψ_j is assumed to be in $L^1(\mathbb{R})$). We will show that when $a(0)$ is invertible, each local refinable FSI subspace of $S(\psi)$ corresponds to some $a(0)$-invariant space (for a matrix $a \in \mathbb{R}^{k\times k}$, a space $\Gamma \subset \mathbb{R}^k$ is *a-invariant* if $a\Gamma \subset \Gamma$). Our specific statements will require a few more definitions.

We use $\mathbb{Z}_+$ to denote the set of non-negative integers and $\mathbb{R}^k$ to denote the set of column vectors of length k. For any set $V \subset L^1_{\text{loc}}$, define

$$V^+ := \{\, f \in V \mid \operatorname{supp} f \subset [0,\infty) \,\};$$

and, for $V \subset S(\psi)$, define

$$\Sigma(V) := \{\, \sigma \in \mathbb{R}^m \mid \psi\sigma \in V^+_| \,\}.$$

(By convention, $V^+_| := (V^+)_|$.)

Proposition 2.1. *For any refinable subspace V of $S(\psi)$, $\Sigma(V)$ is $a(0)$-invariant.*

If $\phi = [\phi_1, \ldots, \phi_k]$ is a generator supported on $[0,\infty)$, then the sum

$$\phi *' c := \sum_{j=0}^{\infty} \phi(\cdot - j)c(j)$$

is locally finite for any sequence $c : \mathbb{Z}_+ \to \mathbb{R}^k$. In particular, the set

$$R(\phi) := \{\phi *' c \mid c : \mathbb{Z}_+ \to \mathbb{R}^k\},$$

spanned by the right shifts of ϕ, is a subset of $S(\phi)^+$.

We say a subspace Λ of $\mathbb{R}^m$ is *preserved by* $a(0)$ if $a(0)\Lambda = \Lambda$, and a matrix $\lambda \in \mathbb{R}^{m\times n}$ is *preserved by* $a(0)$ if its columns form a basis for a space that is preserved by $a(0)$. Note that a matrix $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$ if and only if $a(0)\lambda = \lambda\beta_\lambda$ for a unique invertible $\beta_\lambda \in \mathbb{R}^{n\times n}$. Suppose that $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$. Set

$$\text{(2.1)} \qquad \ell(0) := \lambda \quad \text{and} \quad \ell(2j+\varepsilon) := a(\varepsilon)\ell(j)\beta_\lambda^{-1} \text{ for } \varepsilon \in \{0,1\},\ 2j+\varepsilon > 0.$$

We define the *generalized truncated power* e_λ by

$$e_\lambda := \psi *' \ell.$$

Proposition 2.2. *Suppose that $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$. Then*

(i) $e_\lambda = e_\lambda(2\cdot)\beta_\lambda$;
(ii) *if $\lambda' \in \mathbb{R}^{m\times n}$ has the same column space as λ, then $S(e_\lambda) = S(e_{\lambda'})$; and*
(iii) *$S(e_\lambda)$ is a local refinable FSI subspace of $S(\psi)$.*

Property (2.2.ii) allows us to unambiguously define, for any Λ preserved by $a(0)$, the space $S_\Lambda := S(e_\lambda)$ where the columns of λ form a basis for Λ.

Property (2.2.iii) ensures that $S(e_\lambda) = S(\phi)$ for some compactly supported generator ϕ. A procedure for constructing such ϕ can be based on the proofs in [BD].

Theorem 2.3. *Suppose V is a local refinable FSI subspace of $S(\psi)$. If $\Sigma(V)$ is preserved by $a(0)$, then $V = S_{\Sigma(V)}$.*

If $a(0)$ is invertible, every $a(0)$-invariant subspace is, in fact, preserved by $a(0)$ and we have the following corollary—one of the main results of this paper.

Corollary 2.4. *Suppose $a(0)$ is invertible. Then V is a local refinable FSI subspace of $S(\psi)$ if and only if $V = S_\Lambda$ for some $a(0)$-invariant Λ.*

In other words, when $a(0)$ is invertible, every local refinable FSI subspace of $S(\psi)$ is of the form S_Λ for some $a(0)$-invariant space Λ. The $a(0)$-invariant spaces are easily identified from the Jordan-Canonical form of $a(0)$. By Theorem 2.3, $S_\Lambda = S_{\Sigma(S_\Lambda)}$. So, if $a(0)$ is invertible and ψ *is linearly independent* (meaning the entries of ψ are linearly independent), the local refinable subspaces of $S(\psi)$ are in one-to-one correspondence with the $a(0)$-invariant spaces Λ satisfying $\Lambda = \Sigma(S_\Lambda)$. By the definitions of S_Λ and $\Sigma(V)$, Λ is always a subset of $\Sigma(S_\Lambda)$; but, as Example 4.2 illustrates, it is often a proper subset. Our next result offers a means for determining $\Sigma(S_\Lambda)$ from Λ.

First, define

$$A_0 := \begin{bmatrix} a(1) & 0 \\ 0 & a(0) \end{bmatrix} \quad \text{and} \quad A_1 := \begin{bmatrix} 0 & a(0) \\ 0 & a(1) \end{bmatrix}.$$

Let $\mathcal{H}_\Lambda$ be the minimal subspace of $\mathbb{R}^{2m}$ that contains

$$\begin{bmatrix} 0 \\ \Lambda \end{bmatrix} := \{ \begin{bmatrix} 0 \\ v \end{bmatrix} \mid v \in \Lambda \}$$

and is $\{A_0, A_1\}$-invariant, i.e., A_ε-invariant for $\varepsilon = 0, 1$. And define

$$\mathcal{H}_\Lambda^0 := \{v \in \mathbb{R}^m \mid \begin{bmatrix} 0 \\ v \end{bmatrix} \in \mathcal{H}_\Lambda\}.$$

Theorem 2.5. *Suppose Λ is an $a(0)$-invariant subspace of $\mathbb{R}^m$. If $a(0)$ is invertible and ψ is linearly independent, then $\Sigma(S_\Lambda) = \mathcal{H}_\Lambda^0$.*

We have the following corollary.

Corollary 2.6. *Suppose Λ is an $a(0)$-invariant subspace of $\mathbb{R}^m$. Let the columns of $\lambda \in \mathbb{R}^{m\times n}$ form a basis for Λ. If $a(0)$ is invertible and ψ is linearly independent, then the following are equivalent.*

(i) *$\Lambda = \Sigma(V)$ for some local refinable FSI subspace $V \subset S(\psi)$.*
(ii) *$\Lambda = \Sigma(S_\Lambda)$.*
(iii) *$\Lambda = \mathcal{H}_\Lambda^0$.*
(iv) *$S_\Lambda^+ = R(e_\lambda)$.*

It is clear that $S_{\Lambda|}$ is always a subset of $\operatorname{span}\psi := \operatorname{span}\{\psi_1, \dots, \psi_m\}$. We now give a characterization of when these sets are actually equal.

Theorem 2.7. *Suppose ψ is linearly independent. Suppose $\Lambda \subset \mathbb{R}^m$ is preserved by $a(0)$. Define $\mathcal{L}_\Lambda$ to be the minimal $\{a(0), a(1)\}$-invariant subspace of $\mathbb{R}^m$ containing Λ. Then $S_{\Lambda|} = \operatorname{span}\psi$ if and only if $\mathcal{L}_\Lambda = \mathbb{R}^m$.*

The preceding few results assume that ψ is linearly independent. Corollary 2.9 below characterizes this property in terms of the mask a. Corollary 2.9 is an immediate consequence of Theorem 2.8, which identifies all dependency relations among the entries of ψ and provides the mask for a basis for $\operatorname{span}\psi$ (also in terms of a).

To begin with, define $\mathcal{W}$ to be the smallest subspace of $\mathbb{R}^{1\times m}$ satisfying

$$\hat{\psi}(0) \in \mathcal{W}, \quad \mathcal{W}a(0) \subset \mathcal{W}, \quad \text{and} \quad \mathcal{W}a(1) \subset \mathcal{W}. \tag{2.2}$$

Let the rows of $W \in \mathbb{R}^{k\times m}$ form a basis for $\mathcal{W}$. Then there exists a unique $\tilde{v} \in \mathbb{R}^{1\times k}$ such that $\hat{\psi}(0) = \tilde{v}W$, and unique $\tilde{a}(0)$ and $\tilde{a}(1)$ such that $Wa(\varepsilon) = \tilde{a}(\varepsilon)W$.

Now, define

$$T := a(0) + a(1).$$

Then two necessary conditions for the generator ψ to be linearly independent are:

N1) 2 is a simple eigenvalue of the matrix T with left eigenvector $\hat{\psi}(0)$; and

N2) all other eigenvalues have modulus strictly less than 2

(cf., e.g., [DM], [H], [JS]).

Theorem 2.8. *Suppose T satisfies* N1 *and* N2*. Then*

(i) $\psi\sigma = 0 \iff \sigma \in \mathcal{W}^\perp := \{v \in \mathbb{R}^m \mid wv = 0 \text{ for all } w \in \mathcal{W}\}$;

(ii) *the refinement equation with mask $\tilde{a}$ has a unique solution $\tilde{\psi}$ with $\widehat{\tilde{\psi}}(0) = \tilde{v}$;*

(iii) *$\tilde{\psi}$ is a basis for* $\operatorname{span}\psi$*, i.e., $\tilde{\psi}$ is independent and* $\operatorname{span}\psi = \operatorname{span}\tilde{\psi}$.

So, one may assume that ψ is linearly independent without loss of generality. Note, if $M \in \mathbb{R}^{m\times k}$ is any right inverse of W, then $\tilde{v} = \hat{\psi}(0)M$ and $\tilde{a}(\varepsilon) = Wa(\varepsilon)M$.

Corollary 2.9. *The generator ψ is linearly independent if and only if*

(i) *T satisfies* N1 *and* N2*; and*

(ii) $\mathcal{W} = \mathbb{R}^{1\times m}$.

The interested reader is encouraged to see [HJ] for additional results along these lines.

3. Proofs

Throughout this section, we write $\phi \subset V$ to mean that the entries of the generator ϕ are elements of V. We recall some results from [BD].

Lemma 3.1. *For any closed shift-invariant space V of finite local dimension, there exists $r > 0$ such that if $f \in V$ vanishes on $[-r, 0]$, then $f_| \in V_|^+$.*

Lemma 3.2. *For any closed shift-invariant space V of finite local dimension, there is a compactly supported generator $\phi = [\phi_1, \ldots, \phi_k] \subset V$ such that $\phi_|$ is a basis for $V_|^+$ and $V^+ = R(\phi)$.*

The topology used in [BD] is that of uniform convergence on compact sets, but the arguments used there also apply to the topology of L^1_{loc}.

Proof of Proposition 2.1. Suppose $\sigma \in \Sigma(V)$. Then there exists $f \in V^+$ such that $f_| = \psi\sigma$. Since V is refinable, $f(\cdot/2) \in V^+$. But, $f(\cdot/2) = \psi(\cdot/2)\sigma = \psi a(0)\sigma$ on $[0, 1]$. So $a(0)\sigma \in \Sigma(V)$. □

Proof of Proposition 2.2. (i)

$$e_\lambda\left(\frac{\cdot}{2}\right) = \sum_j \psi\left(\frac{\cdot - 2j}{2}\right)\ell(j) = \sum_{j,\varepsilon} \psi(\cdot - 2j - \varepsilon)a(\varepsilon)\ell(j)$$
$$= \sum_{j,\varepsilon} \psi\big(\cdot - (2j+\varepsilon)\big)\ell(2j+\varepsilon)\beta_\lambda = e_\lambda\beta_\lambda.$$

(ii) There exists $\gamma \in \mathbb{R}^{n\times n}$ such that $\lambda = \lambda'\gamma$. Set $\beta := \beta_\lambda$ and $\beta' := \beta_{\lambda'}$. Then

$$\lambda'\gamma\beta = \lambda\beta = a(0)\lambda = a(0)\lambda'\gamma = \lambda'\beta'\gamma.$$

Since the columns of λ' form a basis, $\gamma = \beta'\gamma\beta^{-1}$. Define ℓ by Eqs. (2.1) and ℓ' similarly, but with λ' in place of λ. Then, $\ell(0) = \lambda = \lambda'\gamma = \ell'(0)\gamma$. Now, suppose $2j+\varepsilon > 0$ and $\ell(j) = \ell'(j)\gamma$. Then

$$\ell(2j+\varepsilon) = a(\varepsilon)\ell(j)\beta^{-1} = a(\varepsilon)\ell'(j)\gamma\beta^{-1} = \ell'(2j+\varepsilon)\beta'\gamma\beta^{-1} = \ell'(2j+\varepsilon)\gamma.$$

It follows that $e_\lambda = e_{\lambda'}\gamma$.

(iii) Set $V := S(e_\lambda)$. Let ϕ be as in Lemma 3.2. Since $\phi \subset V$, we have $S(\phi) \subset S(e_\lambda)$. Conversely, since $e_\lambda \in V^+ = R(\phi) \subset S(\phi)$, we have $S(e_\lambda) \subset S(\phi)$. ☐

The proof of Theorem 2.3 will require the following lemma.

Lemma 3.3. *Let V be a local FSI space. Suppose $\phi = [\phi_1, \dots, \phi_n] \subset V^+$ is such that* $\operatorname{span}\{\phi_{1|}, \dots, \phi_{n|}\} = V^+_{|}$. *Then $V^+ = S(\phi)^+ = R(\phi)$.*

Proof. Let $f \in V^+$. We recursively construct a sequence $c : \mathbb{Z}_+ \to \mathbb{R}^n$ so that

$$f = f_N := \sum_{j=0}^{N} \phi(\cdot - j)c(j) \text{ on } [0, N+1].$$

This is the so-called "peeling-off argument" from [BD]. Since $f \in V^+$ and $\phi_|$ spans $V^+_|$, $f = \phi c(0)$ on $[0,1]$ for some $c(0) \in \mathbb{R}^n$. Now suppose we have $c(0), \dots, c(N)$ such that $f = f_N$ on $[0, N+1]$. Then $(f - f_N)(\cdot + N + 1) \in V^+$. So there exists $c(N+1)$ such that $(f - f_N)(\cdot + N + 1) = \phi c(N+1)$ on $[0,1]$. For this value for $c(N+1)$, $f = f_{N+1}$ on $[0, N+2]$. So $V^+ \subset R(\phi) \subset S(\phi)^+$.

Since $\phi \subset V$ and $S(\phi)$ is the smallest closed shift-invariant space containing ϕ, $S(\phi)$ is a subspace of V. This, in turn, implies that $S(\phi)^+ \subset V^+$. ☐

Proof of Theorem 2.3. Let the columns of λ form a basis for $\Sigma(V)$. We first show that $V^+ = S(e_\lambda)^+$. By Lemma 3.3, it is sufficient to show that $e_\lambda \subset V^+$ since $e_{\lambda|} = \psi\lambda$, which spans $V^+_|$.

Let $\phi = [\phi_1, \dots, \phi_n] \subset V^+$ be such that $\phi_|$ is a basis for $V^+_|$. Then $e_{\lambda|} = \phi_|\gamma$ for some $\gamma \in \mathbb{R}^{n\times n}$. Since $e_\lambda = e_\lambda(2\cdot)\beta$, $e_\lambda = e_\lambda(2^{-k}\cdot)\beta^{-k} = \phi(2^{-k}\cdot)\gamma\beta^{-k}$ on $[0, 2^k]$. Since V is refinable, $\phi(2^{-k}\cdot)\gamma\beta^{-k} \subset V^+$. And since $\phi_|$ is a basis for $V^+_|$, it follows that $V^+ = R(\phi)$. So, for each $k \in \mathbb{N}$, there exists a sequence c_k such that

$$e_\lambda = \sum_{j=0}^{2^k} \phi(\cdot - j)c_k(j) \quad \text{on} \quad [0, 2^k].$$

Since $\phi_|$ is a basis, the set $\{\, \phi(\cdot - j)|_{[0,2^k]} \mid j = 0, 1, \dots, 2^k - 1 \,\}$ is linearly independent. It follows that the sequence $c(j) := c_k(j)$ ($j \in \mathbb{Z}_+$, $2^k > j$) is well-defined and satisfies $e_\lambda = \phi *' c$.

Since V is a local FSI space, it follows that $V = S(\nu)$ for some compactly supported generator ν. Without loss of generality, $\operatorname{supp}\nu \subset [0, \infty)$. Then $\nu \in S(e_\lambda)$ and $e_\lambda \in V$, since $V^+ = S(e_\lambda)^+$. Therefore $V = S(e_\lambda)$. ☐

Proof of Theorem 2.5. Define

$$h(0) := \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \quad \text{and} \quad h(2j+\varepsilon) := A_\varepsilon h(j) \text{ for } \varepsilon \in \{0,1\},\ 2j+\varepsilon > 0.$$

Then $\mathcal{H}_\Lambda$ is the column space of $[h(0), h(1), h(2), \ldots]$. Also (with $\ell(-1) := 0$ for consistency), Eqs. (2.1) give

$$h(j) = \begin{bmatrix} \ell(j-1)\beta_\lambda^j \\ \ell(j)\beta_\lambda^j \end{bmatrix} \text{ for all } j \in \mathbb{Z}_+.$$

Since β_λ is invertible, the column space of $h(j)$ is equal to the column space of $\begin{bmatrix} \ell(j-1) \\ \ell(j) \end{bmatrix}$. Also, $e_\lambda(\cdot + j) = \psi(\cdot + 1)\ell(j-1) + \psi\ell(j)$ on $[-1, 1]$. Hence, $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_\Lambda$ if and only if there exists an $f \in S_\Lambda$ which agrees with $\psi(\cdot + 1)u + \psi v$ on $[-1, 1]$, since ψ is independent. It follows that

$$\mathcal{H}_\Lambda^0 = \{\sigma \in \mathbb{R}^m \mid \psi\sigma = f_| \text{ for some } f \in S_\Lambda \text{ s.t. } f|_{[-1,0]} = 0\},$$

making it clear that $\Sigma(S_\Lambda) \subset \mathcal{H}_\Lambda^0$.

Now suppose $\sigma \in \mathcal{H}_\Lambda^0$. Then there exists $f \in S_\Lambda$ such that f vanishes on $[-1, 0]$ and $f_| = \psi\sigma$. By Lemma 3.1, there is a $k \in \mathbb{N}$ such that if $g \in S_\Lambda$ vanishes on $[-2^k, 0]$, then $g_| \in S_{\Lambda|}^+$. We show $a(0)^k\sigma \in \Sigma(S_\Lambda)$ for this k. First, note that

$$f(2^{-k}\cdot)_| = \psi(2^{-k}\cdot)_|\sigma = \psi a(0)^k\sigma.$$

Since $f(2^{-k}\cdot)$ vanishes on $[-2^k, 0]$, $a(0)^k\sigma$ is in $\Sigma(S_\Lambda)$. Therefore σ is in $\Sigma(S_\Lambda)$, because $\Sigma(S_\Lambda)$ is preserved by $a(0)$ (it is $a(0)$-invariant and $a(0)$ is invertible). ☐

Proof of Corollary 2.6. The equivalence of (ii) and (iii) is an immediate consequence of Theorem 2.5. It is also obvious that (iv) $\implies$ (ii) $\implies$ (i). We prove (i) $\implies$ (ii) and (ii) $\implies$ (iv).

First, let V be a local refinable FSI subspace of $S(\psi)$ such that $\Lambda = \Sigma(V)$. By Theorem 2.3, $V = S_\Lambda$. So $\Lambda = \Sigma(S_\Lambda)$, proving (i) $\implies$ (ii). To prove (ii) $\implies$ (iv), by Lemma 3.3, it is enough to point out that $e_{\lambda|} = \psi\lambda$ is a basis for $S_{\Lambda|}^+ = \psi\Lambda$. ☐

Proof of Theorem 2.7. Let the columns of λ form a basis for Λ and recall that $e_\lambda = \psi *' \ell$ where ℓ is given by Eqs. (2.1). Then $e_\lambda(\cdot + j)_| = \psi\ell(j)$. Let L be the column space of $[\ell(0), \ell(1), \ell(2), \ldots]$. Then $S(e_\lambda)_| = \operatorname{span}\psi$ if and only if $L = \mathbb{R}^m$. We show that $L = \mathcal{L}_\Lambda$.

Clearly $\Lambda \subset L$, since $\lambda = \ell(0)$. Also, by Eqs. (2.1) and since β is invertible, $a(\varepsilon)L \subset L$ for $\varepsilon = 0, 1$. So $\mathcal{L}_\Lambda \subset L$.

Now, the columns of $\ell(0) = \lambda$ are in $\mathcal{L}_\Lambda$ and, if $\mathcal{L}_\Lambda$ contains the columns of $\ell(m)$, then it must contain the columns of $\ell(2m + \varepsilon)$ for $\varepsilon = 0, 1$. Hence $L \subset \mathcal{L}_\Lambda$. ☐

Proof of Theorem 2.8. Suppose T satisfies `N1` and `N2`. Define $v := \hat{\psi}(0)$. Let W, $\tilde{v}$, and $\big(\tilde{a}(0), \tilde{a}(1)\big)$ be as defined before Theorem 2.8. Define $\tilde{T} := \tilde{a}(0) + \tilde{a}(1)$. For any left eigenvector $\tilde{u}$ of $\tilde{T}$, $u := \tilde{u}W$ is a left-eigenvector of T with the same eigenvalue. Moreover, $\tilde{v}\tilde{T} = 2\tilde{v} \neq 0$, since the rows of W are linearly independent and

$$\tilde{v}\tilde{T}W = \tilde{v}WT = vT = 2v = 2\tilde{v}W.$$

That is, $\tilde{T}$ satisfies `N1` and `N2`, so there exists a unique $\tilde{\psi} \subset \mathcal{D}'(\mathbb{R})$ satisfying

$$\hat{\tilde{\psi}}(0) = \tilde{v} \quad \text{and} \quad \tilde{\psi} = \tilde{\psi}(2\cdot)\tilde{a}(0) + \tilde{\psi}(2\cdot - 1)\tilde{a}(1).$$

It follows that $\widehat{\tilde{\psi}W}(0) = v$ and $\tilde{\psi}W$ satisfies Eq. (1.1). So $\tilde{\psi}W = \psi$, proving that $\sigma \in \mathcal{W}^\perp \implies \psi\sigma = 0$. Moreover, since W is of full rank, $\operatorname{span}\psi = \operatorname{span}\tilde{\psi}$.

Now, let the entries of $\phi = [\phi_1, \dots, \phi_j]$ form a basis for $\operatorname{span}\psi$. Then there exists $V \in \mathbb{R}^{j\times m}$ such that $\psi = \phi V$. Let $\mathcal{V}$ be the row space of V, that is, $\mathcal{V} := \{uV \mid u \in \mathbb{R}^{1\times j}\}$. Evidently, $v \in \mathcal{V}$. We will show that $\mathcal{V}a(\varepsilon) \subset \mathcal{V}$ for $\varepsilon = 0, 1$. Consequently, $\mathcal{W} \subset \mathcal{V}$. So $\psi\sigma = 0 \implies \sigma \in \mathcal{V}^\perp \subset \mathcal{W}^\perp$.

For any $\sigma \in \mathbb{R}^m$,

$$\phi V\sigma = \psi\sigma = \psi(2\cdot)a(0)\sigma + \psi(2\cdot -1)a(1)\sigma = \phi(2\cdot)Va(0)\sigma + \phi(2\cdot -1)Va(1)\sigma.$$

And, since the entries of ϕ are linearly independent, $V\sigma = 0 \implies Va(\varepsilon)\sigma = 0$ for $\varepsilon = 0, 1$. Since $\sigma \in \mathbb{R}^m$ was arbitrary, it follows that $\mathcal{V}a(\varepsilon) \subset \mathcal{V}$ for $\varepsilon = 0, 1$.

Lastly, let $\tilde{\mathcal{W}}$ be the smallest subset of $\mathbb{R}^{1\times k}$ satisfying $\tilde{v} \in \tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}\tilde{a}(\varepsilon) \subset \tilde{\mathcal{W}}$ $(\varepsilon = 0, 1)$. Then $v = \tilde{v}W \in \tilde{\mathcal{W}}W$ and $\tilde{\mathcal{W}}Wa(\varepsilon) = \tilde{\mathcal{W}}\tilde{a}(\varepsilon)W \subset \tilde{\mathcal{W}}W$. So $\mathcal{W} \subset \tilde{\mathcal{W}}W$. This, together with the fact that $\dim \mathcal{W} = \operatorname{rank} W = k$, implies that $\tilde{\mathcal{W}} = \mathbb{R}^{1\times k}$. Since $\tilde{\psi}\sigma = 0 \iff \sigma \in \tilde{\mathcal{W}}^\perp$, $\tilde{\psi}$ is independent. □

4. Examples

Example 4.1. We present all local refinable FSI spaces of piecewise polynomials with integer breakpoints and prove the list is complete (cf. [GL],[LLS] for related results).

For any $r, m \in \mathbb{Z}$ satisfying $-1 \le r < m$, the space $\mathcal{S}_r^m$ of all r times continuously differentiable piecewise polynomials of degree at most m with integer breakpoints is defined by

$$\mathcal{S}_r^m := \{\, f \in C^r(\mathbb{R}) \;\mid\; f|_{(j,j+1)} \in \Pi_m \text{ for } j \in \mathbb{Z}\,\}.$$

Note that $\mathcal{S}_r^m = \sum_{j=r+1}^m \mathcal{S}_{j-1}^j$. In fact, we will show that every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is of the form

$$\sum_{j\in J} \mathcal{S}_{j-1}^j \text{ for some } J \subset \{0, \dots, m\}.$$

In particular, every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is a sum of refinable *principal shift-invariant (PSI)* spaces (a subspace V of L^1_{loc} is a principal shift-invariant space if $V = S(\phi)$ for some single scalar-valued generator ϕ). This is not true of shift-invariant spaces in general. For example, the only refinable PSI subspace of the space generated by $\chi_{[0,1)}$ and $\chi_{[0,1/2)}$ is (the proper subspace) $\mathcal{S}_{-1}^0$.

Define $\psi := [\pi_{0|}, \dots, \pi_{m|}]$, where $\pi_j(x) := x^j$. Then the entries of ψ are linearly independent, $\mathcal{S}_{-1}^m = S(\psi)$, and ψ is refinable with mask $a(0) = d$, $a(1) = cd$, where

$$c := \left[\binom{j}{i}\right]_{i,j=0}^m \quad \text{and} \quad d := \operatorname{diag}(2^{-j})_{j=0}^m.$$

Since $a(0)$ is diagonal with distinct eigenvalues, the eigenvectors are

$$\lambda_0 := [1, 0, \dots, 0],\ \lambda_1 := [0, 1, 0, \dots, 0],\ \dots,\ \lambda_m := [0, \dots, 0, 1],$$

and the $a(0)$-invariant spaces are $\Lambda_J := \operatorname{span}\{\lambda_j \mid j \in J\}$, $J \subset \{0, \dots, m\}$. It is easy to verify that, for each j, e_{λ_j} is the well-known truncated power function

$$e_{\lambda_j} : x \mapsto x_+^j := \big(\max\{0, x\}\big)^j$$

and $S(e_{\lambda_j}) = \mathcal{S}^j_{j-1}$. It follows that for any $J \subset \{0, \dots, m\}$,

$$S_{\Lambda_J} = \sum_{j \in J} S(e_{\lambda_j}) = \sum_{j \in J} \mathcal{S}^j_{j-1}.$$

Example 4.2. We consider the case of local dimension $m = 2$ with $a(0)$ invertible in order to illustrate the main results of this paper.

Let $\psi = [\psi_1, \psi_2]$ be a linearly independent generator supported in $[0, 1]$ which is refinable with mask $\big(a(0), a(1)\big)$. Then $S(\psi)$ must contain all constant functions and we can assume, without loss of generality, that $\psi_1 = \chi_{[0,1)}$ (cf. [H], [J1], [J2]). It is also assumed that $a(0)$ is invertible.

First, suppose $a(0)$ is diagonalizable, in which case we may assume (by a change of basis for ψ) that $a(0)$ and $a(1)$ are of the form

$$a(0) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}, \qquad a(1) = \begin{bmatrix} 1 & u \\ 0 & t \end{bmatrix},$$

where $s \neq 0$, since $a(0)$ is invertible. Then $T = a(0) + a(1) = \begin{bmatrix} 2 & u \\ 0 & s+t \end{bmatrix}$.

Since ψ is linearly independent, Corollary 2.9 implies $|s + t| < 2$. Then the left 2-eigenspace of T is spanned by $[2 - s - t, u]$. If $u = 0$, then the invariant space $\mathcal{W}$ is spanned by $[1, 0]$; and if $s = 1$, then $\mathcal{W}$ is spanned by $[1 - t, u]$. In either case, Corollary 2.9 implies that ψ is linearly dependent, hence that $S(\psi) = S(\psi_1)$ which has no proper local refinable FSI subspaces. So we assume $s \neq 0$, $s \neq 1$, and $u \neq 0$ (under these assumptions, $\mathcal{W} = \mathbb{R}^{1\times 2}$ and ψ is necessarily independent). By rescaling ψ_2, we may assume that $u = 1$.

There are three possible choices for an $a(0)$-invariant space Λ:

1. $\Lambda := \operatorname{span}\{\lambda := [1, 0]^T\}$. Then $e_\lambda = \chi_{[0,\infty)}$ and $S_\Lambda = S(\psi_1)$ is the space of piecewise constant polynomials with integer breakpoints.
2. $\Lambda := \mathbb{R}^2$. Then $S_\Lambda = S(\psi)$.
3. (The interesting case) $\Lambda := \operatorname{span}\{\lambda := [0, 1]^T\}$. Begin with $h(0) = [0, 0, 0, 1]^T$, and calculate $h(1) = A_1 h(0)$, $h(2) = A_0 h(1)$, and $h(3) = A_1 h(1)$. The span of $h(0), \dots, h(3)$ is $\{A_0, A_1\}$-invariant and so equals $\mathcal{H}_\Lambda$. By a simple reduction, we find that $\mathcal{H}_\Lambda$ is also spanned by the four vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ s+t-1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ s \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence $\mathcal{H}^0_\Lambda$ is $\operatorname{span}\{[0, 1]^T, [s + t - 1, 0]^T\}$. By Theorem 2.5, Λ is a proper subset of $\Sigma(S_\Lambda)$ whenever $s + t \neq 1$. It follows that $S(\psi)$ and $S(\psi_1)$ are the only local refinable subspaces of $S(\psi)$ when $s + t \neq 1$; but, when $s + t = 1$, there is a third local refinable subspace, $S(e_\lambda)$. Lastly, since $a(1)\lambda = [1, t]^T$, we see that $\mathcal{L}_\Lambda = \mathbb{R}^2$ and so, by Theorem 2.7, $S_{\Lambda|} = \operatorname{span}\{\psi_1, \psi_2\}$ for any values of s and t.

When $a(0)$ is not diagonalizable, we may assume (by a change of basis) that

$$a(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only choices for Λ, in this case, are $\Lambda = \operatorname{span}\{[1, 0]^T\}$, and $\Lambda = \mathbb{R}^2$. So the only local refinable FSI spaces are $S(\psi_1)$ (which is the space of all piecewise constant polynomials with integer breakpoints) and $S(\psi)$.

It is assumed in this example, as throughout this paper, that ψ_1 and ψ_2 are in $L^1(\mathbb{R})$. A sufficient condition for this is that $|s|+|t|<1$. In fact, as shown in [JRZ], the subdivision scheme associated with a converges in L^1 in this case.

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