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WEYL'S THEOREM HOLDS FOR ALGEBRAICALLY HYPONORMAL OPERATORS

YOUNG MIN HAN AND WOO YOUNG LEE

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ABSTRACT. In this note it is shown that if T is an "algebraically hyponormal" operator, i.e., $p(T)$ is hyponormal for some nonconstant complex polynomial p, then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$, where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$.

H. Weyl [12] examined the spectra of all compact perturbations $T+K$ of a single hermitian operator T and discovered that $\lambda \in \sigma(T+K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. Today this result is known as Weyl's theorem, and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [4], and to several classes of operators including hyponormal operators by S. Berberian [1], [2]. The aim of this note is to show that Weyl's theorem holds for "algebraically hyponormal" operators.

Throughout this note let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators acting on an infinite dimensional Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is given by

$$\text{ind}\,(T) = \dim N(T) - \dim R(T)^{\perp} \ \ (= \dim N(T) - \dim N(T^*)).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently ([7, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by (cf. [6], [7])

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}, \\ \omega(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}, \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}; \end{aligned}$$

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evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write $\operatorname{acc} \mathbf{K}$ for the accumulation points of $\mathbf{K} \subseteq \mathbb{C}$. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T (cf. [2], [10]). An operator $T \in \mathcal{L}(\mathcal{H})$ will be called *algebraically hyponormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is hyponormal. Evidently the pth roots of hyponormal operators (i.e., the operator T such that T^p is hyponormal for $p \in \mathbb{N}$) are algebraically hyponormal (see [3] for the pth roots of operators). But the converse is not true in general: for example if $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ on $\ell_2 \oplus \ell_2$, then T^p is not hyponormal for any $p \in \mathbb{N}$, whereas $p(T) = 0$ with $p(z) = (z-1)^2$. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *polynomially hyponormal* if $p(T)$ is hyponormal for every complex polynomial p. Evidently,

$$\begin{aligned} \text{polynomially hyponormal} &\subseteq \text{hyponormal} \\ &\subseteq \text{the } p\text{th roots of hyponormal operators} \\ &\subseteq \text{algebraically hyponormal.} \end{aligned}$$

The following facts follow from the above definition and the well-known facts of hyponormal operators.

(a) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically hyponormal, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.
(b) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically hyponormal and $\mathcal{M} \subseteq \mathcal{H}$ is invariant under T, then $T|\mathcal{M}$ is algebraically hyponormal.
(c) Unitary equivalence preserves algebraic hyponormality.

The following lemma gives the essential facts for algebraically hyponormal operators that we will need to prove the main theorem.

Lemma 1. *Suppose $T \in \mathcal{L}(\mathcal{H})$.*

(i) *If T is algebraically hyponormal and quasinilpotent, then T is nilpotent.*
(ii) *If T is algebraically hyponormal, then T is isoloid.*
(iii) *If T is algebraically hyponormal, then T has finite ascent.*

Proof. (i) Suppose $p(T)$ is hyponormal for some nonconstant polynomial p. Since hyponormality is translation-invariant, we may assume $p(0) = 0$. Thus we can write $p(\lambda) \equiv a_0 \lambda^m (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ ($m \neq 0$, $\lambda_i \neq 0$ for every $1 \leq i \leq n$). If T is quasinilpotent, then $\sigma(p(T)) = p(\sigma(T)) = p(\{0\}) = \{0\}$, so that $p(T)$ is also quasinilpotent. Since the only hyponormal quasinilpotent operator is zero, it follows that $a_0 T^m (T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$. Since $T - \lambda_i I$ is invertible for every $1 \leq i \leq n$, we have that $T^m = 0$.

(ii) Suppose $p(T)$ is hyponormal for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso} \sigma(T)$. Then using the spectral decomposition, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. By the preceding remark, $T_1 - \lambda I$ is also algebraically hyponormal. Since $T_1 - \lambda I$ is quasinilpotent it follows from the statement (i) that $T_1 - \lambda I$ is nilpotent. Therefore $\lambda \in \pi_0(T_1)$ and hence $\lambda \in \pi_0(T)$. This shows that T is isoloid.

(iii) Suppose $p(T)$ is hyponormal for some nonconstant polynomial p. We may assume $p(0) = 0$. If $p(\lambda) \equiv a_0 \lambda^m$, then $N(T^m) = N(T^{2m})$ because hyponormal operators are of ascent 1. Thus we write $p(\lambda) \equiv a_0\,\lambda^m(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$ ($m \neq 0$; $\lambda_i \neq 0$ for $1 \le i \le n$). We then claim that

$$N(T^m) = N(T^{m+1}). \tag{1.1}$$

To show (1.1), let $x(\neq 0) \in N(T^{m+1})$. Then we can write

$$p(T)x = (-1)^n\,a_0\,\lambda_1\cdots\lambda_n\,T^m x.$$

Thus we have

$$\begin{aligned} |a_0\lambda_1\cdots\lambda_n|^2 ||T^m x||^2 &= (p(T)x,\ p(T)x) \\ &\le ||p(T)^* p(T)x||\,||x|| \\ &\le ||p(T)^2 x||\,||x|| \quad \text{(because } p(T) \text{ is hyponormal)} \\ &= ||a_0^2\,(T-\lambda_1 I)^2\cdots(T-\lambda_n I)^2 T^{2m}x||\,||x|| \\ &= 0, \end{aligned}$$

which implies $x \in N(T^m)$. Therefore $N(T^{m+1}) \subseteq N(T^m)$ and the reverse inclusion is evident. This completes the proof. $\square$

Lemma 1 shows that algebraically hyponormal operators share several properties with hyponormal operators. However there exists a deep gap between those two classes of operators: for example, in a sense of spectrum, it can be compared as a gap between nilpotent and "normaloid" (i.e., norm equals spectral radius). In spite of it, we have:

Theorem 2. *Weyl's theorem holds for every algebraically hyponormal operator.*

Proof. Suppose $p(T)$ is hyponormal for some nonconstant polynomial p. We first prove that $\pi_{00}(T) \subseteq \sigma(T)\setminus\omega(T)$. Since algebraic hyponormality is translation-invariant, it suffices to show that

$$0 \in \pi_{00}(T) \Longrightarrow T \text{ is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. Now using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial B_0}(\lambda I - T)^{-1}d\lambda$, where B_0 is an open disk of center 0 which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{0\} \text{ and } \sigma(T_2) = \sigma(T)\setminus\{0\}.$$

But then T_1 is also algebraically hyponormal and quasinilpotent. Thus by Lemma 1 (i), T_1 is nilpotent. Thus we should have that $\dim R(P) < \infty$: if it were not so, then $N(T_1)$ would be infinite dimensional, so that $0 \notin \pi_{00}(T)$, giving a contradiction. Therefore T_1 is a finite dimensional operator. Since finite dimensional operators are always Weyl it follows that T_1 is Weyl. But since T_2 is invertible we can conclude that T is Weyl. Therefore $\pi_{00}(T) \subseteq \sigma(T)\setminus\omega(T)$. For the reverse inclusion, suppose $\lambda \in \sigma(T)\setminus\omega(T)$. Thus $T-\lambda I$ is Weyl. Then by the "Index Product Theorem",

$$\dim N\left((T-\lambda I)^n\right) - \dim R\left((T-\lambda I)^n\right)^{\perp} = \operatorname{ind}\left((T-\lambda I)^n\right) = n \operatorname{ind}(T-\lambda I) = 0.$$

Thus if $\dim N\left((T-\lambda I)^n\right)$ is a constant, then so is $\dim R\left((T-\lambda I)^n\right)^{\perp}$. Consequently finite ascent forces finite descent. Therefore by Lemma 1 (iii), $T-\lambda I$ is Weyl of finite ascent and descent, and thus it is Browder. Therefore $\lambda \in \pi_{00}(T)$. This completes the proof. $\square$

It was known ([9, Theorem 1], [11, Theorem 3.6]) that the Weyl spectrum obeys the spectral mapping theorem for hyponormal operators. We can prove more:

Theorem 3. *If $T \in \mathcal{L}(\mathcal{H})$ is algebraically hyponormal, then*

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in H(\sigma(T)), \tag{3.1}$$

where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$.

Proof. First of all we prove the equality (3.1) when f is a polynomial. In view of [8, Theorem 5], it suffices to show that

$$\operatorname{ind}(T-\lambda I)\operatorname{ind}(T-\mu I) \ge 0 \quad \text{for each pair } \lambda,\mu \in \mathbb{C}\setminus\sigma_e(T). \tag{3.2}$$

By Lemma 1 (iii), $T-\lambda I$ has finite ascent for every $\lambda \in \mathbb{C}$. Observe that if $S \in \mathcal{L}(\mathcal{H})$ is Fredholm of finite ascent, then $\operatorname{ind}(S) \le 0$: indeed, either if S has finite descent, then S is Browder and hence $\operatorname{ind}(S)=0$, or if S does not have finite descent, then

$$n\operatorname{ind}(S) = \dim N(S^n) - \dim R(S^n)^\perp \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty,$$

which implies that $\operatorname{ind}(S) < 0$. Thus we can see that (3.2) holds for every algebraically hyponormal operator T. This proves that the equality (3.1) holds for every polynomial f. Now the equality (3.1) for $f \in H(\sigma(T))$ follows at once from an argument of Oberai [10, Theorem 2]. ∎

We now have:

Corollary 4. *If $T \in \mathcal{L}(\mathcal{H})$ is algebraically hyponormal, then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$.*

Proof. Remembering ([9, Lemma]) that if T is isoloid, then

$$f\big(\sigma(T)\setminus\pi_{00}(T)\big) = \sigma(f(T))\setminus\pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T));$$

it follows from Lemma 1 (ii), Theorem 2 and Theorem 3 that

$$\sigma(f(T))\setminus\pi_{00}(f(T)) = f\big(\sigma(T)\setminus\pi_{00}(T)\big) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. ∎

We next consider algebraically hyponormal weighted shifts. Let $\{e_n\}_{n=0}^\infty$ be the canonical orthonormal basis for ℓ_2, let $\{\alpha_n\}_{n=0}^\infty$ be a bounded sequence of nonnegative numbers and let W_α be the (unilateral) weighted shift with the weights $\alpha = \{\alpha_n\}$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ $(n \ge 0)$. It is well known that W_α is hyponormal if and only if the weight sequence $\{\alpha_n\}$ is monotonically increasing. A straightforward calculation shows that W_α^p is hyponormal for $p \in \mathbb{N}$ if and only if the weight sequence $\{\alpha_n\}$ satisfies that for each $m = 0, 1, \cdots, p-1$,

$$\prod_{j=m}^{p+m-1} \alpha_j \le \prod_{j=p+m}^{2p+m-1} \alpha_j \le \prod_{j=2p+m}^{3p+m-1} \alpha_j \le \cdots. \tag{4.1}$$

Hence, in particular, if W_α is a pth root of a hyponormal operator and $\alpha = \{\alpha_n\}$ contains infinitely many zeros, then by (4.1), W_α can be written as an infinite direct sum of finite dimensional nilpotents of nilpotency at most p, which implies that W_α is nilpotent. In general the infinitely many zeros in the weights don't guarantee the nilpotence of the weighted shift W_α (see [5, Solution 98]). It however seems to be difficult to find necessary and sufficient conditions, in terms of weights, for the weighted shift W_α to be algebraically hyponormal. On the other hand, Weyl's theorem is not transmitted to or from adjoint operators: for example Weyl's theorem holds for the weighted shift W_α with the weights $\alpha = \left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$, but fails for its adjoint W_α^*. We however have:

Corollary 5. *If W_α is an algebraically hyponormal weighted shift, then Weyl's theorem holds for W_α and W_α^* both.*

Proof. In view of Theorem 2, it suffices to show Weyl's theorem for W_α^*. It is well known that the spectrum of the weighted shift is a (possibly degenerated) disc with center 0. If $\sigma(W_\alpha) = \{0\}$, then by Lemma 1(i), W_α is nilpotent and therefore Weyl's theorem holds for W_α^*. If instead $\sigma(W_\alpha) \neq \{0\}$, then $\operatorname{iso}\sigma(W_\alpha) = \emptyset$, so that the result follows at once from [1, Example 4]. □

We conclude with a structure theorem for algebraically hyponormal compact operators.

Corollary 6. *If $T \in \mathcal{L}(\mathcal{H})$ is algebraically hyponormal and compact, then T is decomposed into the direct sum*

$$T = A \oplus \left(\bigoplus_{n=1}^{\infty} F_n \right),$$

where

(i) *A is an (possibly infinite dimensional) algebraic operator;*
(ii) *F_n is a finite dimensional operator for every $n = 1, 2, \cdots$;*
(iii) *$p(F_n) = \lambda_n \to 0$ as $n \to \infty$ for a nonconstant polynomial p with $p(A) = 0$.*

Proof. Suppose T is compact and $p(T)$ is hyponormal for some nonconstant polynomial p. Use the Putnam inequality to see that $p(T)$ is a compact normal operator. Thus we can write $p(T) = 0 \oplus S$ with respect to the decomposition $N(p(T)) \oplus \overline{R(p(T))}$, where S is an injective compact normal operator. Write $T = \left(\begin{smallmatrix} A & C \\ D & B \end{smallmatrix}\right)$ with respect to the decomposition $N(p(T)) \oplus \overline{R(p(T))}$. Since T commutes with $p(T)$, a straightforward calculation shows that $CS = SD = 0$. But since S is injective and has dense range we should have that $C = D = 0$. Therefore $T = A \oplus B$ with $BS = SB$. Remember that S is diagonalizable, i.e., $S = \sum_{n=1}^{\infty} \lambda_n P_n$, where $\{\lambda_n\}$ are the distinct nonzero eigenvalues of $p(T)$ and P_n is the orthogonal projection of $\mathcal{H}$ onto $N(p(T) - \lambda_n I)$. Note that P_n is of finite rank and $\lambda_n \to 0$. Therefore T admits a direct sum $T = A \oplus B$, where $p(A) = 0$ and $B = \bigoplus_{n=1}^{\infty} F_n$ with F_n acting on $N(p(T) - \lambda_n I)$ for every $n = 1, 2 \cdots$. In this case $p(F_n) = \lambda_n$. □

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Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail address: wylee@yurim.skku.ac.kr