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THE LENGTH OF C^*-ALGEBRAS OF b-PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We compute the length of the C^*-algebra generated by the algebra of b-pseudodifferential operators of order 0 on compact manifolds with corners.

1. Introduction and statement of the main result

The calculus of *totally characteristic* or briefly *b-pseudodifferential operators* on manifolds with boundary was introduced by Melrose [8] in 1981; it turned out that the definition of b-pseudodifferential operators naturally extends to more singular manifolds, namely manifolds with corners, i.e. manifolds that roughly speaking locally are of the form $\overline{\mathbb{R}}_+^k \times \mathbb{R}^{n-k}$. The corresponding calculus of b-pseudodifferential operators on manifolds with corners was considered, for instance, by Melrose [7], Melrose and Piazza [11], Melrose and Nistor [10], and Loya [6].

By taking the norm closure of the algebra $\Psi^0_{b,cl}(Z, {}^b\Omega^{\frac{1}{2}})$ of b-pseudodifferential operators of order 0 in the C^*-algebra $\mathcal{L}(L^2(Z, {}^b\Omega^{\frac{1}{2}}))$ of all bounded operators on the Hilbert space $L^2(Z, {}^b\Omega^{\frac{1}{2}})$ of all square integrable b-half densities on a compact manifold with corners Z, we attach a C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ to Z, and it is natural to ask which invariants of the singular manifold Z could be recovered by simply studying the C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$.

Certainly, a first measure of how singular the manifold Z is, is the *codimension* of Z, i.e. the largest k necessarily occurring in the local product decompositions $\overline{\mathbb{R}}_+^k \times \mathbb{R}^{n-k}$ of Z, or, more precisely, the largest codimension of a boundary face of Z. Clearly, the codimension of a manifold with boundary is 1, and by considering appropriate products of manifolds with boundary and closed manifolds we can construct compact manifolds with corners of arbitrary dimension n and codimension $k \leq n$.

On the other hand, recall that a C^*-algebra $\mathcal{B}$ is said to be *solvable* if there exists a finite sequence

$$\mathcal{B} = \mathcal{J}_{\ell+2} \supseteq \mathcal{J}_{\ell+1} \supseteq \cdots \supseteq \mathcal{J}_1 = \{0\} \tag{1.1}$$

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of closed ideals such that $\mathcal{J}_{k+1}/\mathcal{J}_k \cong \mathcal{C}_0(T_k, \mathcal{K}(H_k))$ for some locally compact Hausdorff space T_k and some separable Hilbert space H_k [2]. Moreover, the *composition series* (1.1) is said to be *solving of length* ℓ, and the smallest length of a solving composition series is by definition the *length of* $\mathcal{B}$.

In case of a manifold Z with boundary the C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ was shown to be solvable of length $2 = \operatorname{codim} Z + 1$ [4, Corollary 4.2.2], whereas for manifolds with corners of arbitrary codimension the C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ is known to be solvable of length at most $\dim Z + 1$ [10].

In this note we use the Jacobson topology on the spectrum of the Calkin algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})/\mathcal{K}(L^2(Z, {}^b\Omega^{\frac{1}{2}}))$ to determine the length of $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ exactly, and show that the length of $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ is closely related to the codimension of the manifold.

Theorem 1.1. *Let* $n := \dim Z$ *and* $k_0 := \operatorname{codim} Z$. *Then we have*

$$\textit{length } \mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}}) = \begin{cases} k_0 + 1, & k_0 < n, \\ n, & k_0 = n. \end{cases}$$

Furthermore, for an arbitrary C^*-algebra $\mathcal{B}$ let

$$\mathfrak{m}(\mathcal{B}) := \bigcap \{\mathcal{J} \subseteq \mathcal{B} : \mathcal{J} \text{maximal, two-sided ideal in } \mathcal{B}\}$$

be the *maximal radical* of $\mathcal{B}$.

Proposition 1.2. *The* maximal radical series

$$\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}}) \supseteq \mathfrak{m}(\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})) \supseteq \mathfrak{m}(\mathfrak{m}(\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}}))) \supseteq \cdots$$

is solving of minimal length.

We emphasize that the smooth part of the manifold Z is invisible for the C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$. To cover also $\mathcal{C}^\infty$-phenomena instead of $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ we could consider dense subalgebras $\mathcal{A}(Z, {}^b\Omega^{\frac{1}{2}}) \subseteq \mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ stable under holomorphic functional calculus. In this connection the concept of Ψ^*-algebras due to Gramsch [3] can effectively be used. For a compact manifold X with boundary we constructed and investigated in [4] such a Ψ^*-algebra of b-pseudodifferential operators closely related to the smooth structure of X. A similar construction for manifolds with corners will be presented elsewhere.

Let us mention that similar results for pseudodifferential operators with discontinuities in the symbols have been obtained by Plamenevskij and Senichkin—see, for instance, [12], [13], or [14]. Recently they extended their approach also to algebras of pseudodifferential operators on *manifolds with intersecting edges* [15].

As pointed out by J. Cuntz at the workshop *Symplectic Geometry and Microlocal Analysis* (April 24–26, 1998) at State College, PA, alternatively, the length of $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$ could also be computed by localizing near the boundary faces, where the algebra has the structure of a crossed product, using (well-known) results for crossed products, and, finally, which is the tricky part, gluing these local information together.

2. Review on the C^*-algebra $\mathcal{B}(Z, {}^b\Omega^{\frac{1}{2}})$

For the convenience of the reader, we present the relevant material from [5] and [10] without proofs, thus making our exposition self-contained. However, the reader is expected to be familar with the calculus of b-pseudodifferential operators on manifolds with corners—see for instance [6], [8], [9], [10, Appendix], or [11].

In the sequel, Z denotes a compact manifold with corners of dimension n; we write $\mathcal{F}_k(Z)$, $k = 1, \dots, n$, for the set of *boundary faces of codimension* k, and $\mathcal{F}(Z) := \bigcup_{k=1}^n \mathcal{F}_k(Z)$ for the family of all (true) boundary faces. For $F \in \mathcal{F}_k(Z)$ let $\mathcal{E}(F) := \{H \in \mathcal{F}_1(Z) : F \subseteq H\}$, and $M^{\mathcal{E}(F)} := \{(x_H)_{H\in\mathcal{E}(F)} : x_H \in M\}$. Let $\mathcal{B}(Z, {}^b\Omega^{\frac12})$ be the closure of the algebra $\Psi^0_{b,cl}(Z, {}^b\Omega^{\frac12})$ of *b-pseudodifferential operators* of order 0 in the C^*-algebra $\mathcal{L}(L^2(Z, {}^b\Omega^{\frac12}))$ of all bounded operators on the Hilbert space $L^2(Z, {}^b\Omega^{\frac12})$ of *square-integrable b-half-densities*. Then the homogeneous principal symbol map ${}^b\sigma^{(0)}_\psi$ as well as for each $F \in \mathcal{F}(Z)$ the indicial operator I_{FZ} extends to (continuous) *-homomorphisms

$$\begin{aligned} {}^b\sigma^{(0)}_{\mathcal{B}} &: \mathcal{B}(Z, {}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}({}^bS^*Z) \text{ and} \\ I^{\mathcal{B}}_{FZ} &: \mathcal{B}(Z, {}^b\Omega^{\frac12}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac12}; \mathbb{R}^{\mathcal{E}(F)}) \end{aligned}$$

characterizing the Fredholm property of $b \in \mathcal{B}(Z, {}^b\Omega^{\frac12}) \subseteq \mathcal{L}(L^2(Z, {}^b\Omega^{\frac12}))$ by means of their invertibility. Here, $\mathcal{B}(F, {}^b\Omega^{\frac12}; \mathbb{R}^{\mathcal{E}(F)})$ denotes the closure of the algebra $\Psi^0_{b,cl}(F, {}^b\Omega^{\frac12}; \mathbb{R}^{\mathcal{E}(F)})$ of *parameter-dependent b-pseudodifferential operators* on the manifold with corners F in the C^*-algebra $\mathcal{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{L}(L^2(F, {}^b\Omega^{\frac12})))$ of bounded continuous functions $\mathbb{R}^{\mathcal{E}(F)} \longrightarrow \mathcal{L}(L^2(F, {}^b\Omega^{\frac12}))$ [5, Definition 2.1]. Moreover, the *joint symbol map* $\tau_{\mathcal{B}} := {}^b\sigma^{(0)}_{\mathcal{B}} \oplus \bigoplus_{F\in\mathcal{F}(Z)} I^{\mathcal{B}}_{FZ}$ induces a short exact sequence

$$0 \longrightarrow \mathcal{K}(L^2(Z, {}^b\Omega^{\frac12})) \longrightarrow \mathcal{B}(Z, {}^b\Omega^{\frac12}) \xrightarrow{\tau_{\mathcal{B}}} Q_{\mathcal{B}}(Z) \longrightarrow 0$$

where the algebra $Q_{\mathcal{B}}(Z)$ of *continuous joint symbols* consists of all

$$(f, (h_F)_{F\in\mathcal{F}(Z)}) \in \mathcal{C}({}^bS^*Z) \oplus \bigoplus_{F\in\mathcal{F}(Z)} \mathcal{B}(F, {}^b\Omega^{\frac12}; \mathbb{R}^{\mathcal{E}(F)})$$

satisfying the compatibility conditions

$$\begin{aligned} {}^b\widetilde{\sigma}^{(0)}_{\mathcal{B}}(h_F) &= f|_{{}^bS^*Z|_F} \text{ for all } F \in \mathcal{F}(Z), \text{ and} && (2.1) \\ h_G(\lambda) &= I^{\mathcal{B}}_{GF}\left(h_F\left((\lambda_H)_{H\in\mathcal{E}(F)}\right)\right)\left((\lambda_H)_{H\in\mathcal{E}(G)\setminus\mathcal{E}(F)}\right) && (2.2) \end{aligned}$$

for all $\lambda \in \mathbb{R}^{\mathcal{E}(G)}$ and all boundary faces $F, G \in \mathcal{F}(Z)$ with $G \subseteq F$. For the definition of the map ${}^b\widetilde{\sigma}^{(0)}_{\mathcal{B}} : \mathcal{B}(F, {}^b\Omega^{\frac12}; \mathbb{R}^{\mathcal{E}(F)}) \longrightarrow \mathcal{C}({}^bS^*Z|_F)$, the so-called *parameter-dependent, homogeneous principal symbol*, see [5, Section 3] or [10, Eq. (12)].

The nested family $Q_{\mathcal{B}}(Z) \supseteq \mathcal{J}_{n+1} \supseteq \cdots \supseteq \mathcal{J}_1$ of closed ideals of $Q_{\mathcal{B}}(Z)$ defined by

$$\begin{aligned} \mathcal{J}_{n+1} &:= \{(f, (h_F)_{F\in\mathcal{F}(Z)}) \in Q_{\mathcal{B}}(Z) : f = 0\}, \text{ and} \\ \mathcal{J}_k &:= \{(f, (h_F)_{F\in\mathcal{F}(Z)}) \in \mathcal{J}_{k+1} : h_F = 0 \text{ for all } F \in \mathcal{F}_k(Z)\} && (2.3) \end{aligned}$$

for $k = n, n-1, \dots, 1$ leads to a solving composition series of $Q_{\mathcal{B}}(Z)$ with $\mathcal{J}_1 = \{0\}$, and subquotients

$$\begin{aligned} Q_{\mathcal{B}}(Z)/\mathcal{J}_{n+1} &\cong \mathcal{C}({}^bS^*Z) \text{ resp.,} \\ \mathcal{J}_{k+1}/\mathcal{J}_k &\cong \bigoplus_{F\in\mathcal{F}_k(Z)} \mathcal{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L^2(F, {}^b\Omega^{\frac12}))) \text{ for } k = 1, \dots, n\,. \end{aligned}$$

Let $\widehat{Q_{\mathcal{B}}(Z)}$ be the *spectrum* of the C^*-algebra $Q_{\mathcal{B}}(Z)$, i.e. the set of all non-zero irreducible representations of $Q_{\mathcal{B}}(Z)$. Using [1, Proposition 2.11.2] we obtain a

bijective map

$$\Phi : \widehat{Q_{\mathcal{B}}(Z)} \longrightarrow T := {}^bS^*Z \uplus \biguplus_{F\in\mathcal{F}(Z)} \mathbb{R}^{\mathcal{E}(F)} : [\pi_t] \longmapsto t$$

where for $\zeta \in {}^bS^*Z$ and $\lambda \in \mathbb{R}^{\mathcal{E}(F_0)}$, $F_0 \in \mathcal{F}(Z)$ we have

$$\begin{array}{llllll} \pi_\zeta & : & Q_{\mathcal{B}}(Z) \longrightarrow \mathbb{C} & : & (f,(h_F)_{F\in\mathcal{F}(Z)}) \longmapsto f(\zeta)\,, \\ \pi_{F_0,\lambda} & : & Q_{\mathcal{B}}(Z) \longrightarrow \mathcal{L}(L^2(F_0,{}^b\Omega^{\frac12})) & : & (f,(h_F)_{F\in\mathcal{F}(Z)}) \longmapsto h_{F_0}(\lambda)\,. \end{array} \tag{2.4}$$

Now let $p : {}^bS^*Z \longrightarrow Z$ be the canonical projection, and denote for a bounded, nonvoid set $D \subseteq \mathbb{R}^\ell$ and $\gamma_0 \geq 0$ by $\mathfrak{V}^\wedge_{\gamma_0}(D,\mathbb{R}^\ell) := \{\gamma d : \gamma > \gamma_0, d \in D\}$ the γ_0-*conical subset* generated by D in $\mathbb{R}^\ell$. We are going to describe a topology $\mathcal{T}$ on T by characterizing local bases $\mathfrak{U}(t)$ at the points $t \in T$.

In case $\zeta \in {}^bS^*Z$ with $p(\zeta) \notin \partial Z$ let $\mathfrak{U}(\zeta)$ be the family of all open sets $U \subseteq {}^bS^*Z$ with $\zeta \in U$ and $p(U) \cap \partial Z = \emptyset$. For $\zeta_0 \in {}^bS^*Z$ with $p(\zeta_0) \in G \in \mathcal{F}_k(Z)$ and k maximal, let $\zeta = ((x_H)_{H\in\mathcal{E}(G)}, y, (\xi_H)_{H\in\mathcal{E}(G)}, \eta)$ be local coordinates near ζ_0 with $\zeta_0 = (0, y^{(0)}, (\xi_H^{(0)})_{H\in\mathcal{E}(G)}, \eta^{(0)})$. Then $\mathfrak{U}(\zeta_0)$ consists of all sets of the form

$$U \uplus \biguplus_{G\subseteq F\in\mathcal{F}(Z)} \mathfrak{V}^\wedge_{\gamma_F}(D_F, \mathbb{R}^{\mathcal{E}(F)})$$

where $U \subseteq {}^bS^*Z$ is an open set with $\zeta_0 \in U$ and $p(U) \cap H \neq \emptyset$, $H \in \mathcal{F}_1(Z)$ holds only for $H \in \mathcal{E}(G)$, $\gamma_F \geq 0$ and $D_F \subseteq \mathbb{R}^{\mathcal{E}(F)}$ is open and bounded with $(\xi_H^{(0)})_{H\in\mathcal{E}(F)} \in D_F$.

Finally, for $\lambda = (\lambda_H)_{H\in\mathcal{E}(G)} \in \mathbb{R}^{\mathcal{E}(G)}$ let

$$\mathfrak{U}(G,\lambda) := \Big\{ \biguplus_{G\subseteq F\in\mathcal{F}(Z)} \prod_{H\in\mathcal{E}(F)} V_H : V_H \subseteq \mathbb{R} \text{ open with } \lambda_H \in V_H \Big\}\,. \tag{2.5}$$

The following result was proved in [5, Theorem 5.4].

Theorem 2.1. *The canonical map $\Phi : \widehat{Q_{\mathcal{B}}(Z)} \longrightarrow T$ is a homeomorphism provided $\widehat{Q_{\mathcal{B}}(Z)}$ is endowed with the Jacobson topology, and T with the topology $\mathcal{T}$ described above.*

3. Proof of the main results

In a first step we are going to show that we can restrict ourselves to the algebra of joint continuous symbols.

Lemma 3.1. *Let H be a Hilbert space, $\mathcal{B} \subseteq \mathcal{L}(H)$ be a C^*-algebra with $\mathcal{K}(H) \subseteq \mathcal{B}$, and let $Q := \mathcal{B}/\mathcal{K}(H)$ be solvable of length $\ell - 1$. Then $\mathcal{B}$ is solvable of length ℓ.*

Proof. Let $\tau : \mathcal{B} \longrightarrow Q$ be the canonical projection, and $Q \supseteq \mathcal{J}_\ell \supseteq \cdots \supseteq \mathcal{J}_1 = \{0\}$ be a solving composition series of length $\ell - 1$. Then $\mathcal{I}_1 = \{0\}$, $\mathcal{I}_{k+1} := \tau^{-1}(\mathcal{J}_k)$, $k = 1, \ldots, \ell$, is a nested sequence of closed ideals of $\mathcal{B}$ with $\mathcal{I}_2/\mathcal{I}_1 = \mathcal{K}(H)$ and $\mathcal{I}_{k+1}/\mathcal{I}_k \cong \mathcal{J}_k/\mathcal{J}_{k-1}$, $k = 2, \ldots, \ell$, i.e. $\mathcal{B}$ is solvable of length at most ℓ.

On the other hand, let $\mathcal{B} \supseteq \mathcal{I}_{m+1} \supseteq \cdots \supseteq \mathcal{I}_1 = \{0\}$ be a solving composition series of $\mathcal{B}$ of length m. Because of $\mathcal{K}(H) \subseteq \mathcal{B}$ the identity representation id of $\mathcal{B}$ on H is irreducible. Moreover, the set $\widehat{\mathcal{B}}^{\mathcal{K}(H)} = \{[id]\} \subseteq \widehat{\mathcal{B}}$ is dense by the definition of the Jacobson topology, hence $\widehat{\mathcal{B}}^{\mathcal{K}(H)} \subseteq \widehat{\mathcal{B}}^{\mathcal{I}_2} \approx \widehat{\mathcal{I}_2}$ because $\widehat{\mathcal{B}}^{\mathcal{I}_2} \subseteq \widehat{\mathcal{B}}$ is open. Since the composition series is solving, $\widehat{\mathcal{I}_2} = \widehat{\mathcal{I}_2/\mathcal{I}_1}$ is a locally compact Hausdorff space,

thus, the set $\widehat{\mathcal{B}}^{\mathcal{K}(H)} \subseteq \widehat{\mathcal{B}}^{\mathcal{I}_2}$ is dense and relatively closed, and therefore $\widehat{\mathcal{B}}^{\mathcal{K}(H)} = \widehat{\mathcal{B}}^{\mathcal{I}_2}$ which gives $\mathcal{K}(H) = \mathcal{I}_2$ by [1, Proposition 3.2.2].

It is straightforward to check that $\mathcal{J}_k := \mathcal{I}_{k+1}/\mathcal{I}_2$, $k = 1, \dots, m$, is a solving composition series of $Q = \mathcal{B}/\mathcal{I}_2$ of length $m-1$, i.e. the length of $\mathcal{B}$ is at least ℓ. □

Lemma 3.2. *Let* $k_0 := \mathrm{codim} Z := \max\{k : \mathcal{F}_k(Z) \neq \emptyset\}$. *Then we have*

1. *length* $(Q_{\mathcal{B}}(Z)) \leq k_0$ *if* $k_0 < n$, *and*
2. *length* $(Q_{\mathcal{B}}(Z)) \leq k_0 - 1 = n - 1$ *if* $k_0 = n$.

Proof. Consider the composition series $Q_{\mathcal{B}}(Z) \supseteq \mathcal{J}_{n+1} \supseteq \cdots \supseteq \mathcal{J}_1 = \{0\}$ of Section 2. By the definition of k_0, we have $\mathcal{J}_{n+1} = \mathcal{J}_n = \cdots = \mathcal{J}_{k_0+1}$, i.e. the length of this series is k_0 which gives (1). In case $k_0 = n$, note that

$$\mathcal{J}_n = \bigcap \{N(\pi) : [\pi] \in \widehat{Q_{\mathcal{B}}(Z)}, \dim[\pi] = 1\} = \mathrm{com}(Q_{\mathcal{B}}(Z)),$$

where $\mathrm{com}(Q_{\mathcal{B}}(Z))$ denotes the commutator ideal. Thus, the C^*-algebra $Q_{\mathcal{B}}(Z)/\mathcal{J}_n$ is commutative and unital, i.e. of the form $\mathcal{C}(\mathfrak{M})$ for some compact Hausdorff space $\mathfrak{M}$, and the series $Q_{\mathcal{B}}(Z) \supseteq \mathcal{J}_n \supseteq \cdots \supseteq \mathcal{J}_1 = \{0\}$ is solving of length $n-1$. □

Remark 3.3. It is straightforward to check that $\mathfrak{M} \approx {}^bS^*Z \uplus \biguplus_{F \in \mathcal{F}_n(Z)} \mathbb{R}^{\mathcal{E}(F)}$ where the compact Hausdorff topology on the latter space is the topology induced by the inclusion ${}^bS^*Z \uplus \biguplus_{F \in \mathcal{F}_n(Z)} \mathbb{R}^{\mathcal{E}(F)} \hookrightarrow (T, \mathcal{T})$.

Let Q be a C^*-algebra, and $\mathcal{J}_1, \mathcal{J}_2 \subseteq Q$ be two closed ideals of Q with $\mathcal{J}_1 \subseteq \mathcal{J}_2$. For simplicity, let us write

$$(\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1} := \Big\{[\pi] \in \widehat{Q} : \pi(\mathcal{J}_1) = \{0\} \text{ and } \pi(\mathcal{J}_2) \neq \{0\}\Big\} = \widehat{Q}^{\mathcal{J}_2} \cap \widehat{Q}_{\mathcal{J}_1} = \widehat{Q}^{\mathcal{J}_2} \setminus \widehat{Q}^{\mathcal{J}_1}.$$

Since $[\pi|_{\mathcal{J}_2}] \in \widehat{\mathcal{J}_2}$ for all $[\pi] \in \widehat{Q}^{\mathcal{J}_2}$, we obtain a map

$$\varphi : (\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1} \longrightarrow \widehat{\mathcal{J}_2/\mathcal{J}_1} : [\pi] \longmapsto [\mathcal{J}_2/\mathcal{J}_1 \ni x + \mathcal{J}_1 \longmapsto \pi(x)].$$

Lemma 3.4. *The map* $\varphi : (\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1} \longrightarrow \widehat{\mathcal{J}_2/\mathcal{J}_1}$ *is a homeomorphism provided* $(\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1}$ *is endowed with the topology induced by the inclusion* $(\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1} \hookrightarrow \widehat{Q}$.

Proof. By [1, Proposition 3.2.1] restriction to the ideal $\mathcal{J}_2$ induces a homeomorphism $\psi : \widehat{Q}^{\mathcal{J}_2} \longrightarrow \widehat{\mathcal{J}_2}$ with $\psi((\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1}) = \widehat{\mathcal{J}_2}_{\mathcal{J}_1} \approx \widehat{\mathcal{J}_2/\mathcal{J}_1}$. Since $\varphi = \psi|_{(\widehat{Q}^{\mathcal{J}_2})_{\mathcal{J}_1}}$, this completes the proof. □

Lemma 3.5. *Let* $k_0 := \mathrm{codim} Z$ *be as above. Then we have*

1. *length* $(Q_{\mathcal{B}}(Z)) \geq k_0$ *if* $k_0 < n$, *and*
2. *length* $(Q_{\mathcal{B}}(Z)) \geq k_0 - 1 = n - 1$ *if* $k_0 = n$.

Proof. Let $Q_{\mathcal{B}}(Z) =: \mathcal{I}_{\ell+2} \supseteq \mathcal{I}_{\ell+1} \supseteq \cdots \supseteq \mathcal{I}_1 = \{0\}$ be a solving composition series of length ℓ with subquotients $\mathcal{I}_{j+1}/\mathcal{I}_j \cong \mathcal{C}_0(T_j, \mathcal{K}(H_j))$, $j = 1, \dots, \ell+1$. Then

$$\emptyset = \widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_1} \subseteq \widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_2} \subseteq \ldots \subseteq \widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{\ell+2}} = \widehat{Q_{\mathcal{B}}(Z)}$$

is an increasing sequence of open subsets of $\widehat{Q_{\mathcal{B}}(Z)}$ [1, Proposition 3.2.2], and we have a canonical bijective map $\psi : \widehat{Q_{\mathcal{B}}(Z)} \longrightarrow \biguplus_{j=1}^{\ell+1} T_j$.

Let $G \in \mathcal{F}_{k_0}(Z)$ be arbitrary. Suppose $\mathcal{E}(G) = \{H_1, \dots, H_{k_0}\} = \{1, \dots, k_0\}$, and fix $\lambda = (\lambda_1, \dots, \lambda_{k_0}) \in \mathbb{R}^{\mathcal{E}(G)}$. Because of $\mu_1 := \lambda_1 \in \widehat{Q_{\mathcal{B}}(Z)}$ there exists

$j_1 \in \{2, \dots, \ell+2\}$ with $\mu_1 \in (\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_1}})_{\mathcal{I}_{j_1-1}}$. Similar, to $\mu_2 := (\lambda_1, \lambda_2) \in \widehat{Q_{\mathcal{B}}(Z)}$, there exists $j_2 \in \{2, \dots, \ell+2\}$ with $\mu_2 \in (\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_2}})_{\mathcal{I}_{j_2-1}}$.

Suppose $j_2 < j_1$. Since $\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_2}}$ is open, by (2.5) there exist open sets $V_k \subseteq \mathbb{R}^{H_k}$ with $\lambda_k \in V_k$, $k = 1, 2$, such that $V_1 \times V_2 \uplus V_1 \uplus V_2 \subseteq \widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_2}} \subseteq \widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_1-1}}$ which contradicts our choice of j_1.

Now suppose $j_1 = j_2$, i.e. $\mu_1, \mu_2 \in (\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_1}})_{\mathcal{I}_{j_1-1}}$. By Lemma 3.4 the space $(\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_1}})_{\mathcal{I}_{j_1-1}}$ is Hausdorff in its relative topology, which is impossible because we have $\mu_1 \in W_2 \cap W_1$ for any choice of open sets $W_k \subseteq Q_{\mathcal{B}}(Z)$ with $\mu_k \in W_k$. Consequently, we have $j_1 < j_2$, and iterating this procedure leads to a sequence $2 \le j_1 < \cdots < j_{k_0} \le \ell+2$ which gives $\ell \ge k_0 - 1$, and completes the proof in the case $k_0 = n$. On the other hand, if $k_0 < n$ note that by Lemma 3.4 we have

$$\psi\Big(\biguplus_{k=1}^{k_0} (\widehat{Q_{\mathcal{B}}(Z)}^{\mathcal{I}_{j_k}})_{\mathcal{I}_{j_k-1}} \Big) = \biguplus_{k=1}^{k_0} T_{j_k-1} \subseteq \biguplus_{j=1}^{\ell+1} T_j$$

where each T_{j_k} corresponds to a μ_k, i.e. to an infinite-dimensional irreducible representation. Since $\widehat{Q_{\mathcal{B}}(Z)}$ also has one-dimensional irreducible representations by Theorem 2.1 and (2.4), we have $k_0 + 1 \le \ell + 1$, i.e. the length of $Q_{\mathcal{B}}(Z)$ is at least k_0. □

Of course, a combination of Lemma 3.1, Lemma 3.2, and Lemma 3.5 completes the proof of Theorem 1.1.

It remains to prove Proposition 1.2. Recall that the composition series (1.1) is called *stratified* if for any irreducible representation π of $\mathcal{B}$ with $\pi(\mathcal{J}_k) \neq \{0\}$ we have $\pi(\mathcal{J}_{k+1}) \neq \pi(\mathcal{J}_k)$ [13, Definition 0.3]. Moreover, a solving, stratified composition series coincides with the maximal radical series by [13, Proposition 0.4]. If we abbreviate $\ell := \operatorname{length} Q_{\mathcal{B}}(Z)$, then the series

$$Q_{\mathcal{B}}(Z) \supseteq \mathcal{J}_{\ell+1} \supseteq \mathcal{J}_\ell \supseteq \cdots \supseteq \mathcal{J}_1 = \{0\} \tag{3.1}$$

with $\mathcal{J}_k$ given by (2.3) is solving of minimal length by Theorem 1.1 and the proof of Lemma 3.2. Using the description (2.4) of the irreducible representations of $Q_{\mathcal{B}}(Z)$ and the compatibility conditions (2.1) and (2.2), it is straightforward to check that the series (3.1) is stratified. Thus, it is the maximal radical series for $Q_{\mathcal{B}}(Z)$, hence, an application of Lemma 3.1 completes the proof of Proposition 1.2.

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