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AN ANALOGUE OF HARDY'S THEOREM FOR SEMI-SIMPLE LIE GROUPS

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Abstract. A well known theorem of Hardy on Fourier transform pairs says that a function f on $\mathbf{R}^n$ and its Fourier transform $\hat{f}$ cannot both be "very rapidly decreasing". We prove here an analogue of this result in the case of semi-simple Lie groups.

1. Introduction

In qualitative terms Hardy's theorem for R^n says that a function f on R^n and its Fourier transform $\hat{f}$ cannot both be "very rapidly decreasing ". More precisely, if $| f(x) | \leq Ae^{-\alpha|x|^2}$ and $|\hat{f}(y) | \leq Be^{-\beta|y|^2}$ where $A, B, \alpha, \beta,$ are positive constants and $\alpha\beta > 1/4$, then $f = 0$ almost everywhere (see [2], pp. 155–157). An analogue of this result was established in [4] for connected noncompact semi-simple Lie groups with finite centre having one conjugacy class of Cartan subgroups. It was conjectured in [4] that a result of a similar nature is valid for all noncompact semi-simple Lie groups.

The aim of this article is to prove this conjecture, i.e. establish the analogue of Hardy's theorem for all semi-simple Lie groups, connected and having finite centre. We note that it has been shown in [4] that 1/4 is the best possible constant in this theorem. Thus our result is the best possible one, valid in the broad context of all connected semi-simple Lie groups with finite centre.

Our proof follows the pattern of that in [4] and uses in addition the results of Casselman and Milicic ([1]) on the asymptotic behaviour of the matrix coefficients of admissible representations to handle the case when the group has discrete series.

2. Notation and background material

In this section we set up the notation that we subsequently employ and recall some basic facts from the representation theory of semi-simple Lie groups. Our discussion of the latter will be brief and we refer the reader to [5] for details. If V is a finite dimensional real vector space, V^* will denote its dual and V_C the complexification of V. If $\lambda \in V_C^*$, then Re λ (resp. Im λ) will denote the real (resp. imaginary) part of λ. Furthermore iV^* will denote the vector space of all purely imaginary valued R linear functions on V. Let G be a connected, noncompact,

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semi-simple Lie group with Lie algebra $\mathcal{G}$. Analogous notation will be used for denoting Lie algebras of subgroups of G. We assume that G has finite centre. Let K be a fixed maximal compact subgroup of G with Lie algebra $\mathcal{K}$ and let θ denote the associated Cartan involution of G and of $\mathcal{G}$ as well, by abuse of notation. Let $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the corresponding Cartan decomposition of $\mathcal{G}$ and let B denote the Killing form of $\mathcal{G}$. It is well known that $B \mid_{\mathcal{P}\times\mathcal{P}}$ is positive definite. Let $\mathcal{A}_o \subset \mathcal{P}$ be a maximal abelian subspace and $A_o = \exp \mathcal{A}_o$. Let $\Sigma(\mathcal{G}, \mathcal{A}_o)$ be the roots of the pair $(\mathcal{G}, \mathcal{A}_o)$, i.e. the set of restricted roots. We choose, once and for all, a set of positive restricted roots which we denote by Σ^+. Let Δ_o be the underlying set of simple roots and let $\mathcal{C}_o$ denote the corresponding positive Weyl chamber in $\mathcal{A}_o$. We have $G = KA_oN_o$ (resp. $\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_o \oplus \mathcal{N}_o$) the Iwasawa decomposition of G (resp. $\mathcal{G}$) corresponding to the data above. Let M_o (resp. M_o') be the centraliser (resp. normaliser) of A_o in K. The quotient M_o'/M_o is a finite group called the (little) Weyl group of (G, A_o) and denoted by W. Let $P_o = M_oA_oN_o$. Then P_o is a minimal parabolic subgroup of G and (P_o, A_o) is a (minimal) p pair. To each proper subset F of Δ_o corresponds a standard p pair (P_F, A_F) where P_F is a standard (proper) parabolic subgroup of G and A_F its split component. $F \subset F' \subset \Delta_o \Rightarrow P_F \subset P_{F'}$ and$P_\emptyset = P_o$. We denote by W_F the Weyl group of (P_F, A_F).

G has a Cartan decomposition, $G = KCl(A_o^+)\ K$, i.e. any $x \in G$ can be written as $x = k_1ak_2$ where $k_1, k_2 \in K$ and $a \in C\ell(A_o^+), A_o^+ = exp\ \mathcal{C}_o$, the element a being uniquely determined by x. Let dk denote the normalised Haar measure on K, i.e. $\int_K dk = 1$. The Haar measure dx on G can be normalised so that $dx = J(a)$ $dk_1da\,dk_2$ where $J(a) = \prod_{\alpha\in\Sigma^+}(e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{m_\alpha}, m_\alpha$ being the multiplicity of the root α.

The quotient space G/K is a Riemannian symmetric space of the noncompact type. Let d be the distance function on G/K induced by the Riemann metric on G/K. Let $\sigma : G \longrightarrow R$ be the function

$$\sigma(x) = d(xK, o) \tag{2.1}$$

where $o = eK$ is the "origin" in G/K. σ is a continuous function on G, nonnegative, K bi-invariant and $\sigma(x) = \sigma(x^{-1})\ \forall x \in G$. Furthermore $\sigma(expH) = \|H\| = B(H,H)^{1/2}\ \forall H \in \mathcal{A}_o$.

$G^\wedge$ will denote the unitary dual of G, i.e. the set of equivalence classes of irreducible unitary representations of G. We will use the same symbol, say π, for an irreducible unitary representation and its equivalence class. We hope that no confusion will arise from this abuse of notation. If π is any unitary representation of G in a Hilbert space H_π say and if $f \in L^1(G), \pi(f)$ is the bounded linear operator on H_π given by

$$\pi(f)v = \int_G f(x)\pi(x)v dx\ \forall v \in H_\pi. \tag{2.2}$$

An irreducible unitary representation π of G is said to belong to the discrete series if for some nonzero $v_1, v_2 \in H_\pi$ we have $\int_G \mid \langle\pi(x)v_1, v_2\rangle \mid^2 dx < \infty$. If this is the case, then we have $\int_G \mid \langle\pi(x)v_1, v_2\rangle \mid^2 dx < \infty$ for any $v_1, v_2 \in H_\pi$. Furthermore $\exists$ a positive number d_π such that, for unit vectors v_1, v_2 in H_π,

$$\int_G \mid \langle\pi(x)v_1, v_2\rangle \mid^2 dx = d_\pi^{-1}\ . \tag{2.3}$$

We will denote the discrete series of G by $\mathcal{E}_2(G)$. A famous theorem of Harish-Chandra says that $\mathcal{E}_2(G)$ is nonempty iff G has a compact Cartan subgroup, i.e. iff rank $G =$ rank K.

Let (P_F, A_F) be a standard p pair with Langlands decomposition

$$P_F = M_F A_F N_F.$$

N_F is the unipotent radical of P_F and M_F is reductive. P_F is said to be a cuspidal parabolic subgroup if M_F has a compact Cartan subgroup, i.e. if $\mathcal{E}_2(M_F)$ is nonempty. We denote by $\mathcal{P}_{cusp}$ the set of standard (proper) cuspidal parabolic subgroups of G. To each $P_F \in \mathcal{P}_{cusp}$ we associate a series of admissible representations of G as follows. Let $\xi \in \mathcal{E}_2(M_F)$ and $\lambda \in \mathcal{A}^*_{F,C}$. Then $\xi \otimes e^\lambda \otimes 1$ is a representation of P_F on the Hilbert space H_ξ of ξ. Here 1 denotes the trivial representation of N_F. We define $\pi_{\xi,\lambda} = Ind^G_{P_F} \ \ \xi \otimes e^\lambda \otimes 1$, the representation of G induced from the representation $\xi \otimes e^\lambda \otimes 1$ of P_F. We use normalised induction so that if λ is purely imaginary, then $\pi_{\xi,\lambda}$ is unitary. If λ is purely imaginary, $\pi_{\xi,\lambda}$ is "generically" irreducible and if not, it decomposes into a finite number of irreducible unitary representations of G. If π is any irreducible tempered representation of G, then either π is a member of the discrete series of G (if the latter is nonempty) or it is equivalent to an irreducible subrepresentation of some $\pi_{\xi,\lambda}$, λ purely imaginary, the two possibilities being mutually exclusive. We denote the Hilbert space of the representation $\pi_{\xi,\lambda}$ by $H_{\xi,\lambda}$. We refer the reader to [5] for an explicit description of $\pi_{\xi,\lambda}$. Let $Y_F = [(\xi,\lambda) \mid \xi \in \mathcal{E}_2(M_F),\ \lambda \in i\mathcal{A}^*_F]$ and let $Y = \bigcup_F Y_F$. If $f \in L^1(G)$, then its group-theoretic Fourier transform, $\hat{f}$, is an operator valued function on $Y \cup \mathcal{E}_2(G)$ given by

$$\hat{f}\,(\xi,\lambda) = \pi_{\xi,\lambda}(f) \text{ for } (\xi,\lambda) \in Y_F, \tag{2.4}$$

$$\hat{f}\,(\pi) = \pi(f) \text{ for } \pi \in \mathcal{E}_2(G). \tag{2.5}$$

The Plancherel theorem for G says that there is a positive Borel measure $m(\xi,\lambda)$ on Y such that for $f \in L^1(G) \cap L^2(G)$ we have

(2.6)

$$\int_G \mid f(x) \mid^2 dx = \Sigma_F (\#W_F)^{-1} \int_{Y_F} \parallel \pi_{\xi,\lambda}(f) \parallel_2^2 dm(\xi,\lambda) + \Sigma_{\pi \in \mathcal{E}_2(G)} \parallel \pi(f) \parallel_2^2 d_\pi$$

where $\parallel T \parallel_2$ denotes the Hilbert-Schmidt norm of the operator T; note that if the discrete series of G is empty, then the second term in the right-hand side of (2.6) is absent.

3. Statement of Hardy's theorem

Theorem. *Suppose f is a measurable function on G satisfying the following estimates:*

$$\mid f(x) \mid \leq C e^{-\alpha\sigma^2(x)}, x \in G; \tag{3.1}$$

$$\textit{For } P_F = M_F A_F N_F \in \mathcal{P}_{\text{cusp}} \ (\textit{cf.} \ \ \textit{section}(2)) \tag{3.2}$$

$$\parallel \pi_{\xi,\lambda}(f) \parallel_2 \leq C_\xi e^{-\beta\|\lambda\|^2}, (\xi,\lambda) \in \mathcal{E}_2(M_F) \times i\mathcal{A}^*_F,$$

where C, C_ξ, α, β are positive constants with C_ξ depending on ξ and $\parallel \ \parallel_2$ denotes the Hilbert-Schmidt norm. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ almost everywhere.

Remark. The estimate (3.1) implies that $f \in L^1(G) \cap L^2(G)$ and therefore $\pi_{\xi,\lambda}(f)$ is a well-defined bounded linear operator which is Hilbert-Schmidt for almost every (ξ, λ).

4. Proof of the theorem

In order to keep the exposition simple we divide the proof into two cases.

Case 1. rank $G >$ rank K, i.e. G has no discrete series.

We note that $H_{\xi,\lambda}$ has an orthonormal basis consisting of K finite vectors. Let u (resp. v) be an element of this basis which transforms under K according to δ (resp. μ) $\in \hat{K}$. Then we have for $(\xi, \lambda) \in \mathcal{E}_2(M_F) \times \mathcal{A}^*_{F,\mathbf{C}}$

$$\langle \pi_{\xi,\lambda}(f)u, v\rangle = \int_G f(x)\langle \pi_{\xi,\lambda}(x)u, v\rangle dx. \tag{4.1}$$

Hence

$$| \langle \pi_{\xi,\lambda}(f)u, v\rangle | \leq C \int_G e^{-\alpha\sigma^2(x)} | \langle \pi_{\xi,\lambda}(x)u, v\rangle | dx.$$

Now $P_F = M_F A_F N_F \supset P_0 = M_0 A_0 N_0$ (section 2). We have $\mathcal{A}_F \subset \mathcal{A}_0$ and we write $\mathcal{A}_0 = \mathcal{A}_F \oplus \mathcal{A}_F^{\perp}$ where $\mathcal{A}_F^{\perp}$ is the orthocomplement of $\mathcal{A}_F$ in $\mathcal{A}_0$ with respect to B. Let ρ_{P_F} (resp. ρ_0) be the half-sum of the roots of P_F (resp. P_0). Then, the arguments of Milicic ([3], pp. 82–83) show that

$$| \langle \pi_{\xi,\lambda}(x)u, v\rangle | \leq \sqrt{d(\delta)}\sqrt{d(\mu)}\ \phi_{Re\lambda+\rho_{P_F}-\rho_0}(x)\ \forall x \in G \tag{4.2}$$

where $d(\delta)$ (resp. $d(\mu)$) is the degree of δ (resp. μ) and $\phi_{(.)}$ is the zonal spherical function with parameter (.). Here $Re\,\lambda$ and ρ_{P_F} are extended to $\mathcal{A}_0$ by defining them to be zero on $\mathcal{A}_F^{\perp}$. For brevity we denote $\sqrt{d(\delta)}\sqrt{d(\mu)}$ by $C(\delta, \mu)$. Now if $x = k_1\, exp\, H k_2$ where $k_1, k_2 \in K$ and $H \in \overline{\mathcal{C}}_0$, then, by K bi-invariance,

$$\phi_{Re\lambda+\rho_{P_F}-\rho_0}(x) = \phi_{Re\lambda+\rho_{P_F}-\rho_0}(expH) \leq e^{(Re\lambda+\rho_{P_F}-\rho_0)^+(H)} \tag{4.3}$$

where $(Re\,\lambda+\rho_{P_F}-\rho_0)^+$ is the Weyl translate of $(Re\,\lambda+\rho_{P_F}-\rho_0)$ which is dominant i.e., $(Re\,\lambda + \rho_{P_F} - \rho_0)^+ = s.(Re\lambda + \rho_{P_F} - \rho_0)$ for suitable $s \in W$. Now

$$e^{s.(Re\lambda+\rho_{P_F}-\rho_0)(H)} = e^{s.Re\lambda(H)}e^{s.(\rho_{P_F}-\rho_0)(H)} \leq e^{s.Re\lambda(H)}e^{\|\rho_{P_F}-\rho_0\|\ \|H\|}$$

where $\|\ \|$ is the norm on $\mathcal{A}_0$ (and $\mathcal{A}_0^*$) induced by B. $\|\ \|$ is W invariant. Let $H_{s.Re\lambda} \in \mathcal{A}_0$ correspond to $s.\ Re\,\lambda \in \mathcal{A}_0^*$ under the isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_0^*$ given by B. Then, we obtain, for $x = k_1\, exp\, H k_2$ as before,

$$| \langle \pi_{\xi,\lambda}(x)u, v\rangle | \leq C(\delta, \mu)e^{D\|H\|}e^{\langle H, H_{s.Re\lambda}\rangle},\ D = \| \rho_{P_F} - \rho_0 \| . \tag{4.4}$$

Consequently,

$$| \langle \pi_{\xi,\lambda}(f)u, v\rangle | \leq CC(\delta, \mu)\int_{\overline{\mathcal{C}}_0} e^{-\alpha\|H\|^2}e^{D\|H\|}e^{\langle H, H_{s.Re\lambda}\rangle}J(expH)dH. \tag{4.5}$$

We can now proceed exactly as in [4]. Suppose $\alpha, \beta > 0$ satisfy $\alpha\beta > \frac{1}{4}$. Then we can choose $0 < \alpha^1 < \alpha$ such that $\alpha^1\beta > \frac{1}{4}$ and

$$e^{-\alpha\|H\|^2+D\|H\|}J(expH) \leq\ const.e^{-\alpha^1\|H\|^2}\forall H \in \overline{\mathcal{C}}_0$$

since there exists a constant $E > 0 \ni J(exp\, H) \leq e^{E\|H\|}\ \forall H \in \overline{\mathcal{C}}_0$. Hence the integral in (4.5) is

$$\leq \text{const.} \int_{\overline{\mathcal{C}}_0} e^{-\alpha^1\|H\|^2} e^{\langle H, H_{s.Re\lambda}\rangle} dH \leq \text{const}. \int_{\mathcal{A}_0} e^{-\alpha^1\|H\|^2 + \langle H, H_{s.Re\lambda}\rangle} dH$$

since the integrand is positive.

Now

$$\int_{\mathcal{A}_0} e^{-\alpha^1\|H\|^2+\langle H,H_{s.Re\lambda}\rangle} dH = e^{\frac{1}{4\alpha^1}\|H_{s.Re\lambda}\|^2} \int_{\mathcal{A}_0} e^{-\alpha^1\langle H-H_{\frac{s.Re\lambda}{2\alpha^1}}, H-H_{\frac{s.Re\lambda}{2\alpha^1}}\rangle} dH$$

$$= e^{\frac{1}{4\alpha^1}\|H_{s.Re\lambda}\|^2} \int_{\mathcal{A}_0} e^{-\alpha^1\|H\|^2} dH$$

because dH is translation invariant. Now $\| H_{s.Re\lambda} \|^2 = \| H_{Re\lambda} \|^2 = \| Re\lambda \|^2 \leq \| \lambda \|^2$.

(4.6) $$\text{Therefore } | \langle \pi_{\xi,\lambda}(f)u, v\rangle | \leq const.e^{\frac{1}{4\alpha^1}\|\lambda\|^2}.$$

As in [4] (4.6) gives $\langle \pi_{\xi,\lambda}(f)u, v\rangle = 0$ and hence the operator $\pi_{\xi,\lambda}(f) = 0$, for $(\xi, \lambda) \in Y_F$. The Plancherel theorem then implies that $\| f \|_{L^2(G)} = 0$, i.e. $f = 0$ almost everywhere.

Case 2. Rank G = Rank K, i.e. G has discrete series.

Step 1. We will show below that it is enough to prove the theorem in the case when f is $(bi)K$ finite. We first observe that if f satisfies the estimate (3.1) of the theorem, then so does $L_{k_1}R_{k_2}f$ for $(k_1, k_2) \in K \times K$ where $L_{k_1}f$ (resp. $R_{k_2 f}$) is the left (resp. right) translate of f by k_1 (resp. k_2). It follows that if $(\delta, \mu) \in \hat{K} \times \hat{K}$, then the function $f_{\delta\mu}$ given by

$$f_{\delta\mu}(x) = d(\delta)d(\mu) \int_K \int_K f(k_1^{-1}xk_2)\overline{\chi}_\delta(k_1)\overline{\chi_\mu(k_2)} dk_1 dk_2$$

also satisfies the estimate (3.1) of the theorem, with the constant C depending on (δ, μ).$f_{\delta\mu}$ is $(bi)K$ finite. Let $(\xi, \lambda) \in Y_F$. If we compute the matrix of the Fourier transform, $\hat{f}_{\delta\mu}\ (\xi, \lambda)$, with respect to the orthonormal basis of $H_{\xi,\lambda}$ described in case 1, we find that this matrix has only finitely many nonzero entries and any such entry is a matrix entry of $\hat{f}\ (\xi, \lambda)$. Hence $\hat{f}_{\delta\mu}\ (\xi, \lambda)$ is a Hilbert-Schmidt operator and $\|\hat{f}_{\delta\mu}\ (\xi, \lambda) \|_2 \leq \|\hat{f}\ (\xi, \lambda) \|_2$. Therefore $f_{\delta\mu}$ satisfies the estimate (3.2) since f satisfies (3.2) by assumption. Now f can be expanded in the sense of distributions on G as

(4.7) $$f = \sum_{(\delta,\mu)\in\hat{K}\times\hat{K}} f_{\delta\mu}.$$

If $f_{\delta\mu} = 0 \forall (\delta, \mu) \in \hat{K} \times \hat{K}$, then clearly $f = 0$. We can therefore assume that f is $(bi)K$ finite.

Step 2. The arguments given in case 1 show that $\pi_{\xi,\lambda}(f) = 0$ for $(\xi, \lambda) \in Y_F$, i.e. the Fourier transform of f is supported on the discrete series of G. Now since f is $(bi)K$ finite, it must be a finite linear combination of $(bi)K$ finite matrix coefficients of the discrete series of G; in particular f is $z(\mathcal{G})$ finite where $z(\mathcal{G})$ is the commutative algebra of bi-invariant differential operators on G.

We will now show that $f = 0$ by using the results of Casselman-Milicic (cf. [1]). Let $\phi : G \to L^2(K \times K)$ be given by

$$\phi(x)(k_1, k_2) = f(k_1^{-1} x k_2^{-1}),\ x \in G, k_1, k_2 \in K.$$

Then, since f is $(bi)K$-finite, the linear span of $\{\phi(x) \mid x \in G\}$ is a finite dimensional subspace, V say, of $L^2(K \times K)$. We define a double representation τ of K on $L^2(K \times K)$ by

$$[\tau(\overline{k}_1)\psi](k_1, k_2) = \psi(\overline{k}_1^{-1} k_1, k_2),$$
$$[\psi\tau(\overline{k}_2)](k_1, k_2) = \psi(k_1, k_2 \overline{k}_2^{-1})$$

for ψ in $L^2(K \times K)$ and $k_1, k_2, \overline{k}_1, \overline{k}_2$ in K. Then (τ, V) is a finite dimensional, smooth, unitary, K bi-module, the norm in V being the norm in $L^2(K \times K)$. Now $\phi : G \to V$ is C^∞, τ spherical and $z(\mathcal{G})$ finite and (3.1) implies

$$\| \phi(x) \| \leq C e^{-\alpha\sigma^2(x)}\ \forall x \in G. \tag{4.8}$$

Let A_0^- denote the negative Weyl Chamber in A_0, i.e. $A_0^- = \exp(-\mathcal{C}_0)$. We define an order relation on positive characters of A_0, i.e. continuous homomorphisms of A_0 into the group of positive real numbers as follows. Let $\chi_1, \chi_2 : A_0 \to \mathbf{R}^+$, be as above. Then we say $\chi_1 \leq \chi_2$ if $\chi_1(a) \leq \chi_2(a) \forall a \in A_0^-$. Let (τ, E) be any finite dimensional, smooth, unitary, double representation of K. The theory of Casselman-Milicic determines the asymptotic behaviour of $z(\mathcal{G})$ finite, C^∞, τ spherical, functions $F : G \to E$ when the group variable tends to infinity in terms of certain characters, called leading characters, depending on F, of A_0. We are now in a position to state the theorem of Casselman-Milicic.

Theorem ([1])**.** *Let $F : G \to E$ be C^∞, τ spherical and $z(\mathcal{G})$ finite and let $\omega : A_0 \to \mathbf{R}^+$ be a positive character of A_0. Then the following conditions are equivalent.*

(i) For every leading character ν of F we have

$$| \nu | \leq \omega.$$

(ii) There exists $M > 0$ and $m \geq 0$ such that

$$\| F(a) \| \leq M\omega(a)(1 + \| \log a \|)^m$$

for all $a \in C\ell(A_0^-)$.

We apply this theorem to $F = \phi$. Now $\| \phi(a) \| \leq Ce^{-\alpha\sigma^2(a)} \forall a \in C\ell(A_0^-)$. Now if ω is *any* positive character of A_0, we have $\omega^{-1}(a) \| \phi(a) \| \leq Ce^{\|d\omega\|\sigma(a)} e^{-\alpha\sigma^2(a)}$ $\forall a \in A_0$ where $\| d\omega \|$ is the norm of the linear functional $d\omega \in \mathcal{A}_0^*$. Now $\sup_{a \in A_0} e^{\|d\omega\|\sigma(a) - \alpha\sigma^2(a)} < \infty \Rightarrow \phi$ satisfies (ii) for any ω. The theorem above then shows that $| \nu | \leq \omega$ for *any* leading character ν of ϕ and for *any* positive character ω of A_0. Hence the set of leading characters of ϕ is empty and therefore ϕ must be zero which implies that $f = 0$. This completes the proof of the theorem.

Remark. Our proof carries over to the case when G is a real reductive Lie group in Harish-Chandra class.

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References

[1] W. Casselman and D. Milicic, Asymptotic behavior of matrix coefficients of admissible representations, Duke. Math. Jl, 49 (1982), 869-930. MR **85a:**22024

[2] H. Dym and H. P. McKean, Fourier series and integrals, Academic Press, New York, 1972. MR **56:**945

[3] D. Milicic, Asymptotic behaviour of matrix coefficients of the discrete series, Duke Math. Jl, 44 (1977), 59-88. MR **55:**3171

[4] A. Sitaram and M. Sundari, An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple groups, Pacific. Jl. Math, 177 (1997) 187-200. MR **99a:**22018

[5] N. Wallach, Real reductive groups, vol 1 and 2, Academic Press, New York. MR **89i:**22029; MR **93m:**22018

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