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RELATIVE BOUNDEDNESS AND RELATIVE COMPACTNESS FOR LINEAR OPERATORS IN BANACH SPACES

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Abstract. If A and B are linear operators acting between Banach spaces, we show that compactness of B relative to A does not in general imply that B has A-bound zero. We do, however, give conditions under which the above implication is valid.

Let X, Y, and Z be Banach spaces, and $A : \mathcal{D}(A) \subset X \to Y$, $B : \mathcal{D}(B) \subset X \to Z$ be densely defined linear operators. B is said to be *A-bounded* if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there are nonnegative constants α and β so that

$$\|Bx\| \le \alpha\|x\| + \beta\|Ax\| \qquad \text{for all} \qquad x \in \mathcal{D}(A). \tag{1}$$

This standard definition may be found in [1, Ch. IV] together with the following stronger notion: B is *compact relative to* A if for any sequence $x_n \in \mathcal{D}(A)$, such that x_n and Ax_n are bounded, Bx_n must have a convergent subsequence.

We call the greatest lower bound β_0 of all possible β for which (1) holds the *A-bound* of B. It is a folk theorem of the subject (see, e.g., [2, 3, 4] for special cases) that B is compact relative to A implies that B has A-bound zero. Unfortunately this is false in general, as the following demonstrates (see also [1, Example IV.1.8]).

Example 1. Let $X = Y = L_1[0,1]$, $Z = \mathbb{C}$, $\mathcal{D}(A) = AC[0,1]$, $Ax = x'$, $Bx = x(0)$.

We note that if $x \in \mathcal{D}(A)$, then $|x|$ attains its average at some $t \in [0,1]$, i.e., $\int_0^1 |x| = |x(t)|$. Thus

$$\|Bx\| = |x(0)| \le |x(t)| + |x(0) - x(t)| \le \|x\| + \|Ax\|, \tag{2}$$

which proves that the A-bound of B is not greater than 1. Moreover (2) also shows that if x_n and Ax_n are bounded, then so is Bx_n and hence B is compact relative to A. On the other hand, the sequence given by $y_n(t) = 1 - tn$ on $[0, n^{-1}]$, $y_n(t) = 0$ on $[n^{-1}, 1]$ satisfies

$$\|y_n\| = (2n)^{-1}, \qquad \|Ay_n\| = \|By_n\| = 1,$$

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so if

$$\|By_n\| \le \alpha\|y_n\| + \beta\|Ay_n\| \qquad \text{for all} \qquad n \in \mathbb{N},$$

then $1 \le \beta$, and thus 1 is in fact the A-bound of B.

Folk theorems are usually valid if one imposes "reasonable" assumptions, and several possibilities are given in the following. Such results are frequently used, e.g., in studying boundary value problems in L_p spaces and deriving eigenvalue asymptotics for corresponding linear differential operators; cf. [2]–[5].

Recall ([1, § III.5.2]) that if A is closed, then by definition its domain $\mathcal{D}(A)$ equipped with the norm $\|x\|_X + \|Ax\|_Y$ becomes a Banach space which we denote by $\mathcal{D}_A$. In this case relative boundedness (compactness) of B is equivalent to boundedness (compactness) of B considered as an operator from $\mathcal{D}_A$ to Z ([1, § IV.1.1–3]).

Theorem 2. *If B is compact relative to A, then B has A-bound zero if either*

(a) *B is closable, or*
(b) *A is closed and $\mathcal{D}_A$ is reflexive.*

Remark. Note that for closed A, $\mathcal{D}_A$ is certainly reflexive when X and Y are so. Indeed, by definition $\mathcal{D}_A$ is isometrically isomorphic to the subspace $\Gamma(A) = \{(x, Ax) \mid x \in \mathcal{D}(A)\}$ of the space $X \times Y$. If both X and Y are reflexive and A is closed, then $X \times Y$ is reflexive and $\Gamma(A)$ is a closed subspace, and the claim follows.

Proof. Assume that the statement is not true. Then for some $\beta_0 > 0$ and any $n \in \mathbb{N}$ there exists $x_n \in \mathcal{D}(A)$ such that $\|x_n\| + \|Ax_n\| = 1$ and

$$\|Bx_n\| \ge n\|x_n\| + \beta_0\|Ax_n\|. \tag{3}$$

Since operators compact relative to A are also A-bounded, for some $C > 0$ we have

$$\|Bx_n\| \le C(\|x_n\| + \|Ax_n\|) = C,$$

which together with (3) shows that $\|x_n\| \to 0$ as $n \to \infty$. Moreover, by definition Bx_n should contain a convergent subsequence, say By_n.

(a) Since B is closable, $y_n \to 0$, and By_n is convergent, it follows that $By_n \to 0$, yielding a contradiction with (3) for all n large enough.

(b) Since $\mathcal{D}_A$ is reflexive, y_n has a subsequence z_n weakly convergent in $\mathcal{D}_A$, and it follows that Az_n is weakly convergent in Y. Being closed, the operator A is weakly closed as well so $z_n \to 0$ in X (weakly) shows that z_n converges weakly to zero in $\mathcal{D}_A$. Then we deduce that $Bz_n \to 0$ weakly, and since Bz_n is strongly convergent by construction, in fact $Bz_n \to 0$ strongly, which again contradicts (3). The theorem is proved. □

We mentioned earlier that if A is closed and B is compact relative to A, then B is compact as an operator from $\mathcal{D}_A$ to Z. If, in addition, Z has the *approximation property* (see [6, §VII.3]), then B is a limit in the uniform operator topology of finite-rank operators, i.e., for any given $\varepsilon > 0$ there exists a finite-rank operator[1] $F_\varepsilon \in \mathfrak{B}(\mathcal{D}_A, Z)$ such that

$$\|Bx - F_\varepsilon x\| \le \varepsilon\big(\|x\| + \|Ax\|\big) \qquad \text{for all} \qquad x \in \mathcal{D}(A).$$

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[1] $\mathfrak{B}(X, Y)$ denotes the Banach space of bounded linear transformations from a Banach space X to a Banach space Y.

From this we derive the inequality

$$\|Bx\| \le \varepsilon\bigl(\|x\| + \|Ax\|\bigr) + \|F_\varepsilon x\|, \tag{4_ε}$$

which is actually of limited applicability since F_ε might not even be bounded as a mapping from X to Z. This inconvenience can be overcome in the following situation.

Lemma 3. *Suppose that A is closed and that $\mathcal{D}(A^*)$ is dense in Y^*. If B is compact relative to A and Z has the approximation property, then for any $\varepsilon > 0$ there exists a finite-rank operator $F'_\varepsilon \in \mathfrak{B}(X, Z)$ such that, for all $x \in \mathcal{D}(A)$,*

$$\|Bx\| \le \varepsilon\bigl(\|x\| + \|Ax\|\bigr) + \|F'_\varepsilon x\|. \tag{5}$$

Remark. $\mathcal{D}(A^*)$ is dense in Y^* if, say, A is closable and X and Y are reflexive (see [1, Ch. III.5]).

Proof. The operator $F_{\varepsilon/2}$ from $(4_{\varepsilon/2})$ has the form

$$F_{\varepsilon/2}x = \sum_{k=1}^{N} \bigl(\langle x, f_k\rangle + \langle Ax, g_k\rangle\bigr) z_k,$$

where $f_k \in X^*$, $g_k \in Y^*$, and $z_k \in Z$, $k = 1, \dots, N$, are vectors depending on ε but not x and the brackets denote duality between the corresponding spaces. Since $\mathcal{D}(A^*)$ is dense in Y^*, for any $\delta_k > 0$ there exists $g'_k \in \mathcal{D}(A^*)$ such that $\|g_k - g'_k\| \le \delta_k$. Take $\delta_k := \varepsilon(2^{k+1}\|z_k\|)^{-1}$ and with g'_k chosen as explained above set

$$F'_\varepsilon x = \sum_{k=1}^{N} \bigl(\langle x, f_k\rangle + \langle Ax, g'_k\rangle\bigr) z_k = \sum_{k=1}^{N} \langle x, f_k + A^* g'_k\rangle z_k.$$

Then $F'_\varepsilon \in \mathfrak{B}(X, Z)$ and

$$\|F_{\varepsilon/2}x - F'_\varepsilon x\| \le \|Ax\| \sum_{k=1}^{N} \|g_k - g'_k\| \|z_k\| \le \frac{\varepsilon}{2}\|Ax\|.$$

Combining this with $(4_{\varepsilon/2})$ finishes the proof. □

Corollary 4. *Suppose that A is bounded away from zero, i.e., that $\|Ax\| \ge \delta\|x\|$ for some $\delta > 0$ and all $x \in \mathcal{D}(A)$. Then, under the conditions of Lemma 3, for any $\varepsilon > 0$ there exists a finite-rank operator $F''_\varepsilon \in \mathfrak{B}(X, Z)$ such that, for all $x \in \mathcal{D}(A)$,*

$$\|Bx\| \le \varepsilon\|Ax\| + \|F''_\varepsilon x\|. \tag{6}$$

Inequalities (5), (6) are frequently used, e.g., while studying negative spectra of Schrödinger-type operators (see [3, 7]).

References

[1] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1976. MR **53:**11389

[2] Schechter, M., *Spectra of Partial Differential Operators*, North-Holland Publ. Company, Amsterdam-London, 1971. MR **56:**6144

[3] Birman, M. Š., On the spectrum of singular boundary-value problems (Russian), *Mat. Sb. (N.S.)*, **55(97)**(1961), no. 2, 125–174. MR **26:**463

[4] Lancaster, P., Shkalikov, A., Damped vibration of beams and related spectral problems, *Canad. Appl. Math. Quart.*, **2**(1994), no. 1, 45–90. MR **95m:**47090

[5] Binding, P., Hryniv, R., Langer, H., Najman, B. Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, *to appear.*

[6] Diestel, J., Uhl, J. J., *Vector Measures*, Mathematical Surveys, Vol. 15, AMS, Providence-Rhode Island, 1977. MR **56:**12216

[7] Glazman, I. M. *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Daniel Davey & Co., New York, 1966. MR **32:**8210

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