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CARLESON MEASURES AND SOME CLASSES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. For $|a| < 1$ let φ_a be the Möbius transformation defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, and let $g(z,a) = \log|\frac{1-\bar{a}z}{z-a}|$ be the Green's function of the unit disk $\mathcal{D}$. We construct an analytic function f belonging to $M_p^\# = \{f : f$ meromorphic in $\mathcal{D}$ and $\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2(1-|\varphi_a(z)|^2)^p\, dA(z) < \infty\}$ for all p, $0 < p < \infty$, but not belonging to $Q_p^\# = \{f : f$ meromorphic in $\mathcal{D}$ and $\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} (f^\#(z))^2(g(z,a))^p\, dA(z) < \infty\}$ for any p, $0 < p < \infty$. This gives a clear difference as compared to the analytic case where the corresponding function spaces (M_p and Q_p) are same.

Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and denote by $\partial\mathcal{D}$ the boundary of $\mathcal{D}$. For $a \in \mathcal{D}$, let the Möbius transformation $\varphi_a : \mathcal{D} \to \mathcal{D}$ be defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathcal{D}.$$

For $0 < r < 1$, let $\mathcal{D}(a,r) = \{z \in \mathcal{D} : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center a and radius r, and let $g(z,a) = \log|\frac{1-\bar{a}z}{z-a}|$ be the Green's function on $\mathcal{D}$ with logarithmic singularity at $a \in \mathcal{D}$.

For a subarc $I \subset \partial\mathcal{D}$, let

$$S(I) = \{z \in \mathcal{D} : 1 - |I| < |z| < 1, z/|z| \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = \mathcal{D}$. For $0 < p < \infty$, we say that a measure μ defined on $\mathcal{D}$ is a bounded p-Carleson measure if

$$\sup\{\mu(S(I))/|I|^p : I \subset \partial\mathcal{D}\} < \infty.$$

If $p = 1$, we get the classical Carleson measure (see [8, p. 238]).

For $0 < p < \infty$, the authors in [1] and [4] considered the space Q_p as the following:

$$Q_p = \{f : f \text{ analytic in } \mathcal{D} \text{ and } \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2(g(z,a))^p\, dA(z) < \infty\},$$

where $dA(z)$ is the usual element of Euclidean area on $\mathcal{D}$. It is easy to see that $Q_1 = BMOA$, where $BMOA$ is the space of analytic functions of bounded mean oscillation (cf. [5] and [8]). By [1] we know that the spaces Q_p are the same and

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equal to the Bloch space $\mathcal{B}$ for all p, $1 < p < \infty$. Let M_p denote the space of all functions f analytic in $\mathcal{D}$ for which

$$\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)^p \, dA(z) < \infty.$$

The results in [3] show that $Q_p = M_p$ for all p, $0 < p < \infty$, or, equivalently, that $f \in Q_p$ if and only if $|f'(z)|^2(1-|z|^2)^p dA(z)$ is a bounded p-Carleson measure.

For a meromorphic function f, a natural extension of $|f'(z)|$ is the spherical derivative $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$, and in the corresponding way to the analytic case we define

$$Q_p^{\#} = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} (f^{\#}(z))^2(g(z,a))^p \, dA(z) < \infty\}.$$

From [1] we know that $Q_p^{\#} = N$ for all p, $1 < p < \infty$, where N is the class of normal functions. In the special case $p = 1$, the class $Q_1^{\#}$ coincides with the class UBC of meromorphic functions of uniformly bounded characteristic in $\mathcal{D}$ [13].

Also, for $0 < p < \infty$, we consider the following meromorphic analogue of the space M_p:

$$M_p^{\#} = \{f : f \text{ meromorphic in } \mathcal{D} \text{ and } \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} (f^{\#}(z))^2(1-|\varphi_a(z)|^2)^p \, dA(z) < \infty\}.$$

Notice that the inclusion $Q_p^{\#} \subset M_p^{\#}$ holds for all p, $0 < p < \infty$. This follows from the inequality $1-|\varphi_a(z)|^2 \le 2g(z,a)$, $z, a \in \mathcal{D}$. A natural idea is, corresponding to the analytic case, to show $Q_p^{\#} = M_p^{\#}$ for all p, $0 < p < \infty$. Unfortunately, this judgment cannot be proved, and the reason may be complex. However, it is clear that for a meromorphic function f, $(f^{\#}(z))^2$ may not be subharmonic in general and the expression $1-|\varphi_a(z)|^2$, sometimes, cannot be replaced by the Green's function $g(z,a)$ in an integration of f for the meromorphic case.

In this paper we will show that the class $Q_p^{\#}$ is not equal to the class $M_p^{\#}$ for any p, $0 < p < \infty$, or, equivalently, that for any p, $0 < p < \infty$, there exists a meromorphic function $f \notin Q_p^{\#}$ such that $(f^{\#}(z))^2(1-|z|^2)^p dA(z)$ is a bounded p-Carleson measure. This corrects Proposition 2 (i) in [4] where "equality" between $Q_p^{\#}$ and $M_p^{\#}$ was "proved". The proof of Proposition 2 (i) was based on [9, Corollary]. The following theorem shows very strongly that some results for the spaces of analytic functions do not remain true for the corresponding classes of meromorphic functions.

Theorem. *There exists an analytic function f such that $f \in \bigcap_{0<p<\infty} M_p^{\#}$ but $f \notin \bigcup_{0<p<\infty} Q_p^{\#}$.*

Before embarking into the proof of the Theorem, let us state a result which has been shown to us in a private communication with Professors O. Reséndiz and L. M. Tovar.

Theorem [11]**.** *Let $0 < p < \infty$ and let $\{z_n\} \subset \mathcal{D}$ be a sequence. If there exists a constant $M > 0$ such that*

$$\sum_{j=k+1}^{\infty} (1-|z_j|^2)^p \le M(1-|z_k|^2)^p, \quad k = 1, 2, \dots,$$

then $d\mu(z) = \sum_{n=1}^{\infty}(1-|z_n|^2)^p \delta_{z_n}$ is a bounded p-Carleson measure, where δ_ξ is a Dirac measure at $\xi \in \mathcal{D}$.

The following theorem is due to Matts Essén and Jie Xiao.

Theorem [7]**.** *Let $B(z)$ be a Blaschke product with zeros in $\{z_n\}, n \in \mathbb{N}$, and $0 < p < 1$. Then $B(z)$ belongs to the space Q_p if and only if $\sum_{n=1}^{\infty}(1-|z_n|^2)^p \delta_{z_n}$ is a bounded p-Carleson measure.*

Proof of the Theorem. Let $0 < \beta < 1$. We take the sequence $\{z_n\} = \{1-\beta^n\}$ in $\mathcal{D}$ and consider the Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{z_n - z}{1-\overline{z}_n z}$ associated with the sequence $\{z_n\}$. By [6, Theorem 9.2] we know that, for $n = 1, 2, \dots$,

$$\prod_{\substack{k=1\\k\neq n}}^{\infty} \left| \frac{z_n - z_k}{1-\overline{z}_n z_k} \right| \geq \prod_{k=1}^{\infty} \left(\frac{1-\beta^k}{1+\beta^k} \right)^2 = \delta > 0. \tag{1}$$

Let us consider the function

$$f(z) = B(z) \log \frac{1}{1-z} \quad (\log 1 = 0).$$

From Lemma 1 in [12] we know that $f \notin N$. Since $Q_p^{\#} \subset N$ holds for all p, $0 < p < \infty$, we get $f \notin \bigcup_{0<p<\infty} Q_p^{\#}$. Now we turn to prove that $f \in \bigcap_{0<p<\infty} M_p^{\#}$. Since $M_p^{\#} \subset M_q^{\#}$ holds for $0 < p < q < \infty$, we need only prove that $f \in \bigcap_{0<p<1} M_p^{\#}$. Notice that $|B(z)| < 1$ and for each p, $0 < p < 1$, we have

$$\begin{aligned} &\iint_{\mathcal{D}} (f^{\#}(z))^2 (1-|\varphi_a(z)|^2)^p \, dA(z) \\ &\quad \leq 2 \iint_{\mathcal{D}} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|f(z)|^2)^2} (1-|\varphi_a(z)|^2)^p \, dA(z) \\ &\qquad + 2 \iint_{\mathcal{D}} \left|(\log \frac{1}{1-z})'\right|^2 (1-|\varphi_a(z)|^2)^p \, dA(z) \\ &\quad = 2\{I(a) + J(a)\}. \end{aligned} \tag{2}$$

Since $\log \frac{1}{1-z} \in Q_p$ from [2], we get that $\sup_{a \in \mathcal{D}} J(a) < \infty$. Thus we need only prove that $\sup_{a\in\mathcal{D}} I(a) < \infty$.

We take $\delta_1 = \delta/4$ and consider all those pseudohyperbolic disks with centers z_k and radii $\delta_1, k = 1, 2, \dots$. It is easy to see that $\mathcal{D}_k \cap \mathcal{D}_n = \varnothing$, if $k \neq n$, where $\mathcal{D}_k = \mathcal{D}(z_k, \delta_1), k = 1, 2, \dots$. We now estimate the integral $I(a)$ over the regions $E_1 = \bigcup_{k=1}^{\infty} \mathcal{D}_k$ and $E_2 = \mathcal{D} \setminus E_1$. We set $I(a) = I_1(a) + I_2(a)$, where

$$I_i(a) = \iint_{E_i} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|f(z)|^2)^2} (1-|\varphi_a(z)|^2)^p \, dA(z), \quad i = 1, 2.$$

(i) For any $z \in \partial\mathcal{D}_k$, by a simple computation we get

$$B(z) = -\frac{z - z_k}{1-\overline{z}_k z} \prod_{\substack{n=1\\n\neq k}}^{\infty} \frac{\frac{z_n - z_k}{1-\overline{z}_k z_n} - \frac{z-z_k}{1-\overline{z}_k z}}{1 - \overline{(\frac{z_n - z_k}{1-\overline{z}_k z_n})} \frac{z-z_k}{1-\overline{z}_k z}}, \tag{3}$$

and then it follows from [8, Lemma 1.4, p. 4 and Lemma 5.2, p. 309] and (1) that

$$
\begin{aligned}
|B(z)| &= \delta_1 \prod_{\substack{n=1\\ n\neq k}}^{\infty} \left| \frac{\frac{z_n-z_k}{1-\overline{z}_k z_n} - \frac{z-z_k}{1-\overline{z}_k z}}{1-\left(\overline{\frac{z_n-z_k}{1-\overline{z}_k z_n}}\right)\frac{z-z_k}{1-\overline{z}_k z}} \right| \geq \delta_1 \prod_{\substack{n=1\\ n\neq k}}^{\infty} \frac{\left|\frac{z_n-z_k}{1-\overline{z}_k z_n}\right| - \delta_1}{1-\delta_1\left|\frac{z_n-z_k}{1-\overline{z}_k z_n}\right|} \\
&\geq \delta_1 \frac{\prod_{\substack{n=1\\ n\neq k}}^{\infty} \left|\frac{z_n-z_k}{1-\overline{z}_k z_n}\right| - \delta_1}{1-\delta_1 \prod_{\substack{n=1\\ n\neq k}}^{\infty} \left|\frac{z_n-z_k}{1-\overline{z}_k z_n}\right|} > \frac{3\delta^2}{16}.
\end{aligned} \tag{4}
$$

Set $B_m(z) = \prod_{n=1}^{m} \frac{z_n-z}{1-\overline{z}_n z}, m \in \mathbb{N}$. Then $B_m(z)$ is an analytic function on $\overline{E}_2$. It follows from (4) that, for $z \in \partial\mathcal{D}_k$,

$$
|B_m(z)| \geq |B(z)| > \frac{3\delta^2}{16}\,. \tag{5}
$$

We also know that $|B_m(z)| = 1$ on $\partial\mathcal{D}$ and $B_m(z)$ has no zero points inside E_2. By the minimum principle and (5) we get

$$
|B_m(z)| \geq \frac{3\delta^2}{16}, \quad z \in E_2.
$$

Taking $m \to \infty$ we obtain that $|B(z)| \geq \frac{3\delta^2}{16}$ for any $z \in E_2$. Thus we have

$$
\begin{aligned}
I_2(a) &= \iint_{E_2} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|f(z)|^2)^2} (1-|\varphi_a(z)|^2)^p \, dA(z) \\
&\leq \iint_{E_2} \frac{|B'(z)|^2 |\log(1-z)|^2}{4|B(z)|^2|\log(1-z)|^2} (1-|\varphi_a(z)|^2)^p \, dA(z) \\
&\leq \frac{64}{9\delta^4} \iint_{E_2} |B'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dA(z).
\end{aligned} \tag{6}
$$

Since the zero points of $B(z)$ for $0 < p < 1$ satisfy

$$
\sum_{j=k+1}^{\infty} (1-|z_j|^2)^p \leq \frac{2^p\beta^p}{1-\beta^p} (1-|z_k|^2)^p, \quad k = 1, 2, \ldots,
$$

it follows from Theorem [11] above that $d\mu(z) = \sum_{n=1}^{\infty} (1-|z_n|^2)^p \delta_{z_n}$ is a bounded p-Carleson measure and that $B(z) \in Q_p$ by Theorem [7]. Thus, from (6), we have

$$
\sup_{a\in\mathcal{D}} I_2(a) \leq \sup_{a\in\mathcal{D}} \frac{64}{9\delta^4} \iint_{\mathcal{D}} |B'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dA(z) < \infty. \tag{7}
$$

(ii) Now we turn to estimate the integral $I_1(a)$. Define

$$
I_1^{(k)}(a) = \iint_{\mathcal{D}_k} \frac{|B'(z)|^2 |\log(1-z)|^2}{(1+|B(z)|^2|\log(1-z)|^2)^2} (1-|\varphi_a(z)|^2)^p \, dA(z)\,, \quad k = 1, 2, \ldots. \tag{8}
$$

Let $w = \frac{z-z_k}{1-\overline{z}_k z}$ and from (3) we have that $B(z) = -wP_k(w)$, where

$$
P_k(w) = \prod_{\substack{n=1\\ n\neq k}}^{\infty} \frac{\frac{z_n-z_k}{1-\overline{z}_k z_n} - w}{1-\left(\overline{\frac{z_n-z_k}{1-\overline{z}_k z_n}}\right)w}\,.
$$

By (4) and the minimum principle we obtain

$$
\frac{3\delta}{4} \leq \frac{\delta-\delta_1}{1-\delta\delta_1} \leq |P_k(w)| < 1, \quad |w| < \delta_1. \tag{9}
$$

Since $z_k, k = 1, 2, \dots,$ are real numbers, we have

$$1 - z = (1 - z_k)\frac{1 - w}{1 + \overline{z}_k w}. \tag{10}$$

Using the inequalities

$$\frac{1-\delta}{1+\delta} < \left|\frac{1 - w}{1 + \overline{z}_k w}\right| < \frac{1+\delta}{1-\delta}$$

for $|w| < \delta_1$ and, by (10), we get that there exist positive numbers $c_1 = c_1(\delta), c_2 = c_2(\delta)$ and $c(\delta)$ such that

$$|\log c_1(1 - z_k)| \leq |\log(1 - z(w))| \leq |\log c_2(1 - z_k)|, \quad |w| < \delta_1, \tag{11}$$

and

$$\left|\frac{\log c_2(1 - z_k)}{\log c_1(1 - z_k)}\right| \leq c(\delta), \quad k = 1, 2, \dots, \tag{12}$$

where the constant $c(\delta)$ depends only on δ. A computation shows that

$$\begin{aligned}(1 - |\varphi_a(z)|^2)^p &= \left(1 - \left|\frac{z_k - a}{1 - \bar{a}z_k}\right|^2\right)^p \frac{(1 - |w|^2)^p}{\left|1 + \overline{(\frac{z_k - a}{1 - a\bar{z}_k})}w\right|^{2p}} \\ &\leq (5/3)\left(1 - \left|\frac{z_k - a}{1 - \bar{a}z_k}\right|^2\right)^p \\ &= C(z_k, a, p),\end{aligned} \tag{13}$$

where $w = \frac{z - z_k}{1 - \overline{z}_k z}$ and $|w| < \delta_1$. From $B'(z) = -w'(z)(P_k(w) + wP_k'(w))$ we get

$$|B'(z)|^2 \leq 2|w'(z)|^2(|P_k(w)|^2 + |w|^2|P_k'(w)|^2),$$

which, with (8) and (13), gives

(14)

$$\begin{aligned}I_1^{(k)}(a) &= \iint_{\mathcal{D}_k} \frac{|B'(z)\log(1 - z)|^2}{(1 + |B(z)\log(1 - z)|^2)^2}(1 - |\varphi_a(z)|^2)^p \, dA(z) \\ &\leq C(z_k, a, p)\iint_{\mathcal{D}_k} \frac{(|P_k(w)|^2 + |wP_k'(w)|^2)|\log(1 - z)|^2}{(1 + |wP_k(w)\log(1 - z)|^2)^2}|w'(z)|^2 \, dA(z) \\ &\leq C(z_k, a, p)\left\{\iint_{|w|<\delta_1} \frac{|\log(1 - z(w))|^2}{(1 + |wP_k(w)\log(1 - z(w))|^2)^2} \, dA(w)\right\} \\ &\quad + C(z_k, a, p)\left\{\iint_{|w|<\delta_1} \frac{|wP_k'(w)\log(1 - z(w))|^2}{(1 + |wP_k(w)\log(1 - z(w))|^2)^2} \, dA(w)\right\} \\ &= C(z_k, a, p)\left\{I_{11}^{(k)} + I_{12}^{(k)}\right\}.\end{aligned}$$

By (9), (11) and (12) we obtain

$$
\begin{aligned}
I_{11}^{(k)} &= \iint_{|w|<\delta_1} \frac{|\log(1-z(w))|^2}{(1+|wP_k(w)\log(1-z(w))|^2)^2}\, dA(w) \\
&\le \iint_{|w|<\delta_1} \frac{|\log c_2(1-z_k)|^2}{(1+\frac{9}{16}\delta^2|w|^2|\log c_1(1-z_k)|^2)^2}\, dA(w) \\
&\le 2\pi \int_0^{\delta_1} \frac{|\log c_2(1-z_k)|^2}{(1+\frac{9}{16}\delta^2|\log c_1(1-z_k)|^2 t^2)^2}\, tdt \\
&\le \frac{16\pi}{9\delta^2}(c(\delta))^2\,.
\end{aligned} \tag{15}
$$

Since $P_k(w)$ is continuous in $|w| \le \delta_1$, from (9) we have

$$
\begin{aligned}
I_{12}^{(k)} &= \iint_{|w|<\delta_1} \frac{|wP_k'(w)\log(1-z(w))|^2}{(1+|wP_k(w)\log(1-z(w))|^2)^2}\, dA(w) \\
&\le \frac{4}{9\delta^2}\iint_{|w|<\delta_1} |P_k'(w)|^2\, dA(w) \\
&\le \frac{1}{2\delta^2}\iint_{\mathcal{D}} |P_k'(w)|^2(1-|w|^2)\, dA(w) \\
&\le b(\delta)\|P_k\|_{H^2}^2,
\end{aligned} \tag{16}
$$

where $b(\delta)$ is a constant depending only on δ and

$$\|P_k\|_{H^2}^2 = \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|P_k(re^{i\theta})|^2 d\theta \le 1.$$

Thus, from (14), (15) and (16), we have

$$
\begin{aligned}
I_1^{(k)}(a) &\le (10/3)\left(1-\left|\frac{z_k-a}{1-\bar{a}z_k}\right|^2\right)^p \left\{I_{11}^{(k)}+I_{12}^{(k)}\right\} \\
&\le C(\delta)\left(1-\left|\frac{z_k-a}{1-\bar{a}z_k}\right|^2\right)^p,
\end{aligned} \tag{17}
$$

where the constant $C(\delta)$ depends only on δ.

Since $d\mu(z) = \sum_{n=1}^{\infty}(1-|z_n|^2)^p\delta_{z_n}$ is a bounded p-Carleson measure in $\mathcal{D}$, from (8), (17) and Lemma 2.1 in [3] we have

$$\sup_{a\in\mathcal{D}} I_1(a) = \sup_{a\in\mathcal{D}}\sum_{k=1}^{\infty} I_1^{(k)}(a) \le C(\delta)\sup_{a\in\mathcal{D}}\sum_{k=1}^{\infty}\left(1-\left|\frac{z_k-a}{1-\bar{a}z_k}\right|^2\right)^p < \infty. \tag{18}$$

Combining (2), (7) and (18) we get

$$\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}}(f^{\#}(z))^2(1-|\varphi_a(z)|^2)^p\, dA(z) < \infty,$$

that is, $f \in M_p^{\#}$. Thus $f \in \bigcap_{0<p<\infty} M_p^{\#}$. This completes the proof of the Theorem. □

Corollary 1. *$Q_p^{\#} \subsetneqq M_p^{\#}$ for any p, $0<p<\infty$.*

Corollary 2. *There exists a non-normal function f such that*

$$(f^{\#}(z))^2(1-|z|^2)dA(z)$$

is a Carleson measure.

Remark 1. The theorem above shows that the class $Q_p^{\#}$ cannot be characterized by a general bounded p-Carleson measure for any p, $0 < p < \infty$, and this is an obvious difference between the analytic and meromorphic case. It is necessary to point out that our proof uses some ideas employed in [10].

Remark 2. Recently, the classes $Q_p^{\#}$ have been characterized by other Carleson measures that are called Carleson type measures, which were introduced by the second author in his paper [14]. Also, some further relations between the classes $Q_p^{\#}$ and $M_p^{\#}$ are described there; one of them is that $Q_p^{\#} = N \cap M_p^{\#}$ for all p, $0 < p < \infty$.

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