OBLOQUE MULTIWAVELETS IN HILBERT SPACES

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(Communicated by David R. Larson)

Abstract. In this paper, we elucidate the relationship between two consecutive levels of a multiresolution in the general setting of a Hilbert space. We first prove a result on an extendability problem and then derive, as a consequence, characterizations of oblique multiwavelets in a Hilbert space.

1. Introduction

This is a sequel to our earlier papers [4, 7, 5, 10, 8]. Here, we again consider wavelet-type problems in a general Hilbert space setting, rather than in the usual concrete case of $L^2(\mathbb{R}^d)$, the Hilbert space of square integrable complex-valued functions on $\mathbb{R}^d$.

It is well known that one of the major issues in the study of a multiresolution $\{V_n\}$ of $L^2(\mathbb{R}^d)$ (see e.g. [3, 6]) is the relationship between any two consecutive multiresolution spaces $V_n$ and $V_{n+1}$ (between only $V_0$ and $V_1$ in the stationary case). This paper elucidates this relationship in the setting of a Hilbert space.

In [1], wavelet-type objects called oblique multiwavelets in $L^2(\mathbb{R})$ are introduced. These are generalizations of biorthogonal multiwavelets and it is noted in [1] that they have more flexible properties. Characterizations of oblique multiwavelets in $L^2(\mathbb{R})$ are obtained in [1] and [2], using the machinery of the Fourier transform on $L^2(\mathbb{R})$. This paper extends these characterizations to the general setting of oblique multiwavelets in a Hilbert space $H$, where the Fourier transform is no longer available and the roles of the translation operator and the dilation operator on $L^2(\mathbb{R})$ are replaced by certain unitary operators on $H$. Indeed, we first prove a result on a related extendability problem and then obtain, as a consequence, the above results on oblique multiwavelets in a Hilbert space.

Throughout this paper, $H$ denotes a complex Hilbert space. A sequence $\{v_n\}$ in $H$ is a Riesz basis for its closed linear span $\text{span}\{v_n\}$ if there exist positive constants $A$ and $B$ such that

$$A \sum |a_n|^2 \leq \| \sum a_n v_n \|^2 \leq B \sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2. \quad (1.1)$$

Two sequences $\{v_n\}$ and $\{\tilde{v}_n\}$ in $H$ are biorthogonal if

$$\langle v_n, \tilde{v}_m \rangle = \delta_{n,m} \quad \forall n, m, \quad (1.2)$$

where $\langle x, y \rangle$ denotes the inner product of two vectors $x$ and $y$ in $H$. (See [11].)

Received by the editors August 24, 1998.
2000 Mathematics Subject Classification. Primary 46C99, 47B99, 46B15.
Key words and phrases. Riesz basis, biorthogonal system, oblique projection, multiwavelets.

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Let $U = (U_1, ..., U_d)$ be an ordered $d$-tuple of distinct unitary operators on a Hilbert space $H$ such that $U_kU_j = U_jU_k$, $k, j = 1, \ldots, d$. We shall use the multi-index notation $U^m = U_1^{m_1} \cdots U_d^{m_d}$ for $m = (m_1, ..., m_d) \in \mathbb{Z}^d$, with the convention that $U_j^0$ is the identity operator on $H$, $j = 1, ..., d$. We also assume that $U^m$ is the identity operator only if $m = 0$.

For a subset $S$ of $H$, let $\langle S \rangle$ denote the closed linear span of $S$, and

$$U^d(S) := \{U^n s : n \in \mathbb{Z}^d, s \in S\}.$$

If $V = \{v_1, ..., v_r\}$ and $W = \{w_1, ..., w_p\}$ are finite subsets of $H$ such that

$$\{(v_k, U^n w_j)\}_{n \in \mathbb{Z}^d} \in L^2(\mathbb{Z}^d), \quad k = 1, \ldots, r, \quad j = 1, \ldots, p,$$

then the function $\Phi_{V,W}$ defined almost everywhere on $\mathbb{R}^d$ by

$$\Phi_{V,W}(u) := \left( \sum_{n \in \mathbb{Z}^d} (v_k, U^n w_j)e^{in\cdot u} \right)_{1 \leq k \leq r, 1 \leq j \leq p}$$

is in $L^2(\mathbb{R}^{\times r \times p})$, the space of all $r \times p$ matrices with entries in $L^2(\mathbb{R}^{d})$ (often identified with $L^2(\mathbb{T}^{d}, \mathbb{C}^{r \times p})$, where $\mathbb{C}^{r \times p}$ is the space of all $r \times p$ complex matrices equipped with the operator norm, or depending on the context, with the Frobenius norm). For simplicity, we write $L^2(\mathbb{T}^{d})^p$ for $L^2(\mathbb{T}^{d})^{1 \times p}$.

If $V$ and $W$ are closed linear subspaces of $H$ such that $V \cap W = \{0\}$ and the vector sum $V_1 = V + W$ is closed, then we write $V_1 = V \oplus W$ and call this a direct sum. In this case, we define the maps $P_{V/V'}$ and $P_{W/W'}$ from $V_1$ to $V_1$ by

$$P_{V/V'}(v + w) = v, \quad P_{W/W'}(v + w) = w, \quad v \in V, w \in W,$$

and call $P_{V/V'}$ the (oblique) projection of $V_1$ on $V$ along $W$ and $P_{W/W'}$ the (oblique) projection of $V_1$ on $W$ along $V$. For the special case when $W = V_1 \cap V^\perp$, the orthogonal complement of $V$ in $V_1$, we write $V_1 = V \oplus V^\perp$ for the orthogonal direct sum.

We now give a summary of the contents of this paper. In section 2, we obtain necessary and sufficient conditions for the existence of solutions to an extendability problem in a Hilbert space that is related to multiresolution. Section 3 summarizes some useful results on the connection between $L^2(\mathbb{T}^{d})$ and dilation matrices. The final section elucidates the relationship between two consecutive levels of a multiresolution-type structure and derives characterizations of oblique multiframelets in a Hilbert space.

2. An extendability problem

Throughout this section, let $Y = \{y_1, ..., y_s\}$ and $\tilde{Y} = \{\tilde{y}_1, ..., \tilde{y}_r\}$ be finite subsets of $H$ such that $U^d(Y)$ and $U^d(\tilde{Y})$ are biorthogonal Riesz bases for $V_1 = \langle U^d(Y) \rangle = \langle U^d(\tilde{Y}) \rangle$. Moreover, let $X = \{x_1, ..., x_r\}$ be a finite subset of $H$, where $r < s$, let $U^d(X)$ be a Riesz basis for its closed linear span $V_0$ and let

$$V_0 \subset V_1.$$

In this case, we have the biorthogonal expansions

$$f = \sum_{j=1}^{s} \sum_{n \in \mathbb{Z}^d} \langle f, U^n y_j \rangle U^n y_j = \sum_{j=1}^{s} \sum_{n \in \mathbb{Z}^d} \langle f, U^n \tilde{y}_j \rangle U^n \tilde{y}_j, \quad f \in V_1,$$
Theorem 2.1. Let \( \Gamma = \{ z_1, ..., z_{s-r} \} \) be a subset of \( V_1 \setminus V_0 \) and let \( S = X \cup \Gamma \). The following conditions are equivalent:

\begin{itemize}
  \item[(i)] \( U^{\mathbb{Z}^d}(S) \) is a Riesz basis for \( V_1 \).
  \item[(ii)] \( U^{\mathbb{Z}^d}(\Gamma) \) is a Riesz basis for \( W_0 := \langle U^{\mathbb{Z}^d}(\Gamma) \rangle \) and \( V_0 \oplus W_0 = V_1 \).
  \item[(iii)] There exist positive constants \( C_1 \) and \( C_2 \) such that
    \[ C_1 \leq \Phi_{S,\tilde{Y}}(u) \Phi_{S,\tilde{Y}}(u)^* \leq C_2 \quad \text{a.e.} \]
  \item[(iv)] The matrices \( \Phi_{S,\tilde{Y}}(u) \) are invertible for almost all \( u \), and the functions \( u \rightarrow \| \Phi_{S,\tilde{Y}}(u) \| \) and \( u \rightarrow \| \Phi_{S,\tilde{Y}}(u)^{-1} \| \) are essentially bounded.
  \item[(v)] The operator \( R_0 : L^2(\mathbb{T}^d)^r \oplus L^2(\mathbb{T}^d)^{s-r} \rightarrow L^2(\mathbb{T}^d)^s \) defined by
    \begin{equation}
    R_0(A, B)(u) = (A(u) \quad B(u)) \begin{pmatrix} \Phi_{X,\tilde{Y}}(u) \\ \Phi_{\Gamma,\tilde{Y}}(u) \end{pmatrix}
    = A(u) \Phi_{X,\tilde{Y}}(u) + B(u) \Phi_{\Gamma,\tilde{Y}}(u)
    
    \end{equation}
    is bounded and invertible.
\end{itemize}

Proof. (i)\( \Rightarrow \)(iii) follows from \cite{10} Proposition 3.4.

(iii)\( \Rightarrow \)(ii): Again by \cite{10} Proposition 3.4, \( U^{\mathbb{Z}^d}(S) \) is a Riesz basis for \( \langle U^{\mathbb{Z}^d}(S) \rangle \), which is a subset of \( V_1 \). In particular, the set \( S \) is linearly independent. \( U^{\mathbb{Z}^d}(\Gamma) \) is a Riesz basis for \( W_0 := \langle U^{\mathbb{Z}^d}(\Gamma) \rangle \) and

\[ V_0 \oplus W_0 = \langle U^{\mathbb{Z}^d}(S) \rangle \subset V_1 = \langle U^{\mathbb{Z}^d}(Y) \rangle. \]

Since \( \#(S) = s = \#(Y) \), by \cite{3} Theorem 2.4, \( \langle U^{\mathbb{Z}^d}(S) \rangle = V_1 \).

By \cite{10} Theorem 2.1, we have (ii)\( \Rightarrow \)(i). Finally, (iii)\( \Rightarrow \)(iv) follows from standard arguments in operator theory, and (iv)\( \Leftrightarrow \)(v) follows from \cite{9} pp. 351-352.

It is shown in \cite{3} Theorem 2.5] (\cite{7} Corollary 3.4] for the case \( d = 1 \)] that there exist \( z_1, ..., z_{s-r} \) in \( V_1 \cap V_0^+ \) such that condition (ii) in Theorem 2.1 holds. In the more general setting of a pair of biorthogonal multiresolutions, \cite{10} Theorem 3.6] shows that again there exist \( z_1, ..., z_{s-r} \) satisfying Theorem 2.1(ii) and some other conditions.

Proposition 2.2. Suppose that the conditions in Theorem 2.1 hold. Let \( \tilde{S} \) be a subset of \( V_1 \) such that \( U^{\mathbb{Z}^d}(\tilde{S}) \) is a Riesz basis for \( V_1 \) biorthogonal to \( U^{\mathbb{Z}^d}(S) \). Write \( \tilde{S} = X' \cup \Gamma' \) such that \( \#(X') = r, \#(\Gamma') = s - r \), \( U^{\mathbb{Z}^d}(X') \) biorthogonal to \( U^{\mathbb{Z}^d}(X) \), \( U^{\mathbb{Z}^d}(\Gamma') \) biorthogonal to \( U^{\mathbb{Z}^d}(\Gamma), U^{\mathbb{Z}^d}(X') \perp U^{\mathbb{Z}^d}(\Gamma) \) and \( U^{\mathbb{Z}^d}(\Gamma') \perp U^{\mathbb{Z}^d}(X) \). Then \( R_0^{-1} : L^2(\mathbb{T}^d)^s \rightarrow L^2(\mathbb{T}^d)^r \oplus L^2(\mathbb{T}^d)^{s-r} \) is given by

\begin{equation}
R_0^{-1}(C)(u) = (C(u) \Phi_{X',\tilde{Y}}(u)^*, \ C(u) \Phi_{\Gamma',\tilde{Y}}(u)^*). \end{equation}

Proof. By \cite{5} Theorem 2.4, \( \Phi_{S,\tilde{Y}}(u)^{-1} = \Phi_{S,\tilde{Y}}(u)^* = (\Phi_{X,\tilde{Y}}(u)^* \quad \Phi_{\Gamma,\tilde{Y}}(u)^*) \). By Theorem 2.1 \( R_0^{-1}(C)(u) = C(u) \Phi_{S,\tilde{Y}}(u)^{-1} = (C(u) \Phi_{X',\tilde{Y}}(u)^* \quad C(u) \Phi_{\Gamma',\tilde{Y}}(u)^*). \)
Remark 2.3. (i) Under the assumptions of Proposition 2.2, let \( V'_0 = (U^Z(X')) \) and \( W'_0 = (U^Z(Y')) \). Then we arrive at the biorthogonal setting
\[
V_0 \oplus W_0 = V_1, \quad V'_0 \oplus W'_0 = V_1, \quad V_0 \perp W'_0, \quad V'_0 \perp W_0.
\]
Note that \( V'_0 \) may not necessarily equal \( V_0 \) and \( W'_0 \) may not necessarily equal \( W_0 \).
(ii) If moreover \( U^Z(X) \perp U^Z(Y) \), then \( V'_0 = V_0 \) and \( W'_0 = W_0 \), and we arrive at the semiorthogonal setting \( V_0 \oplus W_0 = V_1 \).

Theorem 2.4. Suppose that the conditions in Theorem 2.2 hold. Let
\[
A_j(u) = \sum_{n \in \mathbb{Z}^d} a_j(n) e^{iu_n}, \quad a_j \in \ell^2(\mathbb{Z}^d), \quad j = 1, \ldots, r,
\]
\[
B_k(u) = \sum_{n \in \mathbb{Z}^d} b_k(n) e^{iu_n}, \quad b_k \in \ell^2(\mathbb{Z}^d), \quad k = 1, \ldots, s - r,
\]
\[
C_\ell(u) = \sum_{n \in \mathbb{Z}^d} c_\ell(n) e^{iu_n}, \quad c_\ell \in \ell^2(\mathbb{Z}^d), \quad \ell = 1, \ldots, s,
\]
and
\[
A = (A_1 \ldots A_r), \quad B = (B_1 \ldots B_{s-r}), \quad C = (C_1 \ldots C_s).
\]
The following conditions are equivalent:
(i) \( \sum_{\ell=1}^s \sum_{n \in \mathbb{Z}^d} c_\ell(n) U^n y_\ell = \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} a_j(n) U^n x_j + \sum_{k=1}^{s-r} \sum_{n \in \mathbb{Z}^d} b_k(n) U^n z_k \).
(ii) (Reconstruction algorithm)
\[
C(u) = R_0(A, B)(u) = A(u) \Phi_{X, Y}(u) + B(u) \Phi_{Y', Y}(u).
\]
(iii) (Decomposition algorithm)
\[
(A(u), B(u)) = R_0^{-1}(C)(u), \quad i.e.,
A(u) = C(u) \Phi_{X, Y}(u)^*,
B(u) = C(u) \Phi_{Y', Y}(u)^*.
\]

Proof. (i)\(\Rightarrow\)(ii): Suppose that (i) holds. Write (i) as \( y = x + z \) such that \( y \in V_1 \), \( x \in V_0 \) and \( z \in W_0 \). Then by [10] Proposition 3.3,
\[
C(u) = \Phi_{\{\gamma\}, Y}(u) = \Phi_{\{\gamma\}, S}(u) \Phi_{Y, S}(u)^*
= (\Phi_{\{\gamma\}, X}(u) \Phi_{\{\gamma\}, Y}(u)) \Phi_{Y', Y}(u)
= (\Phi_{\{\gamma\}, X'}(u) \Phi_{\{\gamma\}, Y'}(u)) \Phi_{Y', Y}(u)
= (A(u) B(u)) \Phi_{S, Y'}(u),
\]
since \( z \perp U^Z(X') \) and \( x \perp U^Z(Y') \). Hence (ii) holds.

The proof of (ii)\(\iff\)(iii) is obvious. Suppose that (ii) holds. Let
\[
x = \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} a_j(n) U^n x_j \quad \text{and} \quad z = \sum_{k=1}^{s-r} \sum_{n \in \mathbb{Z}^d} b_k(n) U^n z_k.
\]
Let \( y = x + z \) be expressed as \( \sum_{\ell=1}^s \sum_{n \in \mathbb{Z}^d} c_\ell(n) U^n y_\ell \), where \( c_\ell \in \ell^2(\mathbb{Z}^d), \quad \ell = 1, \ldots, s \). If \( C_\ell(u) = \sum_{n \in \mathbb{Z}^d} c_\ell(n) e^{iu_n}, \quad \ell = 1, \ldots, s \), then by the implication (i)\(\Rightarrow\)(ii).
established above and (ii), $(C^t_1(u) \ldots C^t_s(u)) = (A(u) B(u)) \Phi_{S,Y}(u) = C(u)$. Hence $c^t_\ell = c_\ell$, $\ell = 1, \ldots, s$, and so (i) holds.

**Corollary 2.5.** Suppose that the conditions in Theorem 2.4 hold. If

$$y = \sum_{\ell=1}^s \sum_{n \in \mathbb{Z}^d} c_\ell(n) U^n y_\ell$$

and

$$C(u) = \left( \sum_{n \in \mathbb{Z}^d} c_1(n) e^{in \cdot u} \ldots \sum_{n \in \mathbb{Z}^d} c_s(n) e^{in \cdot u} \right),$$

then

$$P_{V_0/W_0}(y) = \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} a_j(n) U^n x_j \quad \text{and} \quad P_{W_0/V_0}(y) = \sum_{k=1}^{s-r} \sum_{n \in \mathbb{Z}^d} b_k(n) U^n z_k,$$

where

$$\left( \sum_{n \in \mathbb{Z}^d} a_1(n) e^{in \cdot u} \ldots \sum_{n \in \mathbb{Z}^d} a_r(n) e^{in \cdot u} \right) = C(u) \Phi_{X',Y}(u)^*$$

and

$$\left( \sum_{n \in \mathbb{Z}^d} b_1(n) e^{in \cdot u} \ldots \sum_{n \in \mathbb{Z}^d} b_{s-r}(n) e^{in \cdot u} \right) = C(u) \Phi_{Y',Y}(u)^*.$$

### 3. Some Connections Between $L^2(\mathbb{T}^d)$ and Dilation Matrices

For the rest of this paper, let $M$ be a $d \times d$ matrix with integer entries such that $m = |\text{det}(M)| \geq 2$. Such a matrix is often called a *dilation* matrix. Let $M^T$ be the transpose of $M$. Let $\mathcal{C}_M = \{\gamma_0, \gamma_1, \ldots, \gamma_m\}$ (respectively $\mathcal{C}_{M^T} = \{\gamma_0, \gamma_1, \ldots, \gamma_{m-1}\}$) be a full set of coset representatives of $\mathbb{Z}^d / M \mathbb{Z}^d$ (respectively $\mathbb{Z}^d / M^T \mathbb{Z}^d$), where $\gamma_0 = \eta_0 = 0$. Then $\mathbb{Z}^d$ is the disjoint union of the cosets $M \mathbb{Z}^d + \gamma_j, j = 0, 1, \ldots, m - 1$ (respectively $M^T \mathbb{Z}^d + \eta_j, j = 0, 1, \ldots, m - 1$).

It is well known that the following orthogonality relations hold:

\begin{equation}
\sum_{k=0}^{m-1} e^{i2\pi(\eta_j - \eta_\ell) \cdot M^{-1} \gamma_k} = m \delta_{j, \ell}, \quad j, \ell = 0, 1, \ldots, m - 1,
\end{equation}

and

\begin{equation}
\sum_{j=0}^{m-1} e^{i2\pi \eta_j \cdot M^{-1} (\gamma_k - \gamma_\ell)} = m \delta_{k, \ell}, \quad k, \ell = 0, 1, \ldots, m - 1.
\end{equation}

Let

\begin{equation}
Q(u) = \frac{1}{\sqrt{m}} \left( e^{i\gamma_k \cdot u} e^{i2\pi \eta_\ell \cdot M^{-1} \gamma_k} I_r \right)_{k,\ell=0}^{m-1}, \quad u \in \mathbb{R}^d,
\end{equation}

where $I_r$ is the $r \times r$ identity matrix. By (3.1) and (3.2), $Q(u)$ is a unitary $mr \times mr$ matrix (indeed the product of two unitary matrices).
Let \( q \) be a fixed positive integer and let the map
\[
J : L^2(\mathbb{T}^d)^{q \times r} \oplus \cdots \oplus L^2(\mathbb{T}^d)^{q \times r} \rightarrow L^2(\mathbb{T}^d)^{q \times r}
\]
be defined by
\[
(3.4) \quad J(G_0, \ldots, G_{m-1})(u) = (G_0(M^T u) \ldots G_{m-1}(M^T u)) \Omega(u)
\]
\[
= G_0(M^T u) + G_1(M^T u)e^{2\pi \imath q_1 u} + \ldots + G_{m-1}(M^T u)e^{2\pi \imath q_{m-1} u}
\]
\[
= \sum_{j=0}^{m-1} \sum_{n \in \mathbb{Z}^d} g_j(n)e^{i(Mn + \gamma_j)u},
\]
where \( G_j(u) = \sum_{n \in \mathbb{Z}^d} g_j(n)e^{inu} \), \( g_j \in L^2(\mathbb{Z}^d)^{q \times r} \), \( j = 0, 1, \ldots, m-1 \), and
\[
(3.5) \quad \Omega(u) = \begin{pmatrix}
I_r \\
e^{2\pi \imath q_1 u}I_r \\
\vdots \\
e^{2\pi \imath q_{m-1} u}I_r
\end{pmatrix}, \quad u \in \mathbb{R}^d.
\]

**Theorem 3.1.** The map \( J \) is a unitary operator, and its inverse
\[
J^* : L^2(\mathbb{T}^d)^{q \times r} \rightarrow L^2(\mathbb{T}^d)^{q \times r} \oplus \cdots \oplus L^2(\mathbb{T}^d)^{q \times r}
\]
is given by \( J^*(G)(u) = (F_0(u), \ldots, F_{m-1}(u)) \), where
\[
(3.6) \quad G(u) = \sum_{n \in \mathbb{Z}^d} g(n)e^{inu}, \quad F_j(u) = \sum_{n \in \mathbb{Z}^d} g(Mn + \gamma_j)e^{inu}, \quad j = 0, 1, \ldots, m-1.
\]

**Proof.** Since
\[
\|J(G_0, \ldots, G_{m-1})\|^2 = \sum_{j=0}^{m-1} \sum_{n \in \mathbb{Z}^d} \|g_j(n)\|^2 = \sum_{j=0}^{m-1} \|G_j\|^2,
\]
the map \( J \) is isometric. For any \( G \) in \( L^2(\mathbb{T}^d)^{q \times r} \),
\[
G(u) = \sum_{n \in \mathbb{Z}^d} g(n)e^{inu} = \sum_{j=0}^{m-1} \sum_{n \in \mathbb{Z}^d} g(Mn + \gamma_j)e^{i(Mn + \gamma_j)u}
\]
\[
= \sum_{j=0}^{m-1} F_j(M^T u)e^{i\gamma_j u},
\]
where \( F_j, j = 0, 1, \ldots, m-1 \), are given by (3.3). By (3.4), \( G = J(F_0, \ldots, F_{m-1}) \). Therefore \( J \) is surjective, and the inverse of \( J \) has the desired form. \( \square \)

For any \( G \) in \( L^2(\mathbb{T}^d)^{q \times r} \), define a function \( G_{mod} \) in \( L^2(\mathbb{T}^d)^{q \times mr} \) by
\[
(3.7) \quad G_{mod}(u) = \frac{1}{\sqrt{m}} (G(u) \ G(u + 2\pi(M^T)^{-1}\eta_1) \ldots G(u + 2\pi(M^T)^{-1}\eta_{m-1})).
\]

**Remark 3.2.** \( Q(u) = \Omega_{mod}(u) \), where \( \Omega(u) \) is given by (3.5).

We summarize below some properties of \( G_{mod} \) which will be needed in the next section. We omit their proofs.
Proposition 3.3. For any \( G \) in \( L^2(\mathbb{T}^d)^{q \times r} \),

(i) \( G_{mod}(u) = (J^* G)(M^T u) Q(u) \),

(ii) \( J^*(G)(u) = G_{mod}((M^T)^{-1} u) Q((M^T)^{-1} u)^* \),

(iii) \( \frac{1}{m} \sum_{j=0}^{m-1} G(u + 2\pi (M^T)^{-1} \eta_j) = \sum_{n \in \mathbb{Z}^d} g(Mn) e^{iMn\cdot u} \), where \( G(u) = \sum_{n \in \mathbb{Z}^d} g(n) e^{in\cdot u} \).

Let \( (\mathbb{M}_d) \) where \( (4.1) \)

Let \( (4.2) \)

and let \( Y \) \( (4.4) \)

Then \( \tilde{G} \) is in \( L^2(\mathbb{T}^d)^{mr \times r} \) and \( G_{mod}(u)^{-1} = \tilde{G}_{mod}(u)^* \).

4. CHARACTERIZATIONS OF OBLIQUE MULTIWAVELETS

We shall follow the same notations as in sections 1 and 3. As before, let \( U = (U_1, \ldots, U_d) \) be an ordered \( d \)-tuple of distinct commuting unitary operators on a Hilbert space \( H \). Let \( D \) be another unitary operator on \( H \) such that

\( U^n D = DU^M n, \quad n \in \mathbb{Z}^d \),

where \( M \) is a \( d \times d \) matrix as in section 3. Then for every \( n \) in \( \mathbb{Z}^d \), there exist a unique \( \ell \) in \( \{0, 1, \ldots, m-1\} \) and a unique \( p \) in \( \mathbb{Z}^d \) such that \( n = Mp + \gamma \ell \). Hence

\( DU^n = U^p DU^{\gamma \ell} \).

Let \( X = \{x_1, \ldots, x_r\} \) be a finite subset of \( H \) such that

\( \{ (x_j, U^n x_k) \}_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \quad j, k = 1, \ldots, r \).

Let \( L = \{ (\ell, j) : \ell = 0, 1, \ldots, m-1, \quad j = 1, \ldots, r \} \) with lexicographical ordering in \((\ell, j), \)

\( \gamma_{\ell,j} = DU^{\gamma \ell} x_j, \quad (\ell, j) \in L, \)

and let \( Y = \{ y_{\ell,j} : (\ell, j) \in L \} \). Let

\( V_0 = \langle U^{\mathbb{Z}^d}(X) \rangle \) and \( V_1 = \langle U^{\mathbb{Z}^d}(Y) \rangle \).

For the time being, we do not assume that \( V_0 \subset V_1 \).

By \( \Phi_{X, \mathbb{X}}(u) \)

\( \Phi_{X, \mathbb{X}}(u) = \left( \sum_{n \in \mathbb{Z}^d} (x_j, U^n x_k) e^{in\cdot u} \right)_{1 \leq j \leq r, 1 \leq k \leq r} \).
Proof. (i) and (iii) follow easily from (4.2) and the assumption that
Using (3.3), the (t;p)

Theorem 4.1. We have the following relations between

(4.8)

\[ \Phi_{X,Y}(u) = \left( \begin{array}{ccc} \Phi_{X,Y}(u) & 0_r & \cdots & 0_r \\ 0_r & \Phi_{X,Y}(u+2\pi(M^T)^{-1}\eta_1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0_r \\ 0_r & \cdots & 0_r & \Phi_{X,Y}(u+2\pi(M^T)^{-1}\eta_{m-1}) \end{array} \right). \]

(i) \( V_1 = D(V_0) \).
(ii) \( \Phi_{Y,Y}(M^Tu) = Q(u)\Phi_{X,X}(u)Q(u)^* \).
(iii) \( U^{Z^d}(X) \) is a Riesz basis for \( V_0 \) if and only if \( U^{Z^d}(Y) \) is a Riesz basis for \( V_1 \).

Proof. (i) and (iii) follow easily from (4.2) and the assumption that \( D \) is unitary.

By (4.1) and (4.4), for \( t, p = 0, 1, \ldots, m-1 \), \( j, k = 1, \ldots, r \),

\[ \langle y_{t,j}, U^n y_{p,k} \rangle = \langle DU^{\gamma_\eta}x_j, DU^{M_n+\gamma_\eta}x_k \rangle = \langle x_j, U^{M_n+\gamma_\eta}x_k \rangle. \]

Hence the \((t,p)\)-block of \( \Phi_{Y,Y}(M^Tu) \) is

\[ \left( \sum_{n\in\mathbb{Z}^d} \langle y_{t,j}, U^n y_{p,k} \rangle e^{inM^Tu} \right)^r_{j,k=1} = \sum_{n\in\mathbb{Z}^d} \left( \langle x_j, U^{M_n+\gamma_\eta}x_k \rangle \right)^r_{j,k=1} e^{iM_nu}. \]

Using (4.3), the \((t,p)\)-block of \( Q(u)\Phi_{X,X}(u)Q(u)^* \) is

\[ (*) = \frac{1}{m} \sum_{\ell=0}^{m-1} e^{i(\gamma_\eta u + 2\pi \eta_\ell M^{-1}\gamma_\eta)} \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_\ell) e^{-i(\eta_\ell u + 2\pi \eta_\ell M^{-1}\gamma_\eta)} \]

\[ = \frac{1}{m} \sum_{\ell=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_\ell), \]

where

\[ G(u) = \Phi_{X,X}(u)e^{i(\gamma_\eta - \gamma_\eta)u} = \left( \sum_{n\in\mathbb{Z}^d} \langle x_j, U^n x_k \rangle e^{i(n+\gamma_\eta - \gamma_\eta)u} \right)^r_{j,k=1} = \sum_{n\in\mathbb{Z}^d} \left( \langle x_j, U^{n+\gamma_\eta - \gamma_\eta}x_k \rangle \right)^r_{j,k=1} e^{inu}. \]

Hence by Proposition 1.3

\[ (*) = \sum_{n\in\mathbb{Z}^d} \left( \langle x_j, U^{M_n+\gamma_\eta - \gamma_\eta}x_k \rangle \right)^r_{j,k=1} e^{iM_nu}, \]

which is the \((t,p)\)-block of \( \Phi_{Y,Y}(M^Tu) \). Hence (ii) holds. \( \square \)
Henceforth, assume that $U^{\mathbb{Z}^d}(X)$ is a Riesz basis for $V_0$ so that $U^{\mathbb{Z}^d}(Y)$ is a Riesz basis for $V_1 = D(V_0)$. Let $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_r\}$ be a subset of $V_0$ such that $U^{\mathbb{Z}^d}(\tilde{X})$ is a Riesz basis for $V_0$ biorthogonal to $U^{\mathbb{Z}^d}(X)$. Since $D$ is unitary, $\{DU^n\tilde{x}_j : n \in \mathbb{Z}^d, j = 1, \ldots, r\}$ and $\{DU^n\tilde{x}_j : n \in \mathbb{Z}^d, j = 1, \ldots, r\}$ are biorthogonal Riesz bases for $V_1$. Let

$$
\tilde{g}_{\ell,j} = DU^{\ell}n\tilde{x}_j, \quad (\ell, j) \in L,
$$

and let $\tilde{Y} = \{\tilde{g}_{\ell,j} : (\ell, j) \in L\}$. It is not difficult to show that $U^{\mathbb{Z}^d}(Y)$ and $U^{\mathbb{Z}^d}(\tilde{Y})$ are biorthogonal Riesz bases for $V_1$.

Let $W = \{w_1, \ldots, w_q\}$ be a subset of $V_1$, for some positive integer $q$. Then

$$
w_k = \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} g_{k,j}(n) DU^n x_j, \quad k = 1, \ldots, q,
$$

where $g_{k,j} \in \ell^2(\mathbb{Z}^d)$, $k = 1, \ldots, q$, $j = 1, \ldots, r$. Let

$$
g(n) = (g_{k,j}(n))_{1 \leq k \leq q, 1 \leq j \leq r}, \quad n \in \mathbb{Z}^d.
$$

Then $g$ is in $\ell^2(\mathbb{Z}^d)^{q \times r}$ and (4.10) can be written in matrix form as

$$
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_q
\end{pmatrix} = \sum_{n \in \mathbb{Z}^d} g(n) \begin{pmatrix}
  DU^n x_1 \\
  \vdots \\
  DU^n x_r
\end{pmatrix}.
$$

Define a function $G$ in $L^2(\mathbb{T}^d)^{q \times r}$ by

$$
G(u) = \sum_{n \in \mathbb{Z}^d} g(n) e^{in \cdot u}, \quad u \in \mathbb{R}^d.
$$

Theorem 4.2. The function $G$ associated with $W$ has the following properties.

(i) $\Phi_{W,\tilde{Y}}(u) = J^*(G)(u)$.

(ii) $G_{mod}(u) = \Phi_{W,\tilde{Y}}(M^T u) Q(u)$.

(iii) $U^{\mathbb{Z}^d}(W)$ is a Riesz basis for $\langle U^{\mathbb{Z}^d}(W) \rangle$ if and only if there exist positive constants $C_1$ and $C_2$ such that

$$
C_1 \leq G_{mod}(u) G_{mod}(u)^* \leq C_2 \quad a.e.
$$

Proof. (i): By (4.10), (4.2), (4.1) and (4.4), for $k = 1, \ldots, q$,

$$
w_k = \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} g_{k,j}(Mn + \gamma \ell) DU^{Mn+\gamma \ell} x_j = \sum_{\ell=0}^{m-1} \sum_{j=1}^{r} g_{k,j}(Mn + \gamma \ell) U^n y_{\ell,j}.
$$

Therefore for $\ell = 0, 1, \ldots, m-1$ and $j = 1, \ldots, r$,

$$
\sum_{n \in \mathbb{Z}^d} \langle w_k, U^n \tilde{g}_{\ell,j} \rangle e^{in \cdot u} = \sum_{n \in \mathbb{Z}^d} g_{k,j}(Mn + \gamma \ell) e^{in \cdot u}.
$$
Hence by Theorem 3.1,

\[ \Phi_{W;Y}(u) = \left( \sum_{n \in \mathbb{Z}^d} g(Mn)e^{in \cdot u} \sum_{n \in \mathbb{Z}^d} g(Mn + \gamma_1)e^{in \cdot u} \cdots \sum_{n \in \mathbb{Z}^d} g(Mn + \gamma_{m-1})e^{in \cdot u} \right) = J^*(G)(u). \]

(ii) follows from (i) and Proposition 3.3.

By [10, Proposition 3.4], \( U^d(W) \) is a Riesz basis for \( \langle U^d(W) \rangle \) if and only if there exist positive constants \( C_1 \) and \( C_2 \) such that

\[ C_1 \leq \Phi_{W;Y}(u) \Phi_{W;Y}(u)^* \leq C_2 \text{ a.e.} \]

Then (iii) follows from the relation

\[ G_{\text{mod}}(u) G_{\text{mod}}(u)^* = \Phi_{W;Y}(u) \Phi_{W;Y}(u)^*, \]

which is obvious from (ii).

\[ \square \]

We shall now see the effect of the change of bases for \( V_1 \).

**Corollary 4.3.** Let a vector \( y \) in \( V_1 \) be expressed as

\[ y = \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} a_j(n)DU^n x_j = \sum_{\ell=1}^{m-1} \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} c_{\ell,j}(n)U^n y_{\ell,j}, \]

where \( a_j \) and \( c_{\ell,j} \) are in \( \ell^2(\mathbb{Z}^d) \), \( j = 1, \ldots, r \), \( \ell = 0, 1, \ldots, m-1 \). Let

\[ A_j(u) = \sum_{n \in \mathbb{Z}^d} a_j(n)e^{in \cdot u}, \quad j = 1, \ldots, r, \]

\[ C_{\ell,j}(u) = \sum_{n \in \mathbb{Z}^d} c_{\ell,j}(n)e^{in \cdot u}, \quad \ell = 0, 1, \ldots, m-1, \quad j = 1, \ldots, r, \]

and

\[ A = (A_1 \cdots A_r), \quad C = ((C_{0,j})_{j=1,\ldots,r} \cdots (C_{m-1,j})_{j=1,\ldots,r}). \]

Then \( C = J^*(A) \).

**Proof.** By Theorem 4.2(i), \( C(u) = \Phi_{\{y\};Y}(u) = J^*(A)(u) \).

\[ \square \]

**Corollary 4.4.** Let \( W = \{w_1, \ldots, w_q\} \) and \( \Gamma = \{z_1, \ldots, z_p\} \) be subsets of \( V_1 \) such that \( \{\langle w_k, U^n z_j \rangle\}_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \quad k = 1, \ldots, q, \quad \ell = 1, \ldots, p, \)

and

\[
\begin{pmatrix}
    w_1 \\
    \vdots \\
    w_q
\end{pmatrix} = \sum_{n \in \mathbb{Z}^d} g(n) \begin{pmatrix}
    DU^n x_1 \\
    \vdots \\
    DU^n x_r
\end{pmatrix}, \\
\begin{pmatrix}
    z_1 \\
    \vdots \\
    z_p
\end{pmatrix} = \sum_{n \in \mathbb{Z}^d} f(n) \begin{pmatrix}
    DU^n x_1 \\
    \vdots \\
    DU^n x_r
\end{pmatrix},
\]

where \( g \) is in \( \ell^2(\mathbb{Z}^d)^{q \times r} \) and \( f \) is in \( \ell^2(\mathbb{Z}^d)^{p \times r} \). Define

\[ G(u) = \sum_{n \in \mathbb{Z}^d} g(n)e^{in \cdot u}, \quad F(u) = \sum_{n \in \mathbb{Z}^d} f(n)e^{in \cdot u}, \quad u \in \mathbb{R}^d. \]

Then

\[ \Phi_{W;1}(M^T u) = G_{\text{mod}}(u) \Phi_{X,X}^{[m]}(u) F_{\text{mod}}(u)^* \]

\[ = \frac{1}{m} \sum_{j=0}^{m-1} \left( G(u + 2\pi(M^T)^{-1}\eta_j) \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_j) F(u + 2\pi(M^T)^{-1}\eta_j)^* \right). \]
If moreover \( U^Z(X) \) is orthonormal, then

(i) \( \Phi_{W,Y}(M^T u) = G_{mod}(u) F_{mod}(u)^* \);

(ii) assuming \( \{\langle w_k, U^n w_j \rangle \}_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \ k,j = 1, \ldots, q \), then \( U^Z(W) \) is orthonormal if and only if \( G_{mod}(u) G_{mod}(u)^* = I_q \) a.e.

**Proof.** By [10, Proposition 3.3], Theorem 4.1 and Theorem 4.2,

\[
\Phi_{W,Y}(M^T u) = \Phi_{W,Y}(M^T u) \Phi_{Y,Y}(M^T u) \Phi_{Y,Y}(M^T u)^* = G_{mod}(u) \Phi_{X,X}(u) Q(u)^* \Phi_{Y,Y}(M^T u)^* = G_{mod}(u) \Phi_{X,X}(u) F_{mod}(u)^*.
\]

The rest of the assertions are straightforward. \( \square \)

We now apply our previous results to the special case when \( V_0 \subset V_1 \). The next theorem, which is a consequence of Theorem 4.1 and Theorem 4.2, gives characterizations of oblique multiwavelets in the general setting of a Hilbert space.

**Theorem 4.5.** Assume that

\[
V_0 \subset V_1.
\]

Let \( p = mr - r, \Gamma = \{z_1, \ldots, z_p\} \subset V_1 \setminus V_0 \),

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_r 
\end{pmatrix} = \sum_{n \in \mathbb{Z}^d} h(n) \begin{pmatrix}
  DU^n x_1 \\
  \vdots \\
  DU^n x_r 
\end{pmatrix},
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_p 
\end{pmatrix} = \sum_{n \in \mathbb{Z}^d} g(n) \begin{pmatrix}
  DU^n z_1 \\
  \vdots \\
  DU^n z_p 
\end{pmatrix},
\]

where \( h \) is in \( \ell^2(\mathbb{Z}^d)^{r \times r} \) and \( g \) is in \( \ell^2(\mathbb{Z}^d)^{p \times r} \). Define

\[
H(u) = \sum_{n \in \mathbb{Z}^d} h(n)e^{inu}, \ G(u) = \sum_{n \in \mathbb{Z}^d} g(n)e^{inu}, \ u \in \mathbb{R}^d.
\]

Let \( S = X \cup \Gamma \) and

\[
F(u) = \begin{pmatrix}
  H(u) \\
  G(u)
\end{pmatrix}, \ u \in \mathbb{R}^d.
\]

The following conditions are equivalent:

(i) \( U^Z(S) \) is a Riesz basis for \( V_1 \).

(ii) \( U^Z(\Gamma) \) is a Riesz basis for \( W_0 := \langle U^Z(\Gamma) \rangle \) and \( V_0 \oplus W_0 = V_1 \).

(iii) There exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq F_{mod}(u) F_{mod}(u)^* \leq C_2 \quad \text{a.e.}
\]

(iv) The matrices \( F_{mod}(u) \) are invertible for almost all \( u \), and the functions \( u \rightarrow \|F_{mod}(u)\| \) and \( u \rightarrow \|F_{mod}(u)^{-1}\| \) are essentially bounded.

(v) The operator \( R : L^2(\mathbb{T}^d)^r \oplus L^2(\mathbb{T}^d)^p \rightarrow L^2(\mathbb{T}^d)^r \) defined by

\[
R(A, B)(u) = (A(M^T u) B(M^T u)) \begin{pmatrix}
  H(u) \\
  G(u)
\end{pmatrix}
\]

\[
= A(M^T u)H(u) + B(M^T u)G(u)
\]

is bounded and invertible.
Proof. By Theorem 4.7.

\begin{equation}
\Phi_{S,Y}(u) = J^*(F)(u),
\end{equation}
\begin{equation}
F_{\text{mod}}(u) = \Phi_{S,Y}(M^T u) Q(u),
\end{equation}
and
\begin{equation}
F_{\text{mod}}(u) F_{\text{mod}}(u)^* = \Phi_{S,Y}(M^T u) \Phi_{S,Y}(M^T u)^*.
\end{equation}

Next, by composing the operator \( R_0 : L^2(T^d)^r \oplus L^2(T^d)^p \rightarrow L^2(T^d)^{mr} \) defined in (2.4) with the unitary operator \( J : L^2(T^d)^{mr} \rightarrow L^2(T^d)^r \) defined in (3.4) with \( q = 1 \), and using (4.18),

\[ J R_0(A, B)(u) = (A(M^T u) B(M^T u)) \begin{pmatrix} \Phi_{X,Y}(M^T u) \\ \Phi_{T,Y}(M^T u) \end{pmatrix} \Omega(u) = (A(M^T u) B(M^T u)) F(u) = R(A, B)(u). \]

Hence
\begin{equation}
R = J R_0.
\end{equation}

By (4.20), (4.19) and (4.21), the equivalence of (i) to (v) in this theorem then follows easily from that in Theorem 2.4. \( \square \)

Remark 4.6. Suppose that \( U^{Z^d}(X) \) is orthonormal. Then \( U^{Z^d}(S) \) is orthonormal if and only if \( F_{\text{mod}}(u) \) are unitary for almost all \( u \in R^d \).

The next two results are analogues of Proposition 2.2 and Theorem 2.4 respectively.

**Proposition 4.7.** Suppose that the conditions in Theorem 4.5 hold and \( p = mr - r \).

Let \( F \in L^2(T^d)^{mr \times r} \) be given as in Proposition 3.4 such that
\begin{equation}
\hat{F}_{\text{mod}}(u)^* = F_{\text{mod}}(u)^{-1}.
\end{equation}

(i) Then \( R^{-1} : L^2(T^d)^r \rightarrow L^2(T^d)^r \oplus L^2(T^d)^p = L^2(T^d)^{mr} \) is given by
\begin{equation}
R^{-1}(C)(M^T u) = C_{\text{mod}}(u) \hat{F}_{\text{mod}}(u)^* \begin{pmatrix} 1 \\ m \sum_{j=0}^{m-1} C(u + 2\pi(M^T)^{-1} \eta_j) \hat{F}(u + 2\pi(M^T)^{-1} \eta_j)^* \end{pmatrix}.
\end{equation}

(ii) Write
\[ \hat{F}(u) = \begin{pmatrix} \hat{H}(u) \\ \hat{G}(u) \end{pmatrix}, \]
where \( \hat{H}(u) = \sum_{n \in Z^d} \hat{h}(n)e^{inu}, \hat{G}(u) = \sum_{n \in Z^d} \hat{g}(n)e^{inu}, u \in R^d, \)

\( \hat{h} \) is in \( \ell^2(Z^d)^{r \times r} \) and \( \hat{g} \) is in \( \ell^2(Z^d)^{p \times r} \). Let \( X' = \{ x_1', \ldots, x_r' \} \) and \( \Gamma' = \{ z_1', \ldots, z_p' \} \) be subsets of \( V_1 \) defined by
\[ \begin{pmatrix} x_1' \\ \vdots \\ x_r' \end{pmatrix} = \sum_{n \in Z^d} \hat{h}(n) \begin{pmatrix} DU^n \hat{x}_1 \\ \vdots \\ DU^n \hat{x}_r \end{pmatrix}, \]
\[ \begin{pmatrix} z_1' \\ \vdots \\ z_p' \end{pmatrix} = \sum_{n \in Z^d} \hat{g}(n) \begin{pmatrix} DU^n \hat{x}_1 \\ \vdots \\ DU^n \hat{x}_r \end{pmatrix}, \]
and let \( \tilde{S} = X' \cup \Gamma' \). Then \( U^{Z^d}(\tilde{S}) \) is a Riesz basis for \( V_1 \) biorthogonal to \( U^{Z^d}(S) \), \( U^{Z^d}(X) \) biorthogonal to \( U^{Z^d}(X) \), \( U^{Z^d}(\Gamma) \) biorthogonal to \( U^{Z^d}(\Gamma) \), \( \bigcup U^{Z^d}(\Gamma) \bigcup U^{Z^d}(\Gamma) \bigcup U^{Z^d}(\Gamma) \).
Proof. (i): Since \( R = JR_0 \), by Proposition 4.3, (4.19) and (4.22),
\[
R^{-1}(C)(u) = R_0^{-1}(J^*C)(u) = C_{mod}((M^T)^{-1}u)Q((M^T)^{-1}u)^* \Phi_{\tilde{S}, \tilde{Y}}(u)^{-1} = C_{mod}((M^T)^{-1}u)F_{mod}((M^T)^{-1}u)^{-1} = C_{mod}((M^T)^{-1}u)\tilde{F}_{mod}((M^T)^{-1}u)^*.
\]

(ii): By Theorem 4.2 and Theorem 4.5 (with \( X \) and \( Y \) there replaced by \( \tilde{X} \) and \( \tilde{Y} \), respectively), \( U^{Z^d}(S) \) is a Riesz basis for \( V_1 \) and
\[
(4.24) \quad \tilde{F}_{mod}(u) = \Phi_{\tilde{S}, \tilde{Y}}(M^T u)Q(u).
\]
By [10] Proposition 3.3, (4.19), (4.24) and (4.22),
\[
\Phi_{\tilde{S}, \tilde{Y}}(u) = \Phi_{S, Y}(u)\Phi_{\tilde{S}, \tilde{Y}}(u)^* = F_{mod}((M^T)^{-1}u)\tilde{F}_{mod}((M^T)^{-1}u)^* = I_{nr} \text{ a.e.}
\]
Hence \( U^{Z^d}(\tilde{S}) \) is biorthogonal to \( U^{Z^d}(S) \). The rest of the assertions are obvious.

**Theorem 4.8.** Suppose that the conditions in Theorem 4.7 hold. Let \( p = nr - r \),
\[
A_j(u) = \sum_{n \in \mathbb{Z}^d} a_j(n)e^{in \cdot u}, \quad a_j \in \ell^2(\mathbb{Z}^d), \quad j = 1, ..., r,
\]
\[
B_k(u) = \sum_{n \in \mathbb{Z}^d} b_k(n)e^{in \cdot u}, \quad b_k \in \ell^2(\mathbb{Z}^d), \quad k = 1, ..., p,
\]
\[
P_j(u) = \sum_{n \in \mathbb{Z}^d} p_j(n)e^{in \cdot u}, \quad p_j \in \ell^2(\mathbb{Z}^d), \quad j = 1, ..., r,
\]
and
\[
A = (A_1 \ldots A_r), \quad B = (B_1 \ldots B_p), \quad P = (P_1 \ldots P_r).
\]
The following conditions are equivalent:
\[
(\text{i)} \quad \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} p_j(n)DU^n x_j = \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} a_j(n)U^n x_j + \sum_{k=1}^{p} \sum_{n \in \mathbb{Z}^d} b_k(n)U^n z_k.
\]
\[
(\text{ii)} \quad (\text{Reconstruction algorithm}) \quad P(u) = R(A, B)(u) = A(M^T u)H(u) + B(M^T u)G(u).
\]
\[
(\text{iii)} \quad (\text{Decomposition algorithm}) \quad (A(u), B(u)) = R^{-1}(P)(u)
\]
\[
= \frac{1}{m} \sum_{j=0}^{m-1} P((M^T)^{-1}(u + 2\pi \eta_j))\tilde{F}((M^T)^{-1}(u + 2\pi \eta_j))^*,
\]
where \( \tilde{F} \) is given in (1.22).

Proof. Write
\[
\sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} p_j(n)DU^n x_j = \sum_{\ell=1}^{m-1} \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} c_{\ell, j}(n)U^n y_{\ell, j},
\]
where $c_{\ell,j}$ are in $\ell^2(\mathbb{Z}^d)$, $j = 1,\ldots,r$, $\ell = 0,1,\ldots,m-1$. Let

$$C_{\ell,j}(u) = \sum_{n \in \mathbb{Z}^d} c_{\ell,j}(n)e^{inu}, \quad \ell = 0,1,\ldots,m-1, \quad j = 1,\ldots,r,$$

and $C = ((C_{0,j})_{1,\ldots,r} \ldots (C_{m-1,j})_{1,\ldots,r})$. By Corollary 4.3, $C = J^*(P)$. Then by Theorem 4.4,

$$(i) \iff C = R_0(A,B) \iff J^*(P) = R_0(A,B) \iff P = JR_0(A,B) = R(A,B) \iff (A,B) = R^{-1}(P).$$

Note that we can also obtain formulae for the projections $P_{V_0/W_0}$ and $P_{W_0/V_0}$, as in Corollary 4.4. We omit the details here.

**Remark 4.9.** (i) The results in this paper can be applied to the special case when $H = L^2(\mathbb{R}^d)$, $(U_kf)(x) = f(x - e_k)$, $k = 1,\ldots,d$, and $(Df)(x) = m^{1/2}f(Mx)$, for $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, where $e_k = (\delta_{k,j})_{j=1,\ldots,d}$, $k = 1,\ldots,d$, $M$ is a $d \times d$ dilation matrix considered in section 3 and $m = |\det(M)| \geq 2$. In this case, $(DU^n f)(x) = m^{1/2}f(Mx - n)$, $n \in \mathbb{Z}^d$, and if $V = \{f_1,\ldots,f_r\}$ and $W = \{g_1,\ldots,g_p\}$ are subsets of $L^2(\mathbb{R}^d)$, then

$$\Phi_{V,W}(u) = \left( \sum_{n \in \mathbb{Z}^d} \hat{f}_k(u + 2\pi n)\hat{g}_j(u + 2\pi n) \right)_{1 \leq k \leq r, 1 \leq j \leq p},$$

where $\hat{f}$ is the Fourier transform of a function $f$ in $L^2(\mathbb{R}^d)$.

(ii) If $d = 1$, and $U$ and $D$ are unitary operators on a Hilbert space $H$ such that $UD = DU^2$, using the notations in Theorem 4.5, then

$$F_{mod}(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} H(u) & H(u + \pi) \\ G(u) & G(u + \pi) \end{pmatrix}.$$ 

In particular, Theorem 4.5 generalizes the characterizations of oblique multiwavelets given in [1] Theorem 3.1 and [2] Proposition 3.1, Theorem 3.2, Corollary 3.4], where the setting $H = L^2(\mathbb{R})$, $(Uf)(x) = f(x - 1)$ and $(Df)(x) = \sqrt{2}f(2x)$ is considered.

**Acknowledgement**

This research was supported by the Wavelets Strategic Research Programme, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Republic of Singapore. We would like to thank Professor S. L. Lee for his helpful comments on this paper.

**References**


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