

CARLESON MEASURES AND SOME CLASSES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. For $|a| < 1$ let φ_a be the Möbius transformation defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, and let $g(z, a) = \log \left| \frac{1-\bar{a}z}{z-a} \right|$ be the Green's function of the unit disk \mathcal{D} . We construct an analytic function f belonging to $M_p^\# = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2 (1 - |\varphi_a(z)|^2)^p dA(z) < \infty\}$ for all p , $0 < p < \infty$, but not belonging to $Q_p^\# = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2 (g(z, a))^p dA(z) < \infty\}$ for any p , $0 < p < \infty$. This gives a clear difference as compared to the analytic case where the corresponding function spaces (M_p and Q_p) are same.

Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and denote by $\partial\mathcal{D}$ the boundary of \mathcal{D} . For $a \in \mathcal{D}$, let the Möbius transformation $\varphi_a : \mathcal{D} \rightarrow \mathcal{D}$ be defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathcal{D}.$$

For $0 < r < 1$, let $\mathcal{D}(a, r) = \{z \in \mathcal{D} : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center a and radius r , and let $g(z, a) = \log \left| \frac{1-\bar{a}z}{z-a} \right|$ be the Green's function on \mathcal{D} with logarithmic singularity at $a \in \mathcal{D}$.

For a subarc $I \subset \partial\mathcal{D}$, let

$$S(I) = \{z \in \mathcal{D} : 1 - |I| < |z| < 1, z/|z| \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = \mathcal{D}$. For $0 < p < \infty$, we say that a measure μ defined on \mathcal{D} is a bounded p -Carleson measure if

$$\sup\{\mu(S(I))/|I|^p : I \subset \partial\mathcal{D}\} < \infty.$$

If $p = 1$, we get the classical Carleson measure (see [8, p. 238]).

For $0 < p < \infty$, the authors in [1] and [4] considered the space Q_p as the following:

$$Q_p = \{f : f \text{ analytic in } \mathcal{D} \text{ and } \sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2 (g(z, a))^p dA(z) < \infty\},$$

where $dA(z)$ is the usual element of Euclidean area on \mathcal{D} . It is easy to see that $Q_1 = BMOA$, where $BMOA$ is the space of analytic functions of bounded mean oscillation (cf. [5] and [8]). By [1] we know that the spaces Q_p are the same and

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equal to the Bloch space \mathcal{B} for all p , $1 < p < \infty$. Let M_p denote the space of all functions f analytic in \mathcal{D} for which

$$\sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dA(z) < \infty.$$

The results in [3] show that $Q_p = M_p$ for all p , $0 < p < \infty$, or, equivalently, that $f \in Q_p$ if and only if $|f'(z)|^2 (1 - |z|^2)^p dA(z)$ is a bounded p -Carleson measure.

For a meromorphic function f , a natural extension of $|f'(z)|$ is the spherical derivative $f^\#(z) = |f'(z)|/(1 + |f(z)|^2)$, and in the corresponding way to the analytic case we define

$$Q_p^\# = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2 (g(z, a))^p dA(z) < \infty\}.$$

From [1] we know that $Q_p^\# = N$ for all p , $1 < p < \infty$, where N is the class of normal functions. In the special case $p = 1$, the class $Q_1^\#$ coincides with the class UBC of meromorphic functions of uniformly bounded characteristic in \mathcal{D} [13].

Also, for $0 < p < \infty$, we consider the following meromorphic analogue of the space M_p :

$$M_p^\# = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2 (1 - |\varphi_a(z)|^2)^p dA(z) < \infty\}.$$

Notice that the inclusion $Q_p^\# \subset M_p^\#$ holds for all p , $0 < p < \infty$. This follows from the inequality $1 - |\varphi_a(z)|^2 \leq 2g(z, a)$, $z, a \in \mathcal{D}$. A natural idea is, corresponding to the analytic case, to show $Q_p^\# = M_p^\#$ for all p , $0 < p < \infty$. Unfortunately, this judgment cannot be proved, and the reason may be complex. However, it is clear that for a meromorphic function f , $(f^\#(z))^2$ may not be subharmonic in general and the expression $1 - |\varphi_a(z)|^2$, sometimes, cannot be replaced by the Green's function $g(z, a)$ in an integration of f for the meromorphic case.

In this paper we will show that the class $Q_p^\#$ is not equal to the class $M_p^\#$ for any p , $0 < p < \infty$, or, equivalently, that for any p , $0 < p < \infty$, there exists a meromorphic function $f \notin Q_p^\#$ such that $(f^\#(z))^2 (1 - |z|^2)^p dA(z)$ is a bounded p -Carleson measure. This corrects Proposition 2 (i) in [4] where "equality" between $Q_p^\#$ and $M_p^\#$ was "proved". The proof of Proposition 2 (i) was based on [9, Corollary]. The following theorem shows very strongly that some results for the spaces of analytic functions do not remain true for the corresponding classes of meromorphic functions.

Theorem. *There exists an analytic function f such that $f \in \bigcap_{0 < p < \infty} M_p^\#$ but $f \notin \bigcup_{0 < p < \infty} Q_p^\#$.*

Before embarking into the proof of the Theorem, let us state a result which has been shown to us in a private communication with Professors O. Reséndiz and L. M. Tovar.

Theorem [11]. *Let $0 < p < \infty$ and let $\{z_n\} \subset \mathcal{D}$ be a sequence. If there exists a constant $M > 0$ such that*

$$\sum_{j=k+1}^{\infty} (1 - |z_j|^2)^p \leq M(1 - |z_k|^2)^p, \quad k = 1, 2, \dots,$$

then $d\mu(z) = \sum_{n=1}^{\infty} (1 - |z_n|^2)^p \delta_{z_n}$ is a bounded p -Carleson measure, where δ_ξ is a Dirac measure at $\xi \in \mathcal{D}$.

The following theorem is due to Matts Essén and Jie Xiao.

Theorem [7]. *Let $B(z)$ be a Blaschke product with zeros in $\{z_n\}, n \in \mathbb{N}$, and $0 < p < 1$. Then $B(z)$ belongs to the space Q_p if and only if $\sum_{n=1}^\infty (1 - |z_n|^2)^p \delta_{z_n}$ is a bounded p -Carleson measure.*

Proof of the Theorem. Let $0 < \beta < 1$. We take the sequence $\{z_n\} = \{1 - \beta^n\}$ in \mathcal{D} and consider the Blaschke product $B(z) = \prod_{n=1}^\infty \frac{z_n - z}{1 - \bar{z}_n z}$ associated with the sequence $\{z_n\}$. By [6, Theorem 9.2] we know that, for $n = 1, 2, \dots$,

$$(1) \quad \prod_{\substack{k=1 \\ k \neq n}}^\infty \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| \geq \prod_{k=1}^\infty \left(\frac{1 - \beta^k}{1 + \beta^k} \right)^2 = \delta > 0.$$

Let us consider the function

$$f(z) = B(z) \log \frac{1}{1 - z} \quad (\log 1 = 0).$$

From Lemma 1 in [12] we know that $f \notin N$. Since $Q_p^\# \subset N$ holds for all $p, 0 < p < \infty$, we get $f \notin \bigcup_{0 < p < \infty} Q_p^\#$. Now we turn to prove that $f \in \bigcap_{0 < p < \infty} M_p^\#$. Since $M_p^\# \subset M_q^\#$ holds for $0 < p < q < \infty$, we need only prove that $f \in \bigcap_{0 < p < 1} M_p^\#$. Notice that $|B(z)| < 1$ and for each $p, 0 < p < 1$, we have

$$(2) \quad \begin{aligned} & \iint_{\mathcal{D}} (f^\#(z))^2 (1 - |\varphi_a(z)|^2)^p dA(z) \\ & \leq 2 \iint_{\mathcal{D}} \frac{|B'(z)|^2 |\log(1 - z)|^2}{(1 + |f(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dA(z) \\ & \quad + 2 \iint_{\mathcal{D}} \left| \left(\log \frac{1}{1 - z} \right)' \right|^2 (1 - |\varphi_a(z)|^2)^p dA(z) \\ & = 2\{I(a) + J(a)\}. \end{aligned}$$

Since $\log \frac{1}{1 - z} \in Q_p$ from [2], we get that $\sup_{a \in \mathcal{D}} J(a) < \infty$. Thus we need only prove that $\sup_{a \in \mathcal{D}} I(a) < \infty$.

We take $\delta_1 = \delta/4$ and consider all those pseudohyperbolic disks with centers z_k and radii $\delta_1, k = 1, 2, \dots$. It is easy to see that $\mathcal{D}_k \cap \mathcal{D}_n = \emptyset$, if $k \neq n$, where $\mathcal{D}_k = \mathcal{D}(z_k, \delta_1), k = 1, 2, \dots$. We now estimate the integral $I(a)$ over the regions $E_1 = \bigcup_{k=1}^\infty \mathcal{D}_k$ and $E_2 = \mathcal{D} \setminus E_1$. We set $I(a) = I_1(a) + I_2(a)$, where

$$I_i(a) = \iint_{E_i} \frac{|B'(z)|^2 |\log(1 - z)|^2}{(1 + |f(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dA(z), \quad i = 1, 2.$$

(i) For any $z \in \partial \mathcal{D}_k$, by a simple computation we get

$$(3) \quad B(z) = -\frac{z - z_k}{1 - \bar{z}_k z} \prod_{\substack{n=1 \\ n \neq k}}^\infty \frac{\frac{z_n - z_k}{1 - \bar{z}_k z_n} - \frac{z - z_k}{1 - \bar{z}_k z}}{1 - \left(\frac{z_n - z_k}{1 - \bar{z}_k z_n} \right) \frac{z - z_k}{1 - \bar{z}_k z}},$$

and then it follows from [8, Lemma 1.4, p. 4 and Lemma 5.2, p. 309] and (1) that

$$(4) \quad \begin{aligned} |B(z)| &= \delta_1 \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| \frac{\frac{z_n - z_k}{1 - \bar{z}_k z_n} - \frac{z - z_k}{1 - \bar{z}_k z}}{1 - \left(\frac{z_n - z_k}{1 - \bar{z}_k z_n} \right) \frac{z - z_k}{1 - \bar{z}_k z}} \right| \geq \delta_1 \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{\left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| - \delta_1}{1 - \delta_1 \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right|} \\ &\geq \delta_1 \frac{\prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| - \delta_1}{1 - \delta_1 \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right|} > \frac{3\delta^2}{16}. \end{aligned}$$

Set $B_m(z) = \prod_{n=1}^m \frac{z_n - z}{1 - \bar{z}_n z}$, $m \in \mathbb{N}$. Then $B_m(z)$ is an analytic function on \bar{E}_2 . It follows from (4) that, for $z \in \partial\mathcal{D}_k$,

$$(5) \quad |B_m(z)| \geq |B(z)| > \frac{3\delta^2}{16}.$$

We also know that $|B_m(z)| = 1$ on $\partial\mathcal{D}$ and $B_m(z)$ has no zero points inside E_2 . By the minimum principle and (5) we get

$$|B_m(z)| \geq \frac{3\delta^2}{16}, \quad z \in E_2.$$

Taking $m \rightarrow \infty$ we obtain that $|B(z)| \geq \frac{3\delta^2}{16}$ for any $z \in E_2$. Thus we have

$$(6) \quad \begin{aligned} I_2(a) &= \iint_{E_2} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|f(z)|^2)^2} (1-|\varphi_a(z)|^2)^p dA(z) \\ &\leq \iint_{E_2} \frac{|B'(z)|^2 |\log(1-z)|^2}{4|B(z)|^2 |\log(1-z)|^2} (1-|\varphi_a(z)|^2)^p dA(z) \\ &\leq \frac{64}{9\delta^4} \iint_{E_2} |B'(z)|^2 (1-|\varphi_a(z)|^2)^p dA(z). \end{aligned}$$

Since the zero points of $B(z)$ for $0 < p < 1$ satisfy

$$\sum_{j=k+1}^{\infty} (1-|z_j|^2)^p \leq \frac{2^p \beta^p}{1-\beta^p} (1-|z_k|^2)^p, \quad k = 1, 2, \dots,$$

it follows from Theorem [11] above that $d\mu(z) = \sum_{n=1}^{\infty} (1-|z_n|^2)^p \delta_{z_n}$ is a bounded p -Carleson measure and that $B(z) \in Q_p$ by Theorem [7]. Thus, from (6), we have

$$(7) \quad \sup_{a \in \mathcal{D}} I_2(a) \leq \sup_{a \in \mathcal{D}} \frac{64}{9\delta^4} \iint_{\mathcal{D}} |B'(z)|^2 (1-|\varphi_a(z)|^2)^p dA(z) < \infty.$$

(ii) Now we turn to estimate the integral $I_1(a)$. Define

$$(8) \quad I_1^{(k)}(a) = \iint_{\mathcal{D}_k} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|B(z)|^2 |\log(1-z)|^2)^2} (1-|\varphi_a(z)|^2)^p dA(z), \quad k = 1, 2, \dots$$

Let $w = \frac{z - z_k}{1 - \bar{z}_k z}$ and from (3) we have that $B(z) = -wP_k(w)$, where

$$P_k(w) = \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{\frac{z_n - z_k}{1 - \bar{z}_k z_n} - w}{1 - \left(\frac{z_n - z_k}{1 - \bar{z}_k z_n} \right) w}.$$

By (4) and the minimum principle we obtain

$$(9) \quad \frac{3\delta}{4} \leq \frac{\delta - \delta_1}{1 - \delta\delta_1} \leq |P_k(w)| < 1, \quad |w| < \delta_1.$$

Since $z_k, k = 1, 2, \dots$, are real numbers, we have

$$(10) \quad 1 - z = (1 - z_k) \frac{1 - w}{1 + \bar{z}_k w}.$$

Using the inequalities

$$\frac{1 - \delta}{1 + \delta} < \left| \frac{1 - w}{1 + \bar{z}_k w} \right| < \frac{1 + \delta}{1 - \delta}$$

for $|w| < \delta_1$ and, by (10), we get that there exist positive numbers $c_1 = c_1(\delta), c_2 = c_2(\delta)$ and $c(\delta)$ such that

$$(11) \quad |\log c_1(1 - z_k)| \leq |\log(1 - z(w))| \leq |\log c_2(1 - z_k)|, \quad |w| < \delta_1,$$

and

$$(12) \quad \left| \frac{\log c_2(1 - z_k)}{\log c_1(1 - z_k)} \right| \leq c(\delta), \quad k = 1, 2, \dots,$$

where the constant $c(\delta)$ depends only on δ . A computation shows that

$$(13) \quad \begin{aligned} (1 - |\varphi_a(z)|^2)^p &= \left(1 - \left| \frac{z_k - a}{1 - \bar{a}z_k} \right|^2 \right)^p \frac{(1 - |w|^2)^p}{\left| 1 + \left(\frac{z_k - a}{1 - \bar{a}z_k} \right) w \right|^{2p}} \\ &\leq (5/3) \left(1 - \left| \frac{z_k - a}{1 - \bar{a}z_k} \right|^2 \right)^p \\ &= C(z_k, a, p), \end{aligned}$$

where $w = \frac{z - z_k}{1 - \bar{z}_k z}$ and $|w| < \delta_1$. From $B'(z) = -w'(z)(P_k(w) + wP'_k(w))$ we get

$$|B'(z)|^2 \leq 2|w'(z)|^2(|P_k(w)|^2 + |w|^2|P'_k(w)|^2),$$

which, with (8) and (13), gives

$$(14) \quad \begin{aligned} I_1^{(k)}(a) &= \iint_{\mathcal{D}_k} \frac{|B'(z) \log(1 - z)|^2}{(1 + |B(z) \log(1 - z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dA(z) \\ &\leq C(z_k, a, p) \iint_{\mathcal{D}_k} \frac{(|P_k(w)|^2 + |wP'_k(w)|^2) |\log(1 - z)|^2}{(1 + |wP_k(w) \log(1 - z)|^2)^2} |w'(z)|^2 dA(z) \\ &\leq C(z_k, a, p) \left\{ \iint_{|w| < \delta_1} \frac{|\log(1 - z(w))|^2}{(1 + |wP_k(w) \log(1 - z(w))|^2)^2} dA(w) \right\} \\ &\quad + C(z_k, a, p) \left\{ \iint_{|w| < \delta_1} \frac{|wP'_k(w) \log(1 - z(w))|^2}{(1 + |wP_k(w) \log(1 - z(w))|^2)^2} dA(w) \right\} \\ &= C(z_k, a, p) \left\{ I_{11}^{(k)} + I_{12}^{(k)} \right\}. \end{aligned}$$

By (9), (11) and (12) we obtain

$$\begin{aligned}
 I_{11}^{(k)} &= \iint_{|w| < \delta_1} \frac{|\log(1 - z(w))|^2}{(1 + |wP_k(w) \log(1 - z(w))|^2)^2} dA(w) \\
 &\leq \iint_{|w| < \delta_1} \frac{|\log c_2(1 - z_k)|^2}{(1 + \frac{9}{16}\delta^2|w|^2|\log c_1(1 - z_k)|^2)^2} dA(w) \\
 (15) \quad &\leq 2\pi \int_0^{\delta_1} \frac{|\log c_2(1 - z_k)|^2}{(1 + \frac{9}{16}\delta^2|\log c_1(1 - z_k)|^2 t^2)^2} t dt \\
 &\leq \frac{16\pi}{9\delta^2}(c(\delta))^2.
 \end{aligned}$$

Since $P_k(w)$ is continuous in $|w| \leq \delta_1$, from (9) we have

$$\begin{aligned}
 I_{12}^{(k)} &= \iint_{|w| < \delta_1} \frac{|wP'_k(w) \log(1 - z(w))|^2}{(1 + |wP_k(w) \log(1 - z(w))|^2)^2} dA(w) \\
 (16) \quad &\leq \frac{4}{9\delta^2} \iint_{|w| < \delta_1} |P'_k(w)|^2 dA(w) \\
 &\leq \frac{1}{2\delta^2} \iint_{\mathcal{D}} |P'_k(w)|^2 (1 - |w|^2) dA(w) \\
 &\leq b(\delta) \|P_k\|_{H^2}^2,
 \end{aligned}$$

where $b(\delta)$ is a constant depending only on δ and

$$\|P_k\|_{H^2}^2 = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |P_k(re^{i\theta})|^2 d\theta \leq 1.$$

Thus, from (14), (15) and (16), we have

$$\begin{aligned}
 (17) \quad I_1^{(k)}(a) &\leq (10/3) \left(1 - \left|\frac{z_k - a}{1 - \bar{a}z_k}\right|^2\right)^p \{I_{11}^{(k)} + I_{12}^{(k)}\} \\
 &\leq C(\delta) \left(1 - \left|\frac{z_k - a}{1 - \bar{a}z_k}\right|^2\right)^p,
 \end{aligned}$$

where the constant $C(\delta)$ depends only on δ .

Since $d\mu(z) = \sum_{n=1}^\infty (1 - |z_n|^2)^p \delta_{z_n}$ is a bounded p -Carleson measure in \mathcal{D} , from (8), (17) and Lemma 2.1 in [3] we have

$$(18) \quad \sup_{a \in \mathcal{D}} I_1(a) = \sup_{a \in \mathcal{D}} \sum_{k=1}^\infty I_1^{(k)}(a) \leq C(\delta) \sup_{a \in \mathcal{D}} \sum_{k=1}^\infty \left(1 - \left|\frac{z_k - a}{1 - \bar{a}z_k}\right|^2\right)^p < \infty.$$

Combining (2), (7) and (18) we get

$$\sup_{a \in \mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2 (1 - |\varphi_a(z)|^2)^p dA(z) < \infty,$$

that is, $f \in M_p^\#$. Thus $f \in \bigcap_{0 < p < \infty} M_p^\#$. This completes the proof of the Theorem. □

Corollary 1. $Q_p^\# \subsetneq M_p^\#$ for any $p, 0 < p < \infty$.

Corollary 2. There exists a non-normal function f such that

$$(f^\#(z))^2 (1 - |z|^2) dA(z)$$

is a Carleson measure.

Remark 1. The theorem above shows that the class $Q_p^\#$ cannot be characterized by a general bounded p -Carleson measure for any p , $0 < p < \infty$, and this is an obvious difference between the analytic and meromorphic case. It is necessary to point out that our proof uses some ideas employed in [10].

Remark 2. Recently, the classes $Q_p^\#$ have been characterized by other Carleson measures that are called Carleson type measures, which were introduced by the second author in his paper [14]. Also, some further relations between the classes $Q_p^\#$ and $M_p^\#$ are described there; one of them is that $Q_p^\# = N \cap M_p^\#$ for all p , $0 < p < \infty$.

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